PROBLEMAS TERMOELASTICOS Y DE LA TEORIA DE DIFUSION: SU SOLUCION POR METODOS ANALITICOS APROXIMADOS Y POR EL METODO DE ELEMENTOS FINITOS

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NUMERICAL EXPERIMENTS ON THE DETERMINATION OF UNSTEADY STATE TEMPERATURE DISTRIBUTION IN DOMAINS OF COMPLICATED BOUNDARY SHAPE

by

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P R O L O G O

En la presente publicación se han compaginado dos trabajos realizados entre investigadores del IMA y del Centro Atómico Bariloche (C.N.E.A.), en un plazo de 60 días[1].

Es de destacar el hecho de que la realización de los dos trabajos se hizo por acuerdo directo entre investigadores de ambas partes, sin mediar un vínculo o convenio oficial. Solamente -y esto es lo más importante- el deseo de realizar mejores y mayores logros para el país y su tecnología básica.

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Director, IMA

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1. En ambos casos los investigadores han cumplido, además, con otras numerosas obligaciones diversas.

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A B S T R A C T

This paper deals with a comparison of analytical and finite element results in the case of an unsteady heat-conduction problem in simply and doubly connected plates of regular polygonal shape.

A numerical solution is obtained by means of the powerful finite element method and the results are shown to agree with an approximate conformal mapping-variational technique previously developed by the first author and coworkers.

Introduction

The popular finite element method permits practically any problem of mechanics to be tackled in a unified mathematical fashion suitable for solution on a digital computer.

Applications of the method to the solution of a wide variety of problems ranging from prostheses design to the dynamic analysis of naval and aerospace structures are known nowadays.

Most of the studies performed to evaluate the relative accuracy of the finite element method deal with comparisons between alternative formulations of the method or between exact analytical results and finite element values obtained in the case of rather simple geometries.

Reference /1/ presents a completely independent comparison for diffusion-type problems in cases where the geometry of the domain is complex. However, the problems attacked in that study are of the steady state type.

The goal of the present paper consists in the comparison of finite element results and analytical solutions in the case of unsteady temperature fields in domains of regular polygonal shape (simply and doubly connected regions).

The analytical approach was presented in References /2/ and /3/, and only a brief treatment will be presented here.

It is important to point out that only the case of the simply connected square plate admits an exact solution[1].

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1. Available in standard textbooks.

It is felt, on the other hand, that comparative studies such as the present one serve a useful purpose since cylindrical or prismatic configurations of odd-shaped cross sections are used in a multitude of engineering applications: a graphite brick of a gas-cooled nuclear reactor, solid propellant rocket motors, waveguides commonly used in microwave engineering, etc.

The Complex Variable-Variational Approach

If the conductivity is constant throughout the body and no heat is generated within the solid, two dimensional heat flow in an isotropic solid is governed by the equation:

$$\alpha \nabla^2 T = \frac{\partial T}{\partial t} \qquad (1)$$

where α : thermal diffusivity and ∇^2: two dimensional Laplacian operator.

Let the functional relations:

$$T\,[L(x,y) = 0, t] = 0 \qquad (2a)$$

$$T\,[x,y,t\,]_{t=0} = T_0 \qquad (2b)$$

where $L(x,y) = 0$ is the functional relation which defines the boundary configuration, represent the boundary and initial conditions, respectively, for the thermo-mechanical system under study.

If

$$z = x + iy \qquad (3)$$

and

$$z = f\,(\xi) \qquad (4)$$

represents the analytic function which maps the given domain in the z-plane onto a unit circle in the ξ-plane (Figure 1), it can be shown /2/

that the temperature field in the ξ-plane can be described by the functional relation:

$$T(\xi,\bar{\xi}) \simeq \sum_{m=1}^{M} A_{om}\, J_o[\beta_{om}\,(\xi.\bar{\xi})^{1/2}]\,.\; e^{-\alpha\gamma_{om}t} \qquad (5)$$

where J_o is the Bessel function of the first kind and order cero; the β_{om}'s are the roots of $J_o(x)$ and the γ_{om}'s are the separation constants obtained when one substitutes $T = T_1\,(x,y)\,\Theta\,(t)$ in (1).

The expansion coefficients A_{om}'s are given by the Fourier-Bessel expansion:

$$A_{om} = \frac{2\,T_o}{\beta_{om}\,.J_1\,(\beta_{om})} \qquad (6)$$

Reference /3/ deals with an approximate analytical formulation applicable to doubly connected domains when the governing differential system is also defined by (1) and (2). Clearly, instead of equation (2a) one has now:

$$T\,[L_i\,(x,y) = 0,t] = 0 \qquad (i=1,2) \qquad (7)$$

where the subscript 1 denotes the inner boundary and 2 the outer.

Numerical results are presented in Reference 3 for square and pentagonal plates with concentric circular holes.

The Finite Element Solution

The results were obtained using a finite element code[1]. The domain is subdivided into triangular elements and a linear variation of the temperature field is assumed inside the element.

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1. Developed at Centro Atómico Bariloche, Comisión Nacional de Energía Atómica.

It was decided, in view of the symmetry, to consider the subdomains defined by

$$0 \leqslant \phi \leqslant \pi/s \tag{8}$$

where $\underline{s}$ is the order of the polygon with the conditions (See Figure 2):

$$T = 0 \quad \text{on "a"} \tag{9a}$$

and

$$\frac{\partial T}{\partial n} = 0 \quad \text{on "}a_p\text{" and "b"} \tag{9b}$$

where $\underline{n}$ denotes the outer normal to the subdomain.

In the case of doubly connected plates (Figure 8) the condition (9a) was also applied to the circular portion of the boundary limited by the relation (8).

Figures 3 and 4 show the element distribution in the case of simply connected shapes, used in the present analysis[1] (256 elements and 153 nodal points).

Calculations were performed for:

a) square, pentagonal and hexagonal simply connected plates.
   and
b) square and pentagonal plates with concentric circular holes.

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1. The element distribution was similar when analyzing the doubly connected plates. The number of elements and nodes was the same.

Comparison of Results and Conclusions.

Table 1 shows a comparison of results obtained[1] by means of:

a) exact solution,
b) complex variable-variational formulation /2/, and
c) the finite element method, in the case of a simply connected square domain.

The agreement is indeed quite good (the mean square error of b) and c) with respect to the exact formulation is less than 1%).

Figure 5 depicts graphically the results of Table 1.

Figures 6 and 7 display comparisons of results for pentagonal and hexagonal shapes. The agreement can be again considered as quite satisfactory (one observes a more marked difference in the case of a hexagonal domain for $t > 0.6$ hr).

Figure 9 thru 12 depict several comparisons between results obtained in Reference /3/ and values calculated using the finite elements technique (results are plotted in terms of the dimensionless parameters r/a_p and $\tau = \alpha t/a_p^2$ in order to compare with results published in Reference /3/.

It may be concluded that the agreement is, in general, quite reasonable.

No claim of originality is made in the present paper. On the other hand the present study probably constitutes one of the first "experimental" evaluations of the relative precision of the finite elements method in the case of unsteady diffusion phenomena in complicated boundary shapes.

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1. In the case of the simply connected shapes: $b = 1\text{ft}$ and $\alpha = 0.45\ \text{ft}^2/\text{h}$ in order to compare with calculations performed in Reference /2/.

References

1. P.A.A.Laura and A.J.Faulstich, Jr., "Unsteady Heat Conduction in Plates of Polygonal Shape", International Journal of Heat and Mass Transfer, Vol. 11, pp. 297-303, 1968.

2. P.A.A.Laura, J.A.Reyes and R.E.Rossi, "A Comparison of Analytical and Numerical Solutions in Heat Conduction Problems", Nuclear Engineering and Design, Vol. 31, pp. 379-382, 1974.

3. P.A.A.Laura and R.Ercoli, "A Solution of the Unsteady Diffusion Equation in an Arbitrary, Doubly Connected Region", Nuclear Engineering and Design, Vol.23, pp. 1-9, 1972.

T / T_0			
t (hours)	Exact Solution	Conformal Mapping-Variational Approach /2/	Finite Elements
0.1	0.928	0.925	0.933
0.2	0.654	0.649	0.659
0.3	0.426	0.422	0.428
0.4	0.274	0.271	0.274
0.5	0.176	0.174	0.175
0.6	0.113	0.112	0.112
0.7	0.0724	0.0716	0.0718
0.8	0.0464	0.0459	0.0459
0.9	0.0298	0.0295	0.0294
1.0	0.0191	0.0189	0.0188

TABLE 1: Values of T/T_0 at the Center of a Square Plate: Comparison of Results.

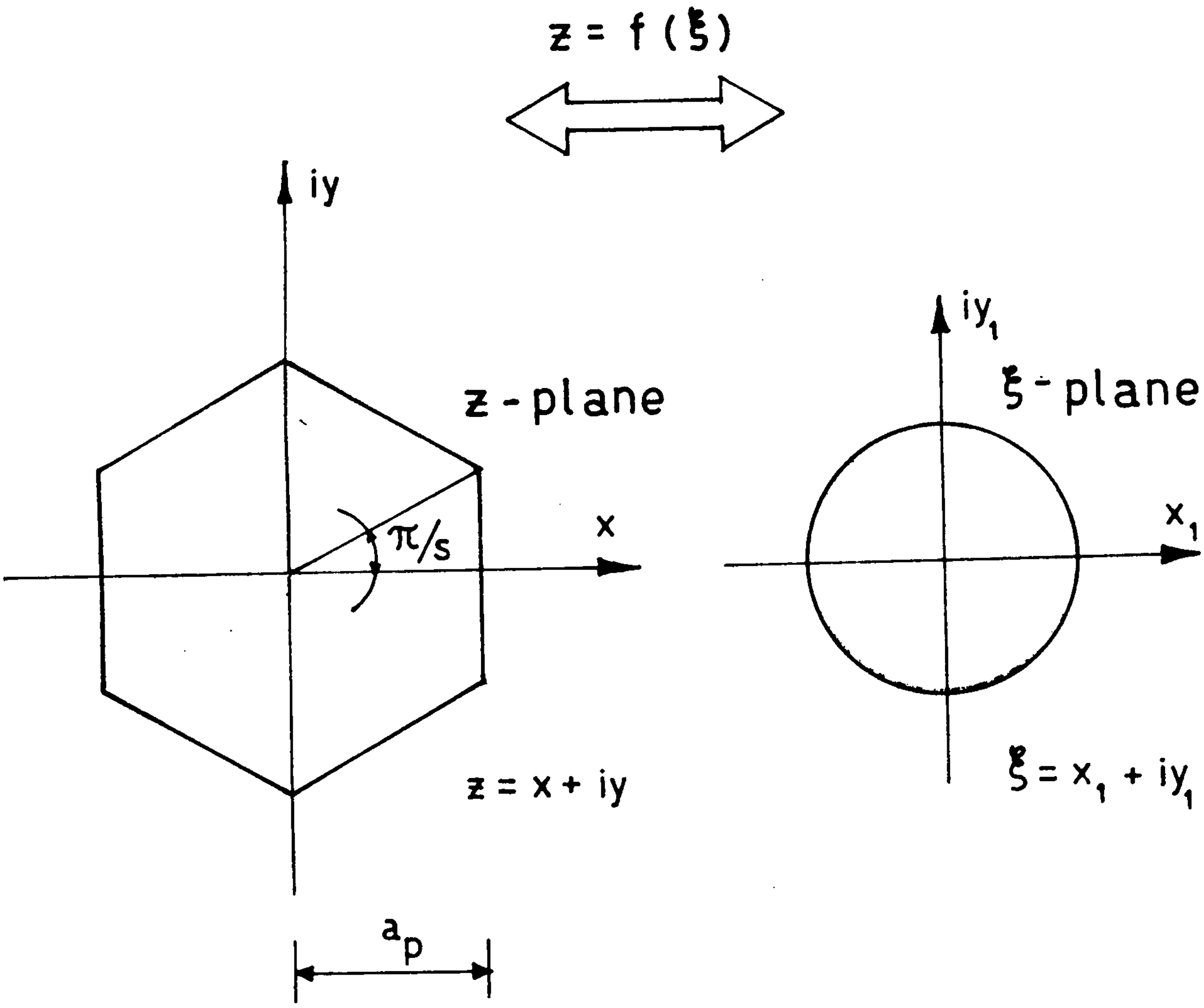


FIGURE 1 - CONFORMAL MAPPING OF A REGULAR POLYGONAL SHAPE.

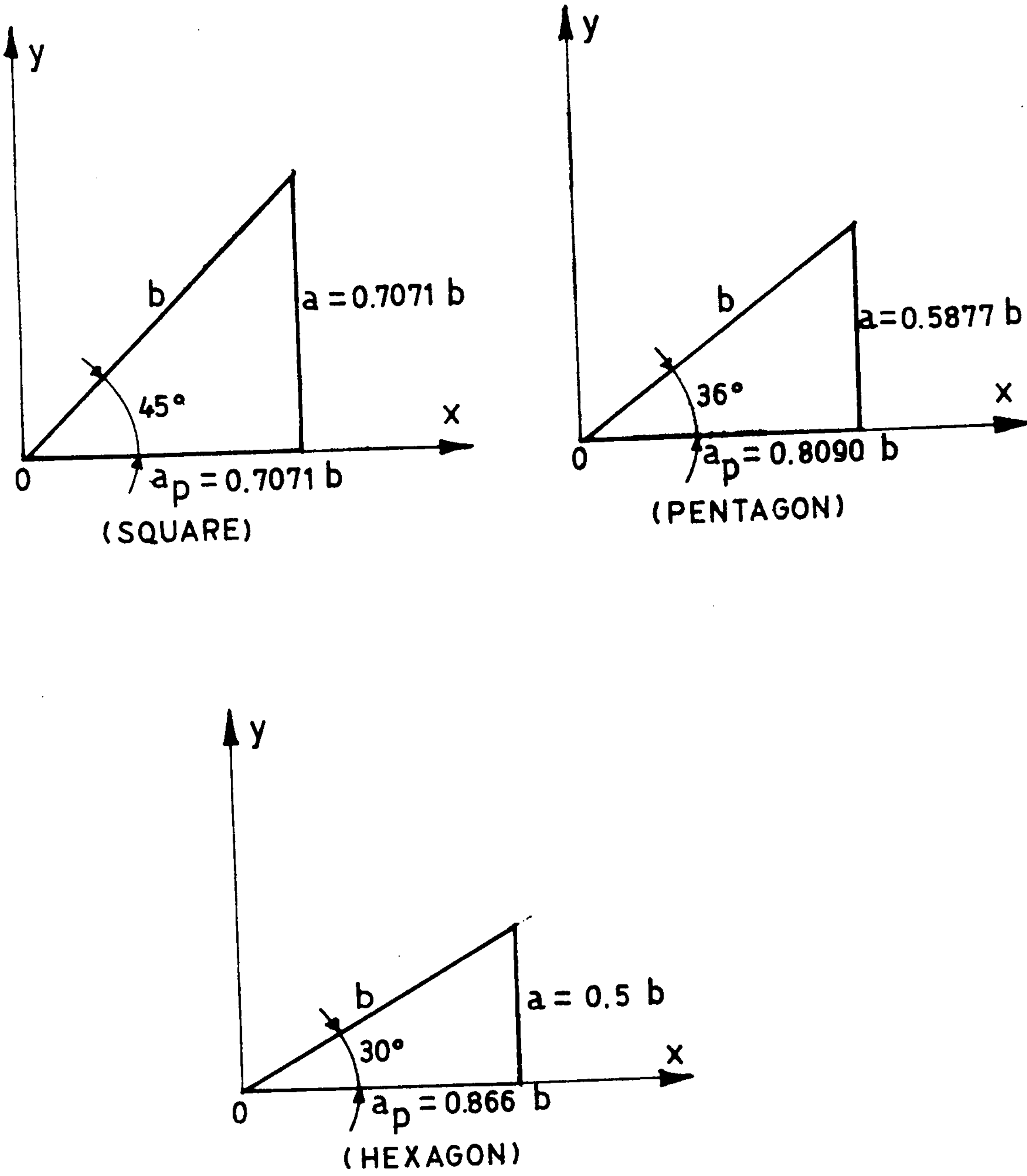
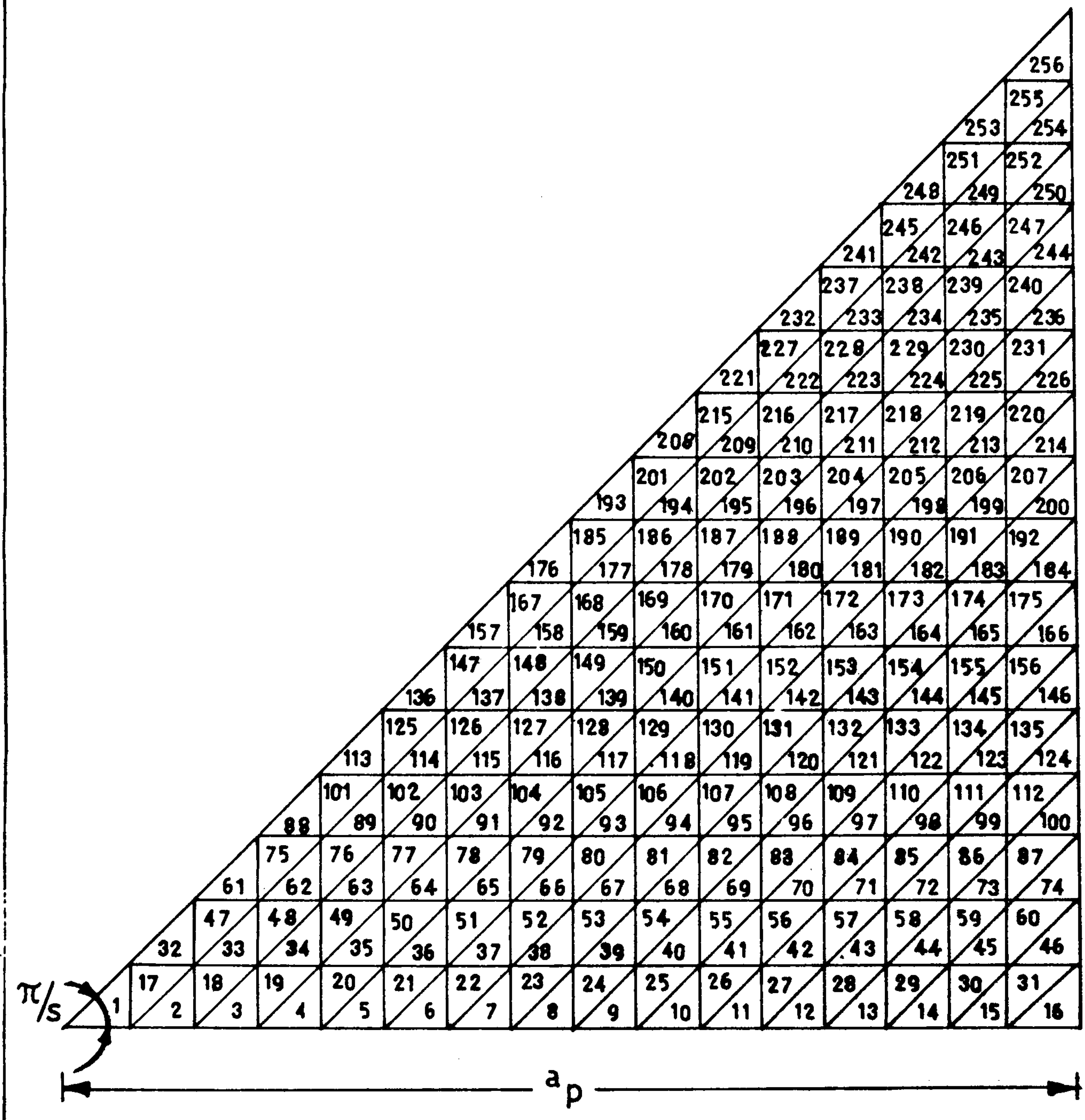


FIGURE 2 - SUBDOMAINS OF THE REGULAR POLYGONAL PLATES.



F I G U R E 3 - ELEMENT DISTRIBUTION.

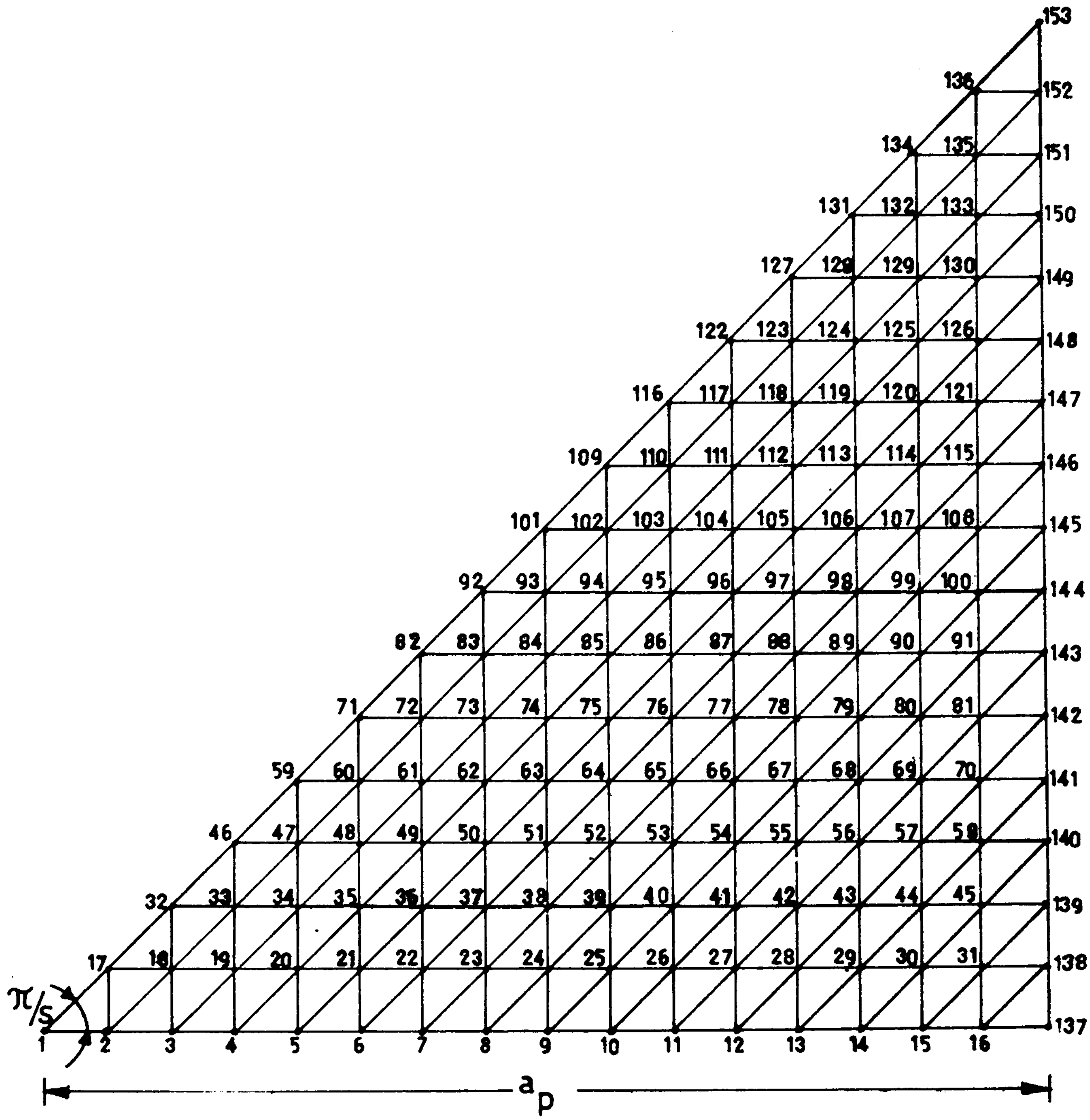


FIGURE 4 – NODAL PATTERN.

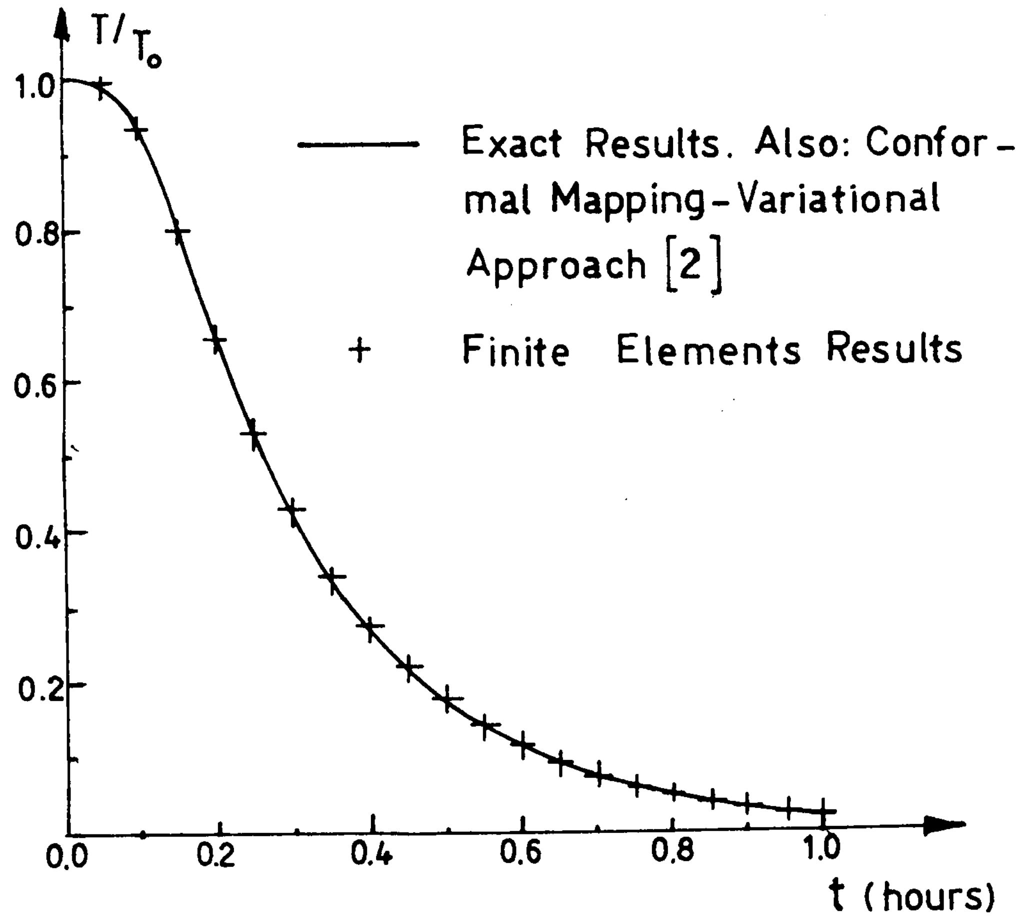


FIGURE 5 - VARIATION OF THE DIMENSIONLESS TEMPERATURE PARAMETER T/T_o AT THE CENTER OF A SQUARE PLATE.

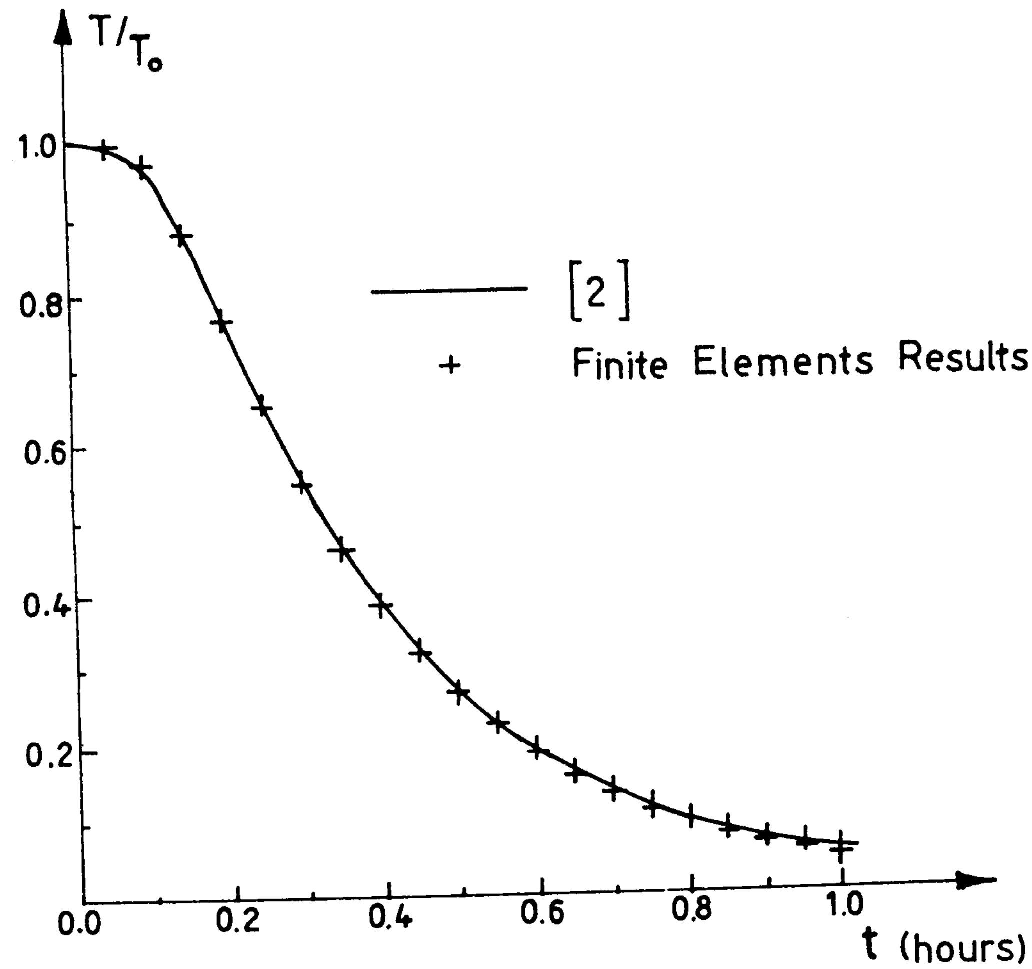


FIGURE 6 - VARIATION OF THE DIMENSIONLESS TEMPERATURE PARAMETER T/T_o AT THE CENTER OF A PENTAGONAL PLATE.

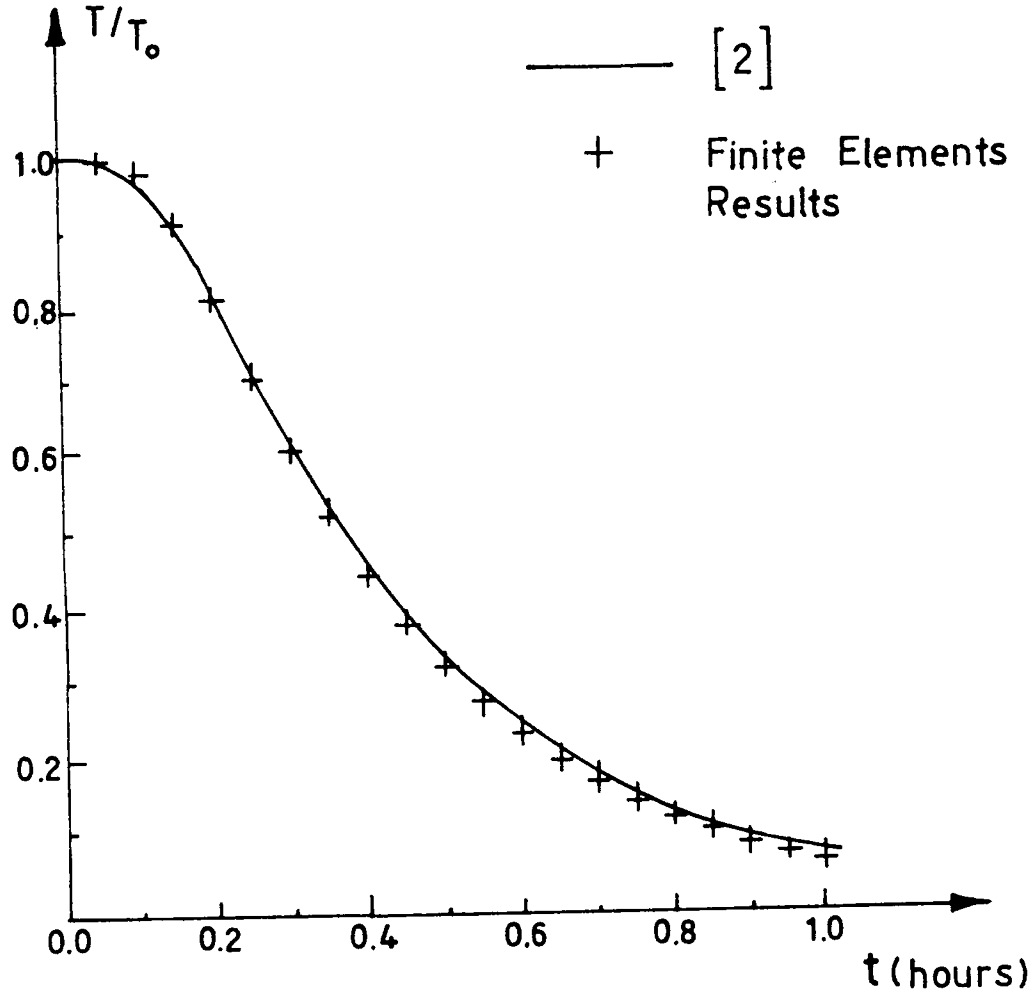
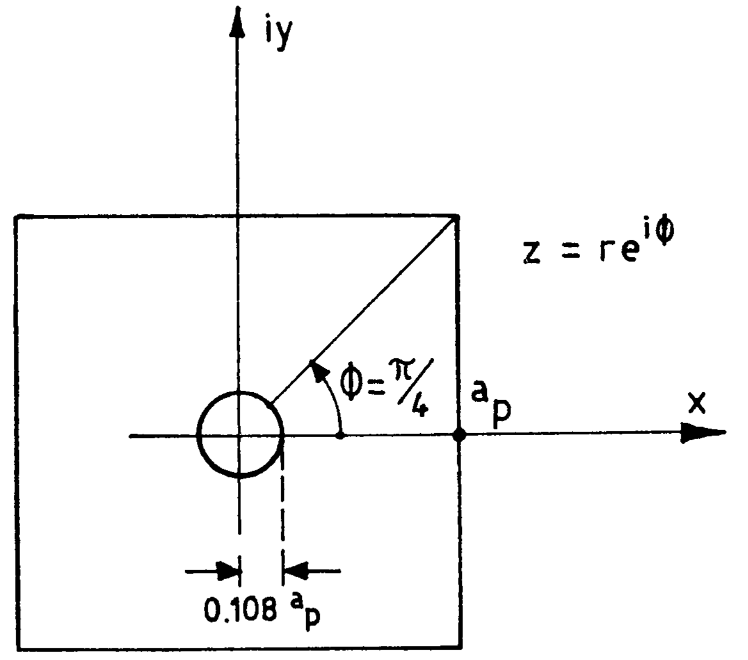
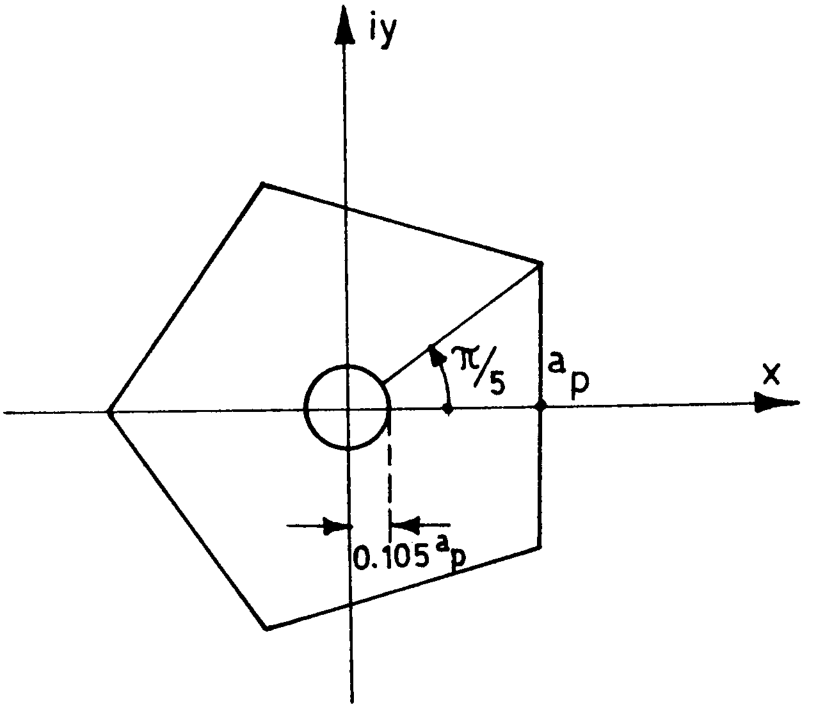


FIGURE 7 - VARIATION OF THE DIMENSIONLESS TEMPERATURE PARAMETER T/T_o AT THE CENTER OF A HEXAGONAL PLATE.



A) CASE I



B) CASE II

FIGURE 8 - DOUBLY CONNECTED REGIONS (a_p : APOTHEM OF THE REGULAR POLYGON)

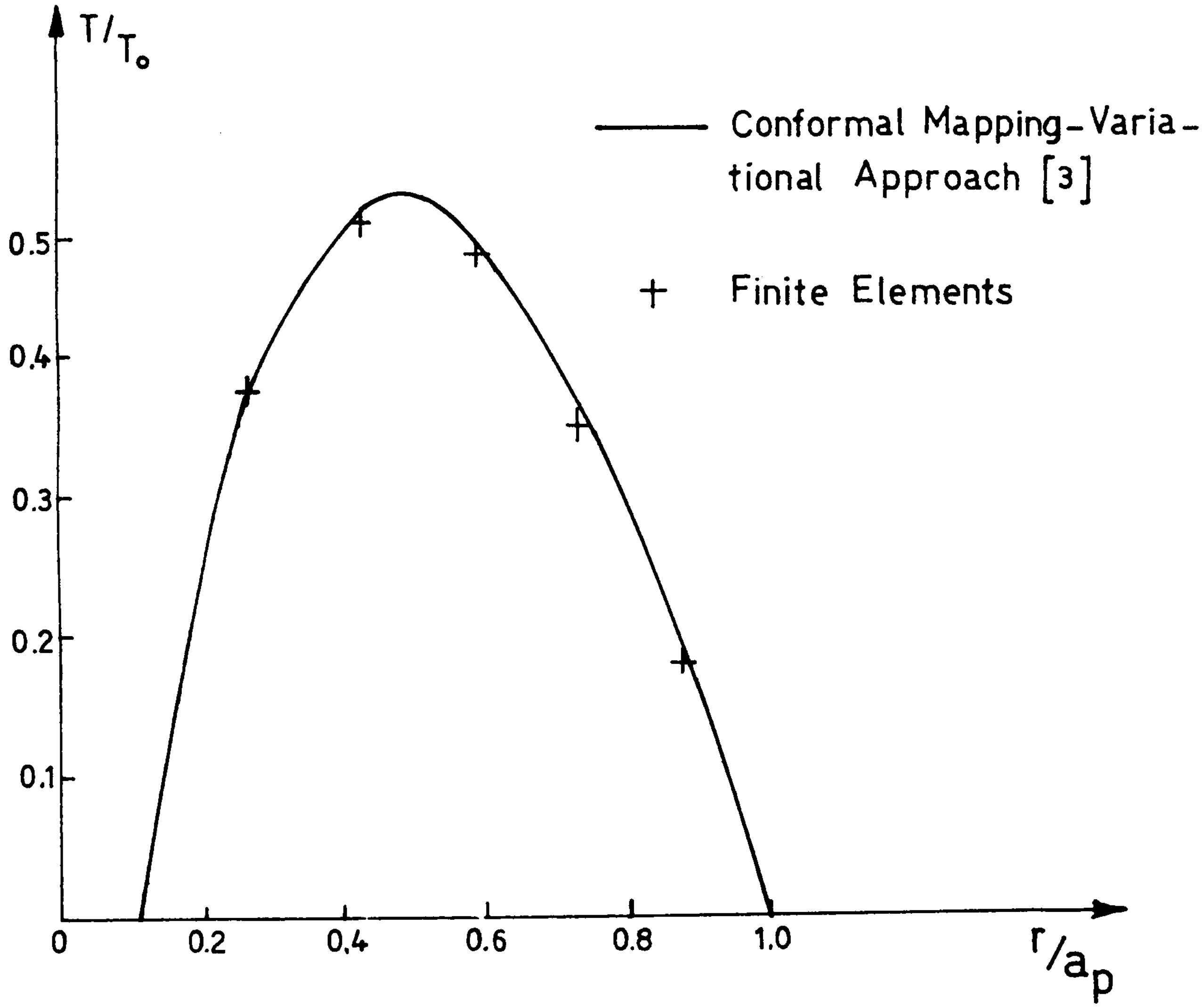


FIGURE 9 - SQUARE SHAPE WITH CIRCULAR PERFORATION: VARIATION OF T/T_o AS A FUNCTION OF r/a_p FOR $\nu = 0$; $\tau = 0.10$

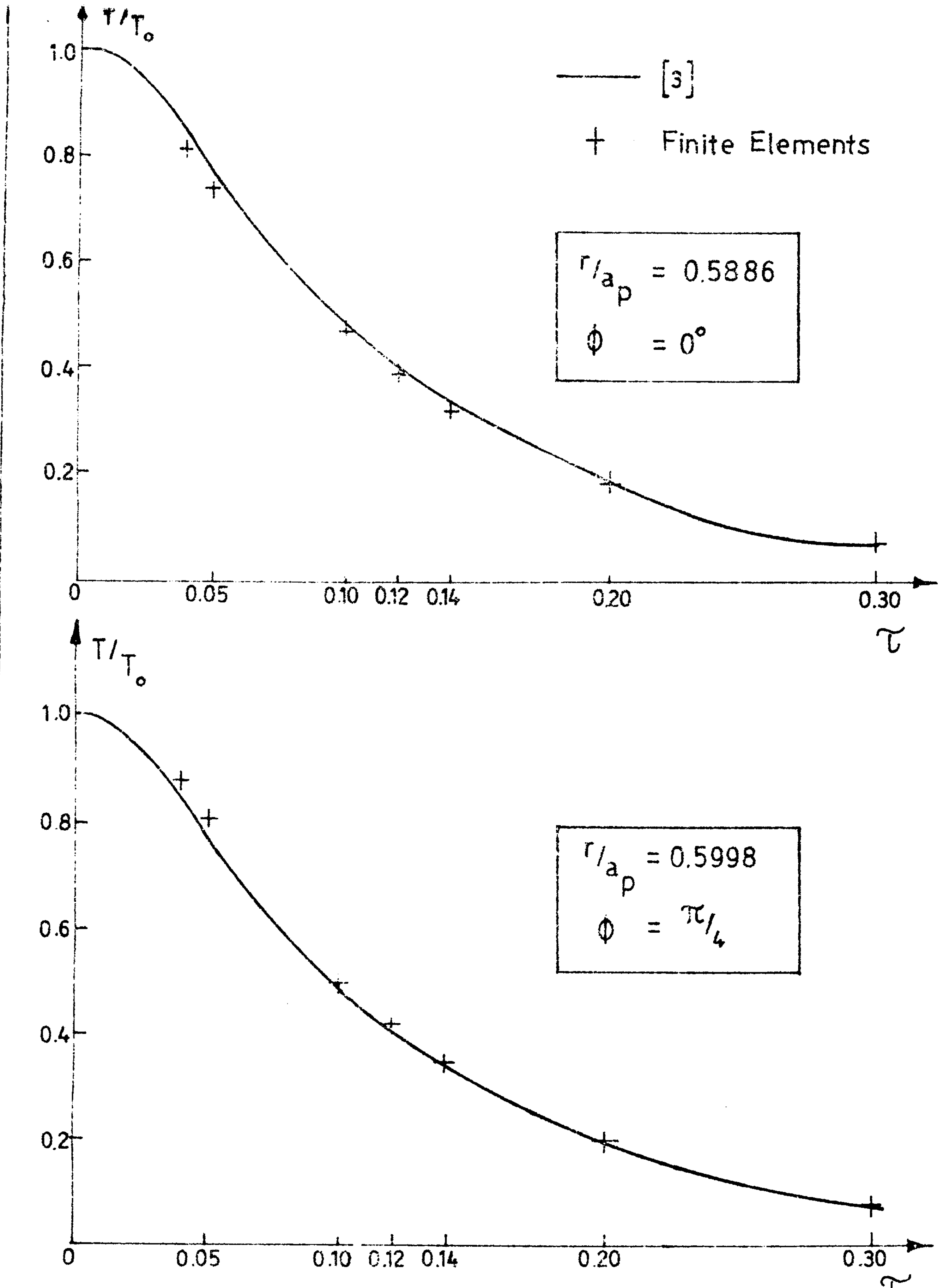


FIGURE 10 - SQUARE SHAPE WITH CIRCULAR PERFORATION: VARIATION OF T/T_0 FOR TWO DIFFERENT POINTS OF THE DOMAIN.

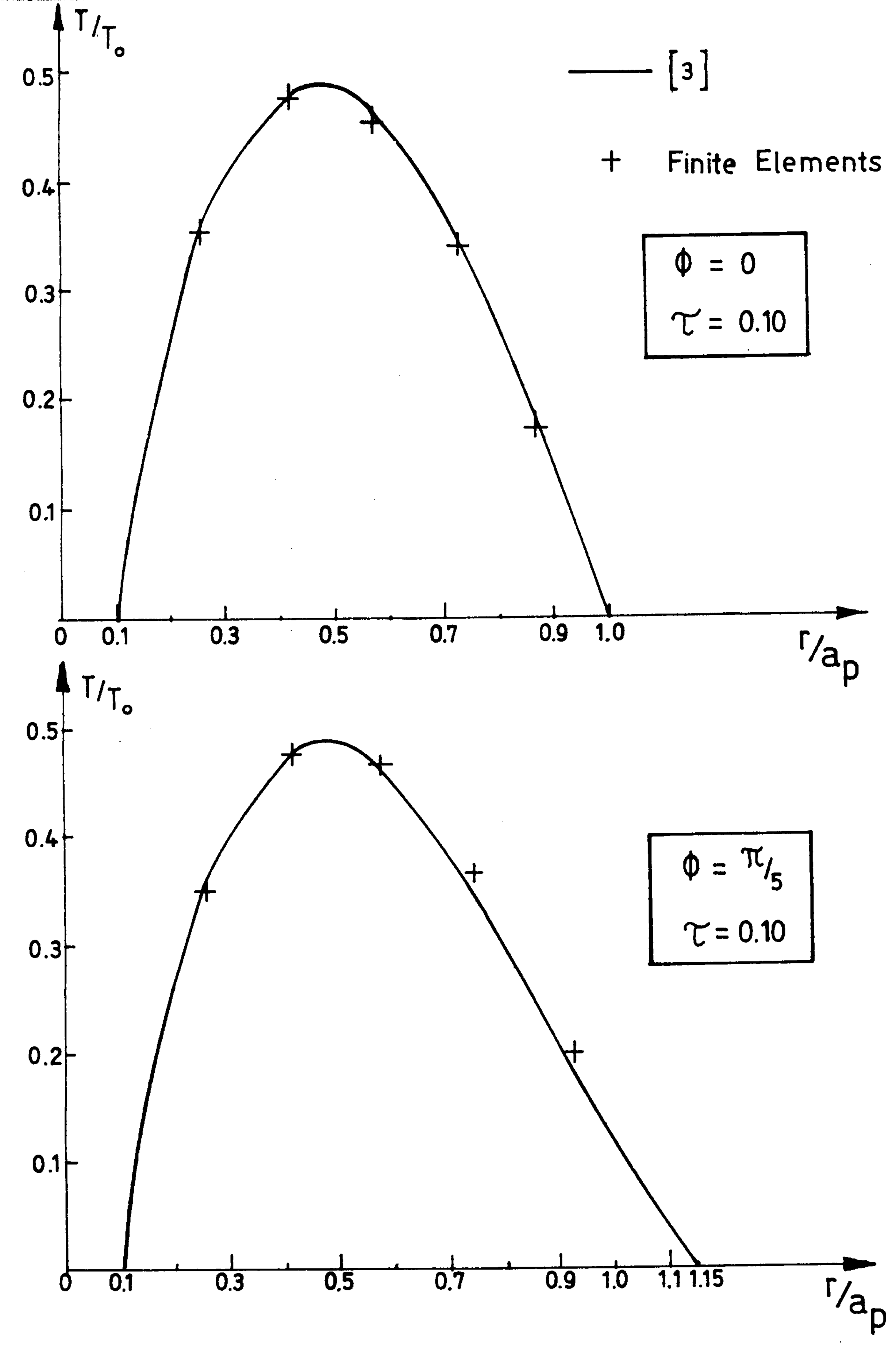


FIGURE 11 - PENTAGONAL SHAPE WITH CIRCULAR PERFORATION: VARIATION OF T/T_o AS A FUNCTION OF r/a_p FOR A PARTICULAR VALUE OF τ