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A Comparative Review of Certain Gauge Theories of the Gravitational Field

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Abstract

A general formal analysis is made trying to obtain a better understanding and greater synthesis of the mathematical structure of the gravitational field's gauge theories. Under this approach, some misstatements appearing in current theories are detected. A theory based on the direct product groups $T(4) \times GL(4)$ and $T(4) \times O(1, 3)$ is suggested (in contrast to those using the Poincaré group, semidirect product). Such a theory corrects the just-mentioned deficiencies possessing the attributes of the preceding ones.

§(1): *Introduction*

The extension of the idea of gauge field to non-Abelian groups due to Yang and Mills awoke general interest in the analysis of the gravitational field as a gauge field employing appropriate non-Abelian groups. From the pioneer work of Utiyama to the present time many different formulations have been proposed {see for example [1, 2, 62, 5-7, 9, 17-21, 33-35, 40, 41, 51-54, 57, 59]}, a lot of them founded on approaches that appear to be different.

The gauge field theories may be advantageously interpreted under a geometrical point of view using the notion of fiber bundle space [4, 8, 10, 12, 36, 37, 39, 42, 43, 52, 56]. Our purpose is to make a comparative study of some of the theories mentioned, analyzing them within such a formal mathematical approach, determining their interrelations, and evaluating how they adhere to the

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suggested scheme. As a corollary, a theory based on the direct product groups $T(4) \times GL(4)$ and $T(4) \times O(1,3)$ is proposed.

Tacitly or not, those theories are based on certain principal fiber bundles with specific structural groups. Different theories would correspond to different fiber spaces which may be linked by natural homomorphisms establishing a relation between the respective theories. The aim of the present approach is then to obtain a more synthetical and unified view of those theories which would permit one, as a beginning, to make evident their basic formal aspects. In a certain sense this approach resembles that of Trautman [11] but with different purposes. The underlying structure corresponds to the mathematical notion of a "category," its "objects" being the theories, and its "morphisms" the natural homomorphisms that may exist between the principal fiber bundles.

In Section 2, the fundamental aspects of the theoretical structure to be applied to the specific cases is described, and similarly in Section 3 concerning the homomorphisms between fiber bundles. In Section 4 these general ideas are applied to some representative gauge theories of the gravitational field. In Section 5 the natural homomorphisms existing between the corresponding fiber spaces are analyzed. Finally in Section 6 we summarize the more significant conclusions to which this approach has led.

§(2): *Geometrical Structure of the Gauge Theories*

In this section we will develop an abstract point of view, essentially similar to that carried out by Cho in [4]. In the final part the notion of "connection adapted to the principal fiber bundle P" will be introduced with the purpose of presenting a special Lagrangian defined on the bundle P with properties different from those usually employed.

2.1. *Some Mathematical Preliminaries*

Let E be a *fiber bundle space* [15, 16, 10, 12] of base M and typical fiber F. Its projection onto the base M will be designated by Π and the fiber over $x \in M$ by $F_x = \Pi^{-1}(x)$. The property of local triviality for an open covering $\{U_\alpha\}$ of M will be materialized through the family of homeomorphisms $\phi_\alpha : \Pi^{-1}(U_\alpha) \to U_\alpha \times F$. For each $x \in U_\alpha$, ϕ_α will induce an homeomorphism $\phi_x^\alpha : F_x \to F$ between the fiber over x and the typical fiber. Let G be the group of homeomorphisms of F such that if $x \in U_\alpha \cap U_\beta$ the homeomorphism $\Psi_{\alpha\beta} = \phi_x^\alpha \circ (\phi_x^\beta)^{-1}$ of F onto itself (transition function) is an element of G depending continuously on x.

A *principal fiber bundle P* is a fiber bundle for which the structural group G coincides with the typical fiber F, the group acting on itself by left translations.

In these conditions it is easy to define [15] a right action $\tilde{R}$ of G on P:

$$P \times G \ni (p, g) \xrightarrow{\tilde{R}} \tilde{R}_g p \in P \tag{2.1}$$

This action conserves the fibers F_x acting on them effectively and freely and is such that for every $p \in P$ and $g \in G$ it holds $\phi_\alpha(p \cdot g) = \phi_\alpha(p) \cdot g$, where $p \cdot g$ means $\tilde{R}_g p$.

An equivalent and direct definition of principal fiber bundle is given in [16].

We endow P with a *connection* given by the *connection form* ω, a $\mathcal{G}$-valued 1-form defined on P, $\mathcal{G}$ being the Lie algebra of G [16, 15]. (We are interested in the case in which M and F are differentiable manifolds, G a Lie group and Π, ϕ_α, ϕ_x^α, $\tilde{R}$ differentiable functions.) At every point $p \in P$ the tangent space $T_p(P)$ is then univocally decomposed in a direct sum of horizontal H and vertical V spaces: $T_p(P) = H_p \oplus V_p$. For any $X \in T_p$ it holds $\omega(X) = 0 \Leftrightarrow X \in H_p$. ω possesses the property of being G adjoint, that is,

$$(\tilde{R}_a)^*\omega(X) \overset{\text{def}}{=} \omega(d\tilde{R}_a \cdot X) = Ad(a^{-1}) \cdot \omega(X) \tag{2.2}$$

A connection form in P may be conveniently expressed for our purposes by a family of $\mathcal{G}$- valued 1-forms [16, 4], each defined on an open set of the base M.

For an open covering $\{U_\alpha\}$ of M and some local trivialization of P, let us define for every α the local cross section σ_α: $U_\alpha \to P$ by $\sigma_\alpha(x) \overset{\text{def}}{=} \phi_\alpha^{-1}(x, e)$, where e is the identity of G.

Let θ be the $\mathcal{G}$-valued left-invariant canonical form of G, that is,

$$\theta(v) \overset{\text{def}}{=} dL_{a^{-1}} \cdot v, \qquad a \in G, \qquad v \in T_a(G) \tag{2.3}$$

where R_a, L_a are right and left actions on G.

For every nonvoid intersection $U_\alpha \cap U_\beta$ the $\mathcal{G}$-valued 1-form is introduced as

$$\theta_{\alpha\beta} \overset{\text{def}}{=} \Psi_{\alpha\beta}^* \theta \tag{2.4}$$

and for each α the 1-form ω_α on U_α by

$$\omega_\alpha = \sigma_\alpha^* \omega \tag{2.5}$$

In these conditions the following proposition holds ([16], p. 66):

The forms $\theta_{\alpha\beta}$ and ω_α are subject to the conditions

$$\omega_\beta = Ad(\Psi_{\alpha\beta}^{-1})\,\omega_\alpha + \theta_{\alpha\beta} \tag{2.6}$$

on $U_\alpha \cap U_\beta$. Conversely, for every family of $\mathcal{G}$-valued 1-forms $\{\omega_\alpha\}$ each defined on U_α and satisfying the preceding conditions, there exists one and only one connection form ω in P that gives rise to the $\{\omega_\alpha\}$ in the manner described.

Finally the definitions of the curvature and torsion of a connection are given. The *curvature form* Ω is the $\mathcal{G}$-valued 2-form on P that is the exterior covariant differential of ω, that is, for every $X, Y \in T_p(P)$:

$$\Omega(X, Y) \overset{\text{def}}{=} D\omega \overset{\text{def}}{=} d\omega(hX, hY) \tag{2.7}$$

where h is the projection operator on H.

It satisfies the Cartan structure equation [15, 16]:

$$d\omega(X, Y) = -\tfrac{1}{2}\,[\omega(X), \omega(Y)] + \Omega(X, Y) \qquad X, Y \in T_p(P) \tag{2.8}$$

The torsion form is defined for linear connections, those for which P is in particular the bundle $L(M)$ of linear frames of M and $G = GL(n, R)$. Let $\hat{\theta}$ be the canonical form of P, the $\mathbb{R}^n$-valued 1-form such that for every $X \in T_p(P)$ it gives the components on the frame p of the projection $d\Pi \cdot X$ on $T_{\Pi(p)}(M)$. Then the *torsion form* Θ is the $\mathbb{R}^n$-valued 2-form obtained as the covariant exterior differential of $\hat{\theta}$:

$$\Theta(X, Y) \overset{\text{def}}{=} D\hat{\theta}\,(X, Y) \overset{\text{def}}{=} d\hat{\theta}\,(hX, hY) \tag{2.9}$$

2.2. *Gauge Fields and Fiber Bundles*

We now proceed to describe the gauge field's geometrical structure [3, 4, 8-10] particularizing M to the space-time manifold, and n being the dimension of G.

2.2.1. *Physical Preliminaries.* The mathematical representations employed by physicists to express the idea of classical fields are usually not determined. The same field generally admits an infinite set of equivalent representations. Each particular representation is associated with the notion of a particular *gauge.* Two members of this equivalence class are then tied by what is called a *gauge transformation.* These gauge transformations present the characteristic of being dependent on the space-time point under consideration. At any space-time point two different fields of the same type are comparable, for instance, through their representation originated by fixing a particular gauge in that point. Nevertheless the comparison of field values at different space-time points through their representation on one gauge loses all its meaning: such a comparison would not be *gauge invariant* since it suffices to consider a gauge transformation different from the identity only on a neighborhood of one point not containing the other to obtain a different result in the comparison. This concerns, in particular, the definition of the derivative as a limit of the incremental quotient.

It becomes necessary then to define an operation that allows us to compare fields at different points and that is gauge invariant.

Such an operation consists of a *parallel transport* of the field value from one point to the other along a path joining them in such a way that the comparison should be finally *made at the same point.* It requires the introduction of an additional *compensating field* of a different type designated the *gauge field* or *gauge potential.* The *covariant differentiation* will be that which is defined by taking into account a particular parallelism.

These ideas admit a natural interpretation by means of the notion of fiber bundle space: (a) A "gauge" principal bundle P, its structural group G being the *group of gauge transformations*, and its base M the space-time manifold. A cross section σ on P will be a particular *gauge.* A *gauge transformation* is then a G-valued function defined on a certain region of M, associated to the corresponding change of the cross section. A connection on P will allow us to establish a *parallel transport*, being in that way linked to the *gauge field.* (b) A bundle associated with the principal bundle P [15, 16] for a specific representation of the structural gauge group G, the "field" bundle. Its cross sections will be the possible fields of that type existing on regions of the space-time M. The connection on P introduces here a *parallel transport of fields*, and in consequence their *covariant derivative.* Two representations of the same field for two different gauges will be called *gauge-related representations.*

2.2.2. *Parallel Transport and Connection.* A connection on P given by the connection form ω will then establish a parallel transport with the preceding interpretation. Let σ be a local cross section of P on the open set $U \subset M$. On U we define a $\mathcal{G}$-valued 1-form $A_{(\sigma)}$ as the reciprocal image of ω by σ:

$$A_{(\sigma)} \stackrel{\text{def}}{=} \sigma^*\omega \tag{2.10}$$

These are similar to the ω_α of (2.5). If ξ_b, $b = 1, \ldots, n$ (n being the dimension of G) is a basis of the Lie algebra $\mathcal{G}$ of G, $A_{(\sigma)}$ admits the decomposition

$$A_{(\sigma)} = \xi_b A^b_{(\sigma)} \tag{2.11}$$

where the $A^b_{(\sigma)}$ are n real 1-forms (usually called gauge potentials or compensating fields).

For any two cross sections σ and σ' on U related by

$$\sigma'(x) = \sigma(x)\,a(x), \qquad a(x) \in G \tag{2.12}$$

the corresponding forms $A_{(\sigma)}$ and $A_{(\sigma')}$ are connected by an expression similar to (2.6) [16]:

$$A_{(\sigma')} = Ad(a^{-1}) \cdot A_{(\sigma)} + a^*\theta \tag{2.13}$$

where $a(x)$ plays the role of the transition function $\Psi_{\alpha\beta}(x)$ of (2.4) and (2.6). For any basis $\{\xi_b\}$ of $\mathcal{G}$, using (2.3) and the definition of a^*, expression (2.13) becomes ([4], gauge transformation of gauge potentials)

$$\xi_b A^b_{(\sigma')} = Ad(a^{-1}) \cdot \xi_b A^b_{(\sigma)} + dL_{a^{-1}(x)} \circ d_x a \tag{2.14}$$

where $d_x a$ is the differential of the mapping $x \to a(x) \in G$.

2.2.3. *Natural Bases.* On each tangent space to P, two types of natural basis useful for later interpretations and calculations may be defined as usual:

(a) "Direct product" basis of $T_p(P)$:

$$\{\lambda_B\} \equiv \{\bar{e}_i, \xi_b^*\}, \qquad B = 1, \ldots, 4, \ldots, 4+n; \quad i = 1, \ldots, 4; \quad b = 1, \ldots, n \tag{2.15}$$

For any trivialization of $\Pi^{-1}(U)$, $\Pi(p) \in U$, let σ be a local trivial cross section containing p. Then, for any basis $\{e_i\}$ of $T_{\Pi(p)}(M)$ we define

$$\bar{e}_i = d\sigma \cdot e_i \tag{2.16}$$

On the other hand ξ_b^* will be the particular values $p \in P$ of the fundamental fields on P (or Killing fields for the action of G on P, relative to a given basis of $\mathcal{G}$).

(b) "Horizontal-vertical" basis of $T_p(P)$

$$\{\varphi_B\} \equiv \{\hat{e}_i, \xi_b^*\}, \qquad B = 1, \ldots, 4, \ldots, 4+n; \quad i = 1, \ldots, 4; \quad b = 1, \ldots, n \tag{2.17}$$

A connection being given on P, $\{\hat{e}_i\}$ is now the horizontal lift of the basis $\{e_i\}$ of $T_{\Pi(p)}(M)$ and the $\{\xi_b^*\}$ have the same meaning as in the direct product basis. $\{\hat{e}_i\}$ is then a basis of H_p and $\{\xi_b^*\}$ a basis of V_p.

The brackets of the direct product basis are

$$[\lambda_i, \lambda_j] \equiv [\bar{e}_i, \bar{e}_j] = c_{ij}^k \bar{e}_k, \qquad ([e_i, e_j] = c_{ij}^k e_k)$$
$$[\lambda_b, \lambda_c] \equiv [\xi_b^*, \xi_c^*] = f_{bc}^d \xi_d^*, \qquad ([\xi_b, \xi_c] = f_{bc}^d \xi_d) \tag{2.18}$$

$\{\xi_b\}$: basis of $\mathcal{G}$; f_{bc}^d: structure constants of $\mathcal{G}$

$$[\lambda_b, \lambda_i] \equiv [\xi_b^*, \bar{e}_i] = 0$$

The last relation is justified by the definition of the $\bar{e}_i$ and because the bracket is the Lie derivative of such fields on the direction of the fields ξ_b^*.

For the horizontal-vertical basis the brackets are

$$[\varphi_i, \varphi_j] \equiv [\hat{e}_i, \hat{e}_j] = c_{ij}^k \hat{e}_k - F_{ij}^b \xi_b^*$$
$$[\varphi_b, \varphi_c] \equiv [\xi_b^*, \xi_c^*] = f_{bc}^d \xi_d^* \tag{2.19}$$
$$[\varphi_b, \varphi_i] \equiv [\xi_b^*, \hat{e}_i] = 0$$

The first is obtained by considering that the projection of the bracket is the bracket of the projection. The third is justified for the same reasons as the last of equations (2.18).

The coefficients of the transformation B_A^C of both bases, $\lambda_A = B_A^C \varphi_C$ are

$$B_A^C: \begin{cases} B_j^i = \delta_j^i \\ B_a^d = \delta_a^d \\ B_a^i = 0 \\ B_j^d = \{A_{(\sigma)}\}_j^d \qquad \text{with } \{A_{(\sigma)}\}_j^d = A_{(\sigma)}^d(e_j) \end{cases} \tag{2.20}$$

The last expression is easily obtained by the use of $\bar{e}_j = B_j^d \xi_d^* + B_j^i \hat{e}_i$, the connection form ω, formula (2.10), and the definition (2.16).

2.2.4. *Covariant Derivative.* The covariant derivative operator D_i is the horizontal lift $\hat{e}_i$ of the basis vector e_i, that is,

$$D_i \overset{\text{def}}{=} \hat{e}_i = \bar{e}_i - \{A_{(\sigma)}\}_i^d \xi_d^* \tag{2.21}$$

2.2.5. *Field Intensities Tensor and Curvature of Connection.* From the following formula, consequence of the structure equation (2.8) for horizontal vectors [16],

$$\omega([X, Y]) = -2\Omega(X, Y) \tag{2.22}$$

and from the first of equations (2.19) we get

$$F_{ij}^b = 2\Omega^b(\hat{e}_i, \hat{e}_j) = 2\Omega^b(\bar{e}_i, \bar{e}_j) \qquad \text{with } \Omega = \xi_b \Omega^b \tag{2.23}$$

A simple calculation of the components of Ω^b on the direct product basis gives the Ω_{ij}^b as the only nonvanishing components, and with (2.23) one obtains (putting A_j^b for $\{A_{(\sigma)}\}_j^b$):

$$F_{ij}^b = 2\Omega_{ij}^b = \partial_i A_j^b - \partial_j A_i^b + A_i^c A_j^d f_{cd}^b - c_{ij}^k A_k^b \tag{2.24}$$

This is the usual expression of the "gauge field intensities tensor" of the current gauge theories, calculated from the potentials $A_i^b = \{A_{(\sigma)}\}_i^b$. They verify the well-known Bianchi identities [15, 16].

2.2.6. *A Metric for the Fiber Bundle.* Let $\overset{M}{g}$ and $\overset{G}{g}$ be two Riemannian structures for M and G, respectively. It is assumed that $\overset{G}{g}$ is also left invariant. We shall endow P with a Riemannian structure g "adapted" to $\overset{M}{g}$ and $\overset{G}{g}$. Given a connection P, this will be the current block diagonal metric making orthogonal the horizontal and vertical subspaces ([3, 4, 8, 9], etc.).

For the decomposition $T_p(P) = H_p \oplus V_p$ let

$$g = g_{H_p} \oplus g_{V_p}, \qquad H_p \perp V_p \tag{2.25}$$

where for $X, Y \in T_p(P)$

$$g_{H_p}(hX, hY) \overset{\text{def}}{=} \overset{M}{g}_{\Pi(p)}(d\Pi \cdot X, d\Pi \cdot Y) \tag{2.26}$$

and

$$g_{V_p}(vX, vY) \stackrel{\text{def}}{=} \overset{G}{g}_e((r_p \circ v) \cdot X, (r_p \circ v) \cdot Y) \tag{2.27}$$

where the r_p are the canonical isomorphisms between the V_p and $\mathcal{G}$, and v is a vertical projection.

For later purposes it will be useful that the metric g be invariant for the right actions of G and P, or similarly that the $\tilde{R}_a$ be isometries:

$$\begin{gathered} g_{pa}(d\tilde{R}_a X, d\tilde{R}_a Y) = g_p(X, Y) \\ \forall X, Y \in T_p, \quad \forall a \in G, \quad \forall p \in P \end{gathered} \tag{2.28}$$

This is equivalent to the requirement that $\overset{G}{g}$ be also right invariant or

$$\begin{gathered} \overset{G}{g}_e(Ad(a^{-1})\gamma_1, Ad(a^{-1})\gamma_2) = \overset{G}{g}_e(\gamma_1, \gamma_2) \\ \forall \gamma_1, \gamma_2 \in T_e(G) = \mathcal{G}, \quad \forall a \in G \end{gathered} \tag{2.29}$$

2.2.7. *Gauge Fields Dynamics.* All that has been just described in this paragraph corresponds to the "kinematics" of the gauge fields. Let us now see to which dynamical laws they must be restricted in order to be physically determined.

Here we shall establish such laws through a variational formulation defined on the fundamental entity of work, that is, in the principal fiber bundle P. This is certainly not the only way to obtain the desired laws. This kind of procedure has been successfully used by many authors [14, 3, 4, 8, 9] etc., and is specially suitable for the comparative study, which is the purpose of this work.

To such end it will be convenient to endow the bundle P as a differentiable manifold with a linear connection, but at the same time trying to make sure that this aggregate does not introduce new artificial variables, from the point of view of their later physical interpretation. In other words we must try to make sure that the linear connection of P be defined only by the available data of the base M and the typical fiber G, such being the respective metrics and/or linear connections.

There are at least two ways to accomplish this. The first one is to assign to P the Christoffel-Cartan metric and torsionless linear connection corresponding to the g defined in Section 2.2.6 [13, 14, 3, 4, 8, 9]. Keeping in mind our purpose—namely, to study the gravitational field as a gauge field—this linear connection produces somewhat restricted results because it excludes the gravitational theories with torsion such as the Einstein-Cartan theory, which has been of great recent interest [26-32, 45-47, 57].

The second way is to endow P with the more flexible linear connection here suggested.

Let M possess a metric $\overset{M}{g}$ and a general linear connection $\overset{M}{\Gamma}$; the group G a biinvariant metric $\overset{G}{g}$, and the corresponding Christoffel-Cartan connection $\overset{G}{\Gamma}$. This implies that in G the parallel transport be invariant right and left.

The proposed linear connection for P (at the same time endowed with a connection form ω) will be designated "the adapted connection" to the linear connections of M, $\overset{M}{\Gamma}$, and G, $\overset{G}{\Gamma}$. It will satisfy the following four natural conditions:

(i) *Conservation of the vertical or horizontal character of vectors:* Any horizontal or vertical vector does not lose its character by parallel transport on any path.

(ii) *Compatibility with the base connection:* The "projection" on the base of the parallel transport of a vector along a path contained on P coincides with the parallel transport on the base M of the projected vector along the projected path.

(iii) *Compatibility with the group connection:* The parallel transport of vertical vectors on vertical paths, taken onto G by the diffeomorphism ϕ_x between the fiber over x and the typical fiber G, coincides with the parallel transport of the initial vector so projected along the equally projected path. Because of the left invariance of the parallel transport on G this property is not dependent on the chosen ϕ_x.

(iv) *Horizontal parallelism of the fundamental fields:* The covariant derivative of the fundamental fields of P on any horizontal direction vanishes.

Note that the condition (ii) implies that the parallel transport of horizontal vectors along vertical paths does not depend on the vertical path.

The foregoing conditions *univocally determine on P a linear connection starting from* $\overset{M}{\Gamma}$ and $\overset{G}{\Gamma}$. We arrive at this conclusion without difficulties by simply analyzing the connection components on suitable bases of the tangent spaces to P. Of course, these are the horizontal-vertical bases of Section 2.2.3. In particular for these bases it holds that the only nonvanishing components of the adapted connection Γ are those corresponding to $\overset{M}{\Gamma}$ and $\overset{G}{\Gamma}$.

Additionally, from the four conditions (i)-(iv) one obtains an important corollary: *the parallel transport on P is invariant by the right action* $\tilde{R}_g$ *of G.* It is also easily verified using components that *the linear adapted connection of P is metric if and only if the starting linear connections* $\overset{M}{\Gamma}$ *and* $\overset{G}{\Gamma}$ *are metric.*

For the Christoffel-Cartan linear connection of P, condition (i) holds exceptionally. As a counterpart it is to be expected that the linear "adapted" connection will possess torsion in general.

On the fiber space P two forms of total dimension $4 + n$ present themselves as natural candidates for a variational principle. The first one is $\overline{R}\eta$, $\overline{R}$ being the scalar curvature of the adapted connection and η the volume element of P corresponding to the metric g ([38], p. 120; [15], p. 92).

The second one is $\Omega\Lambda{*}\Omega$, where $*\Omega$ is the dual of the curvature form by the element of volume η [38, 15]. The exterior product is built up with $\overset{G}{g}_e$, the metric of $\mathcal{G}$.

Both are invariant by the action of G on P. The first because $\tilde{R}_g$ is an isometry and then η and $\overline{R}$ are invariants. The second because Ω and $*\Omega$ are of the adjoint type [15, 16], the definition of the exterior product and the adopted hypothesis (2.29).

The proposed total dimension form for the variational principle is then a linear combination of both:

$$\mathcal{L} = \alpha\overline{R}\eta + \beta\Omega\Lambda{*}\Omega, \qquad \alpha, \beta \in \mathbb{R} \tag{2.30}$$

An easy calculation on the horizontal-vertical basis leads to

$$\overline{R}\eta = (\overline{R}_M + \overline{R}_G)\eta \tag{2.31}$$

where $\overline{R}_M$ is the scalar curvature of M, and $\overline{R}_G$ is the scalar curvature of G (constant), and to

$$\begin{aligned}\Omega\Lambda{*}\Omega &= \tfrac{1}{2}\, \overset{G}{g}_{ab}\, \overset{M}{g}{}^{kp}\, \overset{M}{g}{}^{ep}\, \Omega^a_{ke}\, \Omega^b_{pq}\, \eta \\ &= \tfrac{1}{8}\, \overset{G}{g}_{ab}\, \overset{M}{g}{}^{kp}\, \overset{M}{g}{}^{eq} F^a_{ke} F^b_{pq}\, \eta = \tfrac{1}{8}\, F^2 \eta\end{aligned} \tag{2.32}$$

where use was made of (2.24).

The Lagrangean finally takes the form

$$\begin{aligned} S &= \int_P \mathcal{L} = \int_P \left(\alpha\overline{R}_M + \alpha\overline{R}_G + \frac{\beta}{8}F^2\right)\eta \\ &= V_G \int_M \left[\alpha\overline{R}_M + \alpha\overline{R}_G + \frac{\beta}{8}F^2\right]\overset{M}{\eta}\end{aligned} \tag{2.33}$$

where $\overset{M}{\eta}$ is the volume form of M.

The last equality expresses the invariance of $\mathcal{L}$ with the action of the group. The constant V_G, the volume of the group, may certainly be infinite but its role in (2.33) is only formal.

The action S depends on the adapted connection Γ of P, on the metric g of P, and on the connection form ω (gauge field). The first two are at the same time dependent on the respective metrics and connections of M and G. As usual it is assumed that $\overset{G}{\Gamma}$ and $\overset{G}{g}$ are given (for instance for semisimple groups $\overset{G}{g}$ may be the Killing metric and $\overset{G}{\Gamma}$ its Christoffel-Cartan connection.) Then we get

$$S = \hat{S}(\Gamma, g, \omega) = S(\overset{M}{\Gamma}, \overset{M}{g}, \omega) \tag{2.34}$$

The variations of this action will refer to the last equality.

Expression (2.33) presents the common aspect of a variational principle corresponding to the unified theory of a gauge field interacting with the gravitational field. The last field obeys more general equations than those of Einstein (for example, Einstein-Cartan equations if $\overset{M}{\Gamma}$ is metric). This is not the case for the usual Lagrangians in gauge theories using the fiber bundle formalism, built upon the Christoffel-Cartan connection of P [14, 3, 4, 8, 9]. The second term on the last integrand of (2.33) looks like a cosmological constant. For $\alpha = 1, \beta = -2$, a $\overset{M}{\Gamma}$ metric, and no torsion the variational principles of [4, 8, 9] etc. are obtained.

If the gauge field is in particular the gravitational field, a case to be later considered, (2.33) becomes a Lagrangian with linear and quadratic term on the curvature tensor, with cosmological constant and admitting torsion. In such a way it generalizes the Mansouri-Chang theory [9], which is obtained for $\alpha = 1$, $\beta = -2$ and, $\overset{M}{\Gamma}$ metric, and no torsion. This seems to be a basically sound theory from the physical point of view [25]. Under these hypothesis about $\overset{M}{\Gamma}$, with $\alpha = 1$ and $\beta = 0$ from (2.33) we obtain Einstein's theory with a cosmological term and in the absence of matter; if $\alpha = 0$ and $\beta = 1$ (Weyl-type Lagrangian),[2] (2.34) leads through the Palatini variational procedure to the Stephenson-Yang theory [48, 18], questioned from the point of view of its physical relevance and internal coherence [23, 24]. Lagrangians of this type have also been used by Hamamoto [49] and many other authors.

§(3): *Relations between Theories*

In this section, certain generic relations which may be established between two gauge field theories will be described. Such relations are supported on the concept of homomorphism between the respective principal fiber bundles underlying each theory [16, 15, 11].

Let (P_i, M_i, G_i, Π_i) $i = 1, 2$ be two principal fiber bundles of bases M_i, structural groups G_i, and projections Π_i. A homomorphism between them is given by a pair (f, φ),

$$f: P_1 \longrightarrow P_2: \text{ differentiable mapping}$$

$$\varphi: G_1 \longrightarrow G_2: \text{ Lie groups homomorphism}$$

such that

$$f(p_1 a_1) = f(p_1)\varphi(a_1) \tag{3.1}$$

for every $p_1 \in P_1$ and $a_1 \in G_1$. As under these assumptions f takes each fiber of P_1 onto a fiber of P_2, a differential mapping $\bar{f}: M_1 \to M_2$ is also induced by f

[2] "Yang-Mills type" theories according to Fairchild [50].

between the respective bases. Thus the following diagram commutes:

$$\begin{array}{ccc}
P_1 \times G_1 & \xrightarrow{f \times \varphi} & P_2 \times G_2 \\
\tilde{R}_1 \downarrow & & \downarrow \tilde{R}_2 \\
P_1 & \xrightarrow{f} & P_2 \\
\Pi_1 \downarrow & & \downarrow \Pi_2 \\
M_1 & \xrightarrow{\bar{f}} & M_2
\end{array}$$

On the present cases we shall have $M_1 = M_2 = M$ (space-time) and $\bar{f}$ the identity on M.

If $\bar{f}$ is a diffeomorphism, a connection on P_1 induces univocally a connection on P_2:

(i) Let $p_2 \in P_2$ such that $p_2 = f(p_1)$, $p_1 \in P_1$.

The horizontal subspace on p_2, $H_{p_2} \subset T_{p_2}(P_2)$ will be defined as

$$H_{p_2} \overset{\text{def}}{=} (df)_{p_1} \cdot H_{p_1} \tag{3.2}$$

(ii) Take now an arbitrary $p_2 \in P_2$. As $\bar{f}$ is a diffeomorphism between the bases, the fiber containing p_2 is the image of one fiber of P_1, that is, there exists $p_1 \in P_1$ and $g_2 \in G_2$ such that

$$p_2 = \tilde{R}_{g_2} \cdot f(p_1) \tag{3.3}$$

Then we define

$$H_{p_2} = d\tilde{R}_{g_2} \cdot H_{f(p_1)} \tag{3.4}$$

Without difficulty one proves that this definition does not depend in the particular choice of p_1, and that the so-induced horizontal spaces on P_2 correspond correctly to a connection [15].

Expression (3.1) implies that df takes vertical fields of P_1 on vertical fields of P_2. That means, if $\{\overset{1}{\xi}{}^*_{b_1}\}$ and $\{\overset{2}{\xi}{}^*_{b_2}\}$ are, respectively, the fundamental fields of P_1 and P_2 chosen for their horizontal-vertical basis, we shall have

$$df \cdot \overset{1}{\xi}{}^*_{b_1} = W^{b_2}_{b_1} \overset{2}{\xi}{}^*_{b_2}, \qquad W^{b_2}_{b_1} \in \mathbb{R} \tag{3.5}$$

where $b_i = 1, \ldots, n_i$ one dimensions of G_i; $i = 1.2$.

We now proceed to determine the connection 1-forms $\overset{2}{A}_{(\sigma_2)}$ induced by the $\overset{1}{A}_{(\sigma_1)}$ [$\overset{i}{A}_{(\sigma_i)} = \sigma_i^* \omega_i$ after (2.10); where σ_i is the local cross section of P_i, $i = 1, 2$].

Let $f(\sigma_1)$ be the local cross section on P_2 corresponding to the local section σ_1 on P_1. If $\overset{i}{A}{}^{b_i}_{(\sigma_i)}$, $i = 1, 2$, and the components of the 1-forms on the chosen basis of the respective Lie algebras (real-valued 1-forms on M), we get the following result easily verified from the definitions, formulas (2.21) and (3.5).

Lemma. The real 1-forms $\overset{2}{A}{}^{b_2}_{[f(\sigma_1)]}$ induced for $f(\sigma_1)$ are related to the $\overset{1}{A}{}^{b_1}_{(\sigma_1)}$ in a similar manner as the components of the vertical vectors transfered to P_2 by df, or

$$\overset{2}{A}{}^{b_2}_{[f(\sigma_1)]} = W^{b_2}_{b_1} \overset{1}{A}{}^{b_1}_{(\sigma_1)} \tag{3.6}$$

The W^a_b will be called "transfer coefficients."

Given an arbitrary local section σ_2' of P_2 linked to $f(\sigma_1)$ by $\sigma_2'(x) = f[\sigma_1(x)]\, a(x)$, formulas (2.13), and (2.14) allow us to obtain $\overset{2}{A}{}^{b_2}_{(\sigma_2')}$.

Finally we wish to point out that these relations between gauge theories present a structure of "concrete category." The "objects" are the theories themselves and the "morphisms," the existent homomorphisms between the underlying principal bundles.

A *concrete category* ([53], p. 64) is a class of elements Q, the *objects* of the category, together with two functions U and hom satisfying two axioms. The first function U assigns to each object Q of the category as set $U(Q)$, called the "underlying set" of Q. The second function assigns to every pair of objects Q and R a set $\hom(Q, R)$ whose elements are functions f from the set $U(Q)$ to the set $U(R)$. When $f \in \hom(Q, R)$ we say that f is a *morphism* of the category with domain Q and codomain R. The two axioms are as follows:

(i) For each object Q, the identity function of Q belongs to $\hom(Q, Q)$.
(ii) For any three objects Q, R, T,

$$f \in \hom(Q, R) \quad \text{and } g \in \hom(R, T) \Longrightarrow (g \circ f) \in \hom(Q, T) \tag{3.7}$$

From the definition it is implied that the composition of morphisms is associative.

It becomes clear that a theory T is an object and the corresponding principal bundle P is the underlying set $U(T)$.

§(4): *Some Gauge Theories of the Gravitational Field*

In this section, a comparative synthesis will be made of some gauge theories of the gravitational field ("objects" of the category). According to the general framework developed in Section 2, taking into account the underlying structural group and the corresponding principal fiber bundle, we shall describe critically each theory making a simultaneous comparison with similar theories previously proposed.

4.1. *Theories Based on the Group of Translations T(4)*

The fiber bundle in this case will be the product $P = M \times T(4)$.

Such a class of theories has been discussed for instance by Cho [5] (see also the references quoted there) and by Hayashi and Nakano [17].

The fiber over x of P will be interpreted as the set of all possible tetrads of coordinates assignable to $x \in M$. It is clear that not every local cross section σ on $U \subset M$. will be a "coordinate cross section." To this end $Pr_2\,[\sigma(U)]$ must be diffeomorphic to the open set U.

Let the canonical basis of the Lie algebra of $G = T(4)$ be $\{\xi_b\}$ $b = 1, 2, 3, 4$ and the corresponding Killing vectors $\{\xi_b^*\}$. A connection on P will be conceived through the local 1-forms $\xi_b A^b_{(\sigma)}$. For the present Abelian group the transformation law of their components is, according to (2.14),

$$\{A_{(\sigma')}\}_i^b = \{A_{(\sigma)}\}_i^b + \partial_i \lambda^b(x), \qquad i, b = 1, 2, 3, 4 \tag{4.1}$$

where $\lambda(x)$ is the $T(4)$-valued transition function from σ to σ'.

Equation (4.1) is like the law of transformation of the vierbeins h_i^μ connecting the coordinate basis $\{\partial_\mu\}, \mu = 1, 2, 3, 4$ and the general basis $\{e_i\}$,

$$\begin{aligned} e_i = h_i^\mu \partial_\mu, \qquad \partial_\mu = h_\mu^i \partial_i \\ h_i^\nu h_\nu^j = \delta_i^j, \qquad h_\nu^k h_k^\mu = \delta_\nu^\mu \end{aligned} \tag{4.2}$$

using the standard notation. This allows us to endow P with a connection, being the coefficients $\{A_{(\sigma)}\}_i^b \overset{\text{def}}{=} h_i^b$ for a coordinate cross section on U. Each change of coordinate cross section will then be interpreted as a coordinate transformation on U. For any other cross section the coefficients will be given by (4.1). The definition of the connection is well established according to the mentioned lemma of Section 2.1. Note that the connection does not depend on the basis $\{e_i\}$. This is exactly the flat connection of P [16]. If the $\{e_i\}$ is orthogonal the gauge field is tied to the metric $\overset{M}{g}$ by expression (4.32).

The covariant differentiation field (2.21) now takes the form

$$D_i = \bar{e}_i - h_i^c \xi_c^* \tag{4.3}$$

It is coincident with the covariant differentiation e_i introduced in [2, 17, 5] for the usual fields, that is, for kinds of fields for which the kernel of the representation of $T(4)$ is all $T(4)$ and consequently the representation of the Lie algebra corresponds to the null matrix [5]. (The infinitesimal generators "annihilate the fields".) All this certainly constitutes a difficult aspect for this approach, from the point of view of the associated bundle (fields).

The commutation coefficient of $\{e_i\}$ are

$$c_{ij}^k = (\partial_i h_j^\mu - \partial_j h_i^\mu) h_\mu^k \tag{4.4}$$

Using expression (2.24) one immediately verifies that the form of curvature vanishes as it corresponds to a flat connection [16].

We assign to $T(4)$ the biinvariant metric making orthonormal the canonical fields $\{\xi_b\}$.

The use of the variational principle (2.30), (2.33) formally leads to the

Einstein-Cartan field equations if $\overset{M}{\Gamma}$ is metric. In the present case its use is rather fictitious because it eludes the connection on P, which is a priori fixed but not determined by the condition that the functional (2.32) be stationary.

Such considerations have led other authors [2, 5] to define (in a similar way to the preceding), a nontrivial connection on P at the cost of locally introducing on M additional noncoordinate reference systems on which this time, the new connection will depend. Now let the expression for the coordinate section σ be

$$\{A_\sigma\}_i^b \overset{\text{def}}{=} h_i^b - \delta_i^b \tag{4.5}$$

This depends on the selected reference system and has similar interpretations to that made in the previous case. Without difficulty it is verified that this is a correct definition in the sense that it agrees with the transformation law (2.14).

The covariant derivative is now

$$D_i = \bar{e}_i - (h_i^b - \delta_i^b)\xi_b^* \tag{4.6}$$

For the same reasons as before it coincides with those employed in [2, 17, and 5].

At present the curvature form does not vanish and its components on the direct product basis are

$$F_{ij}^b = (\partial_i h_j^\mu - \partial_j h_i^\mu) h_\mu^k h_k^b = c_{ij}^k \delta_k^b \tag{4.7}$$

The direct application of the variational principle (2.32) would lead to the determination of the additional noncoordinate reference systems as superfluous fields interacting with the gravitational field, meaningless from the physical point of view.

A better variational principle is that usually employed in the vierbein approach of general relativity [44], adopted by Cho in [5], and by means of which the dependence on the fictitious basis is avoided.

4.2. *Theory Based on the Lorentz Group*

Many authors have discussed gauge field gravitational theories related to the Lorentz group. See for example [62, 1, 33, 34, 50, 51, 9, 22, and 8].

Let the present principal fiber bundle, L_0, be that of the orthonormal frames on the pseudo-Riemannian manifold M, the space-time. A basis of the Lie algebra of the Lorentz group $O(1, 3)$ is taken as the vectors $\xi_{kl} = -\xi_{lk}$, $k, l = 1, 2, 3, 4$.

The corresponding fundamental fields on L_0 are $\{\xi_{kl}^*\}$ with the well-known brackets.

A connection on L_0 will be given by the local 1-forms $A_{(\sigma)}$ of components

$$\{A_{(\sigma)}\}_l^{ij} = -\{A_{(\sigma)}\}_l^{ji}$$

obeying the transformation law for a change of cross section $\sigma \to \sigma'$:

$$\{A_{(\sigma')}\}_l^{si} = (\Lambda^{-1})_p^i \{A_{(\sigma)}\}_l^{pk} (\Lambda^{-1})_k^s + (\Lambda^{-1})_p^i (\partial_l \Lambda_k^p) \gamma^{ks} \tag{4.8}$$

where γ_{ks} is the Minkowskian metric.

This is directly obtained from (2.14), where $\Lambda(x)$ is the $O(1,3)$-valued local function on M, Lorentz matrices, related to the change of orthonormal frames.

This connection is naturally interpreted as a metric connection $\overset{M}{\Gamma}$ of M, with the $\{A_{(\sigma)}\}_l^{ij} = \overset{M}{\Gamma}{}_l^{ij}$ being its components on the orthonormal basis constituting σ.

The horizontal covariant differentiation field is

$$D_i = \bar{e}_i - \overset{M}{\Gamma}{}_i^{pq} \xi^*_{pq} \tag{4.9}$$

which for a field based on a certain representation of the Lorentz group becomes the usual expression of its covariant derivative, on the associated fiber bundle.

Formula (2.24) directly gives in this case the components R^{ij}_{kl} of the curvature tensor of $\overset{M}{\Gamma}$ on the vierbein frames σ; and equation (2.9), on the direct product basis, the only nonvanishing components of the torsion:

$$\Theta^k_{ij} = \overset{M}{\Gamma}{}^k_{ij} - \overset{M}{\Gamma}{}^k_{ji} - c^k_{ij} \tag{4.10}$$

As $G = O(1,3)$ is a semisimple group we take for it the Killing metric (with opposite sign). The components of the corresponding Christoffel–Cartan connection on the basis constituted by the chosen left-invariant fields are simply a half of the structure constants of $O(1,3)$. The scalar curvature of the group is

$$\bar{R}_G = -1 \tag{4.11}$$

Finally the Lagrangian (2.33) particularizes to

$$S_{O(1,3)}\left(\overset{M}{\Gamma}, \overset{M}{g}\right) = \text{const} \times \int_M \left[\alpha(\bar{R}_M - 1) + \frac{\beta}{2} R^2\right] \overset{M}{\eta} \tag{4.12}$$

where

$$R^2 = \gamma_{ij}\gamma_{i'j'} R^{ii'}_{pq} R^{jj'}_{rs} \gamma^{pr}\gamma^{qs}$$

because in this case we have $F^2 = 4R^2$, taking into account the Killing metric of $\mathcal{G}_{O(1,3)}$.

The variational formulation (4.12) contains the following particular cases of interest:

$\alpha = 1, \beta = 0$, $\overset{M}{\Gamma}$ torsionless: Einstein's theory with cosmological term.
$\alpha = 1, \beta = 0$: Einstein–Cartan type theory with cosmological term.

4.3. *Theory Based on the GL(4) Group*

This theory is essentially the same as the preceding one with the difference that now we are working with the bundle of general frames $L(M)$, of M. The $GL(4)$ group has been employed by Yang [18]. See also [9].

The chosen basis of the Lie algebra of $GL(4)$ is denoted $\{\xi^i_j\}$, $i, j = 1, 2, 3, 4$.

A connection on $L(M)$ is naturally interpreted as a linear connection on M. As before it is given through its coefficients

$$\{A_{(\sigma)}\}^j_{ik} = \overset{M}{\Gamma}{}^j_{ik} \tag{4.13}$$

with a law (2.14) of transformation for $\sigma' = \sigma a, a \in GL(4)$:

$$\{A_{(\sigma')}\}^j_{ik} = (a^{-1})^j_s \{A_{(\sigma)}\}^s_{il} a^l_k + (a)^j_l (\partial_i a)^l_k \tag{4.14}$$

The horizontal covariant differentiation field is

$$D_i = \bar{e}_i - \overset{M}{\Gamma}{}^m_{ln} \xi^{*n}_m \tag{4.15}$$

From (2.24) and (2.9) one gets the curvature and torsion tensors of $\overset{M}{\Gamma}$ on the basis σ.

$GL(4)$ is not a semisimple group. Nevertheless we may choose a biinvariant metric, and the correspondent Christoffel-Cartan connection.

The Lagrangian (2.33) is then

$$S_{GL(4)}(\overset{M}{\Gamma}, \overset{M}{g}) = \int_M [\alpha(\bar{R}_M - 1) + \beta(R_{ijk}{}^k \; R^{ijl}{}_l - R_{ijl}{}^k \; R^{ijl}{}_k)] \overset{M}{\eta} \tag{4.16}$$

It contains the following particular cases:

$\alpha = 1, \beta = 0$, $\overset{M}{\Gamma}$ metric and torsionless: Einstein's theory, with cosmological term.

$\alpha = 1, \beta = 0$, $\overset{M}{\Gamma}$ metric: Einstein-Cartan type theory, with cosmological term.

$\alpha = 0, \beta = 1$, $\overset{M}{\Gamma}$ torsionless: Stephenson-Yang type theory.

$\alpha = 1, \beta = -2$, $\overset{M}{\Gamma}$ metric and torsionless: Mansouri-Chang type theory.

4.4. *Theory Based on the Poincaré Group*

We treat this case in more detail.

Many attempts exist to found a gauge theory of the gravitational field on the Poincaré group $IO(1, 3)$, semidirect product of the group of translations $T(4)$ and the Lorentz group $O(1, 3)$. See for example [2, 6, 9, 57, 41, 49, 40, 55]. Nevertheless in our opinion many of them correspond more neatly to the formalism founded on the direct product $T(4) \times O(1, 3)$ as we shall later see.

The present theory will be developed using the principal bundle $A_0(M)$, of the affine orthonormal frames (see [16], p. 126) of the space-time bundle M, with structural group $IO(1, 3)$.

The Lie algebra of $IO(1,3)$ is, as a vector space, the direct sum of those of $T(4)$ and $O(1,3)$. A natural basis will be given by the vectors $\{\xi_b, \xi_{kk'}\}$ $b, k, k' = 1, 2, 3, 4$.

The corresponding brackets are

$$[\xi_a, \xi_b] = f^c_{a,b}\xi_c + \tfrac{1}{2} f^{ss'}_{a,b}\xi_{ss'} = 0 \tag{4.17a}$$

$$[\xi_a, \xi_{kk'}] = f^c_{a,kk'}\xi_c + \tfrac{1}{2} f^{ll'}_{a,kk'}\xi_{ll'} \neq 0 \tag{4.17b}$$

$$[\xi_{ii'}, \xi_{jj'}] = f^b_{ii',jj'}\xi_b + \tfrac{1}{2} f^{ll'}_{ii',jj'}\xi_{ll'} \neq 0 \tag{4.17c}$$

with

$$f^a_{b,c} = f^{ss'}_{a,b} = f^{ll'}_{a,kk'} = f^b_{ii',jj'} = 0 \tag{4.18a}$$

$$f^c_{a,kk'} = \delta^c_{k'}\gamma_{ka} - \delta^c_k\gamma_{k'a} \tag{4.18b}$$

$$f^{mn}_{ij,kl} = \gamma_{jk}(\delta^m_i\delta^n_l - \delta^n_l\delta^m_i) - \gamma_{jl}(\delta^m_i\delta^n_k - \delta^n_k\delta^m_i)$$
$$-\gamma_{ik}(\delta^m_j\delta^n_l - \delta^n_l\delta^m_j) - \gamma_{il}(\delta^m_j\delta^n_k - \delta^n_k\delta^m_j) \tag{4.18c}$$

where γ_{ij} is the Minkowskian metric.

A connection on this bundle will as usual be given by the set of local 1-forms $A_{(\sigma)}$ with components $\{A_{(\sigma)}\}^b_i, \{A_{(\sigma)}\}^{jj'}_i$ obeying the transformation law (2.14) for changes of cross section $\sigma'(x) = \sigma a(x)$; $a(x) = (\lambda(x), \Lambda(x)) \in IO(1,3)$:

$$A'^b_i = A^c_i(\Lambda^{-1})^b_c + A^{kk'}_i(\Lambda^{-1})^b_k\gamma_{k'c}\lambda^c + (\Lambda^{-1})^b_c\partial_i\lambda^c \tag{4.19a}$$

$$A'^{kk'}_i = A^{ll'}_i(\Lambda^{-1})^k_l(\Lambda^{-1})^{k'}_{l'} + (\Lambda^{-1})^k_l(\partial_i\Lambda)^l_s\gamma^{sk'} \tag{4.19b}$$

(As in (2.24) we put $A'^b_i = \{A_{(\sigma')}\}^b_i$; $A^c_i = \{A_{(\sigma)}\}^c_i$, etc.)

Equation (4.19b) allows us to interpret the $\{A_{(\sigma)}\}^{kk'}_i$ as the components of the metric connection $\overset{M}{\Gamma}{}^{kk'}_i$ in the orthonormal referential of the affine frame σ. This was just done in Section 2.1. On the contrary (4.19a) presents difficulties for the interpretation of the $\{A_{(\sigma)}\}^b_i$ (or the $\delta^b_i + \{A_{(\sigma)}\}^b_i$) like the components of the vierbeins, if the transformation $a = (\lambda, \Lambda)$ is at the same time to be considered as a coordinate change $x^\mu \to x'^\mu = x^\mu + \lambda^\mu(x)$ and a frame rotation $e_i \to e'_i = \Lambda^k_i e_k$. This kind of interpretation is for instance suggested in [2, 6] and other papers. That is possible for the direct product of $T(4)$ and $O(1,3)$, or as well for the semidirect product $IO(1,3)$ if the transformations are restricted to the subgroups of elements $a = (0, \Lambda)$. More specifically in the work of Kibble [2], formula (4.11) for the transformation of the compensating fields (vierbeins) h^μ_k does not obey the transformation law (2.7) of the same paper for general gauge fields, if the structure constants are those of the Poincaré group. On the other hand in [6] Cho presents the correct transformation formulas of the gauge fields for $IO(1,3)$ but fails when interpreting them as a change of the vierbein coefficients for coordinate changes $[\lambda(x)]$ and rotations $[\Lambda(x)]$ of the reference basis. Such interpretations of the gauge transformation are *typical* *for the direct product but not for the semidirect product* because this last, owing to (4.17b) and (4.18b), mixes two actions that must be independent.

A *generalized affine connection* is by definition a connection on the fiber bundle $A(M)$ of affine frames ([16], Chap. III, Section 3). There exists a one-to-one correspondence between the generalized affine connections and the pairs constituted by a tensor field of type $\binom{1}{1}$ of M (mixed tensor of the second rank) and by a linear connection on M ([16], p. 127). An *affine connection* is a generalized affine connection for which the tensor field is the identity transformation in the vector spaces tangent to M ([16], p. 129 and [52], p. 93). They are then in a bijective correspondence with the linear connections on M.

These results make it possible to interpret the affine connection as a linear connection on M plus the vierbein coefficients [tensor field $\binom{1}{1}$, the identity]. Such an approach has been used by Mansouri and Chang [9]. Formula (4.19a) has no interpretation in the general case.

The covariant derivative operator is now

$$D_i = \bar{e}_i - \{A_{(\sigma)}\}_i^b \xi_b^* - \{A_{(\sigma)}\}_i^{kk'} \xi_{kk'}^* \tag{4.20}$$

where the ξ_b^* and $\xi_{kk'}^*$ are the fundamental fields related to ξ_b and $\xi_{kk'}$. This expression corresponds to that given by Cho ([6] formula (8), noting that there holds $h_i^b = \delta_i^b - \{A_{(\sigma)}\}_i^b$) and by Mausouri and Chang {[9] formula (5.3)} but not to that of Kibble {[2] formula (4.10)}. It has no clear meaning as a covariant differentiation of fields on the associated bundle.

The nonvanishing components of the curvature form (2.24) are

$$F_{ij}^b = \partial_i A_j^b - \partial_j A_i^b + A_i^a A_j^{kb} \gamma_{ka} - A_j^a A_i^{kb} \gamma_{ka} - c_{ij}^k A_k^b \tag{4.21}$$

$$F_{ij}^{kk'} = \partial_i A_j^{kk'} - \partial_j A_i^{kk'} + A_i^{kq} \gamma_{qr} A_j^{rk'} - A_i^{k'q} \gamma_{qr} A_j^{rk} - c_{ij}^s A_s^{kk'} \tag{4.22}$$

For a homogeneous cross section σ of $A_0(M)$, that is, $\sigma = (0, \{e_i\})$ [orthonormal section of $L(M)$], (4.21) takes the form

$$F_{ij}^k = \nabla_i h_j^k - \nabla_j h_i^k + T_{ij}^s h_s^k \tag{4.23}$$

where

$$T_{ij}^s = \overset{M}{\Gamma}{}_{ij}^s - \overset{M}{\Gamma}{}_{ji}^s - c_{ji}^s$$

are components of the torsion tensor on $\{e_i\}$; $\nabla_i h_j^k$ are covariant derivatives of the tensor field $\binom{1}{1}$ associated with the generalized affine connection; and $h_j^k = \{A_{(\sigma)}\}_j^k$.

The components in equation (4.22) are certainly the components of the curvature tensor on the same base:

$$F_{ij}^{kk'} = \overset{M}{R}{}_{ij}^{kk'} \tag{4.24}$$

If the connection is simply an affine connection the covariant derivatives of the $\binom{1}{1}$ field vanish and $h_s^k = \delta_s^k$. Equations (4.23) and (4.24) correspond then to the known result which states that the reciprocal image of the curvature form of the affine bundle by the natural injection γ from $L(M)$ to $A(M)$ is the sum of the linear connection form on $L(M)$ associated to the affine connection of $A(M)$, and its torsion form ([16], p. 130).

The field intensities tensor relative to the "rotational part" of $IO(1,3)$ is then associated with the curvature of M, and the tensor corresponding to the "translational part" is associated with the torsion [33, 40].

For the Poincaré group there is no biinvariant metric, as is easy to verify through the necessary and sufficient condition (2.29) for the mixed components of $\overset{G}{g}$.

This eventuality excludes then the possibility of applying the variational principle proposed in Section 2.2.7, except for exceptional cases, because in general it loses the gauge-invariant character.

This fact, pointed out by Cho [6], was not taken into consideration in [9] and the quadratic term of the Lagrangian suggested there is not gauge invariant for the Poincaré group.

Conditions (5.19) from reference [9], on vanishing covariant derivatives of the $\binom{1}{1}$ tensor field, are automatically fulfilled for affine connections. If $\overset{M}{\Gamma}$ is torsionless, the Lagrangian reduces to that corresponding to the Lorentz group (Section 4.2), as is stated in [9], becoming now gauge invariant because $F^k_{ij} \equiv 0$, and the noninvariant part of the quadratic term just vanishes ($\overset{M}{\Gamma}$ is metric here).

We finally point out another difficulty related to the use of the Poincaré group in gauge theories of gravitation. Still for an affine connection the parallel transport of a homogeneous affine frame along any given path does not coincide in general with the parallel transport of the same frame along the same path, viewed now from the bundle $L(M)$. The connection on $L(M)$ is that associated with the affine connection. This is a consequence of the existence of a tensor field of type $\binom{1}{1}$. Specifically, if η is a horizontal vector in $L(M)$, $d\gamma \cdot \eta$ will not be horizontal on $A(M)$ except in the case in which the tensor field vanishes, that is,

$$\omega(\eta) = 0, \qquad \tilde{\omega}(d\gamma \cdot \eta) = \gamma^*\tilde{\omega}(\eta) = \omega(\eta) + \varphi(\eta) = \varphi(\eta) \tag{4.25}$$

where $\tilde{\omega}$ is the generalized affine connection form on $A(M)$, ω is the form of connection on $L(M)$ associated to $\tilde{\omega}$, and φ is a $\mathbb{R}^n$-valued 1-form of tensorial type associated to the tensor field $\binom{1}{1}$.

Then in general, the parallel transport of fields [bundles associated with $A(M)$] will not coincide with the parallel transport of such fields for the associated connection form ω of $L(M)$ (the usual interpretation).

4.5. *Theory Based on the Affine Group*

This theory is essentially of the same type and has the same properties as the preceding one. The Lie algebra of the affine group $IGL(4)$ has structure constants similar to those of $IO(1,3)$. The adjoint transformation on $\mathfrak{g}_{IGL(4)}$ "mixes" also here its "translation" and "rotational parts."

For identical reasons as before, a biinvariant metric does not exist for $IGL(4)$.

This affine theory of the gravitational field presents the same characteristics and difficulties as the preceding theory. The essential difference is now that $\overset{M}{\Gamma}$ is not necessarily a metric connection.

Lord [35] discusses an affine theory. The work is, in our opinion, susceptible to some of the objections made in section 4.4 concerning the interpretation of the cross section changes on the affine bundle $A(M)$ as changes of coordinates and referential frames.

4.6. *Theory Based on the Group $T(4) \times O(1,3)$*

At each space-time point $x \in M$, let us consider the set of all orthonormal frames and the set of all real tetrads (coordinates to be assigned to x). The last is similar to what was done in Section 4.1. One verifies that this set for every $x \in M$ constitutes a principal fiber bundle $P[T(4) \times O(1,3)]$ of base M and structural group $G = T(4) \times O(1,3)$. The fiber over x has just been described.

The adopted basis for the Lie algebra of G is the same as in Section 4.4, that is, $\{\xi_b, \xi_{kk'}\}$ $b, k, k' = 1, 2, 3, 4$. But now for the brackets (4.17), (4.18) also holds

$$f^{c}_{a,kk'} = 0 \tag{4.26}$$

This relation essentially avoids the difficulties commented on in Section 4.4.

For a change of cross section $\sigma' = \sigma a$ in P, the transformation law (2.24) becomes

$$\begin{aligned} \{A_{(\sigma')}\}^{\alpha}_{i} &= \{A_{(\sigma)}\}^{\alpha}_{i} + \partial_i \lambda^{\alpha} \\ \{A_{(\sigma')}\}^{ij}_{k} &= \{A_{(\sigma)}\}^{pq}_{k} (\Lambda^{-1})^{i}_{p} (\Lambda^{-1})^{j}_{q} + (\Lambda^{-1})^{i}_{s} (\partial_k \Lambda)^{s}_{l} \gamma^{lj} \end{aligned} \tag{4.27}$$

with $a(x) = (\lambda(x), \Lambda(x))$. As previously noted, not every cross section σ corresponds to a "coordinate section."

The (4.27) allow now the interpretations

$$\begin{aligned} \{A_{(\sigma)}\}^{\alpha}_{i} &= h^{\alpha}_{i} \\ \{A_{(\sigma)}\}^{ij}_{k} &= \overset{M}{\Gamma}{}^{ij}_{k} \end{aligned} \tag{4.28}$$

This is justified because expressions (4.27) correspond well to the law of transformation of the bierbein and linear connection components, respectively, for a coordinate change $x^{\mu} \to x'^{\mu} = x^{\mu} + \lambda^{\mu}$, and a vierbein change $e'_i = e_k \Lambda^{k}_{i}$. We have fixed in this way a class of possible connections.

For transformations $a \in G$ close to the identity, relations (4.28) correctly coincide with formulas (4.11) and (4.9) of the classical paper of Kibble [2], if care is also taken to rotate the basis $\{e_i\}$ of $T_x(M)$ by Λ. This suggests that from this point of view the relevant group of that paper is the direct product $T(4) \times O(1,3)$ and not the semidirect product $IO(1,3)$ as is frequently stated in the literature, including the just-mentioned paper. This is a consequence of the simple fact commented on earlier—namely, the rotations of the basis of the $T_x(M)$ commute with the coordinate changes on x, i.e., they are *independent actions.* But the rotations, as is well known, do not commute with translations.

The covariant derivative field is

$$D_i = \bar{e}_i - h_i^\alpha \xi_\alpha^* - \tfrac{1}{2} \overset{M}{\Gamma}{}_i^{ll'} \xi_{ll'}^* \tag{4.29}$$

For the notion of "field" (associated bundle) it is now convenient to think in nonfaithful representations of $T(4) \times O(1,3)$, the kernel being all $T(4)$ (see Section 4.1 and [5]). This may be interpreted by remembering that the components of the field on x remain unaltered, if while keeping fixed the frame of $T_x(M)$ one changes the coordinates of x; on such kind of representations the ξ_α^* are represented by null matrices, and in consequence (4.29) will correctly correspond to the expression of a covariant differentiation.

The components (2.24) of the curvature term are now

$$F_{ij}^\alpha \equiv 0, \qquad F_{ij}^{kk'} = R_{ij}^{kk'} \tag{4.30}$$

where $R_{ij}^{kk'}$ are components of the curvature tensor of $\overset{M}{\Gamma}$. The first relation is a result similar to that of Section 4.1.

In this case, biinvariant metrics for $G = T(4) \times O(1,3)$ are available—for example, the left extension to all G of the direct product of the metric that makes orthonormal the chosen basis ξ_a of $\mathcal{G}_{T(4)}$, and the Killing metric of $\mathcal{G}_{O(1,3)}$.

The variational principle (2.33) reduces directly to

$$S_{T(4)\times O(1,3)}(\overset{M}{\Gamma}, \overset{M}{g}) = \text{const} \times \int_M \left[\alpha(\bar{R}_M + \bar{R}_G) + \frac{\beta}{2} R^2\right] \overset{M}{\eta} \tag{4.31}$$

[where R^2 is as given following equation (4.12)]. This is the same as (4.12) but with the bonus of linking $\overset{M}{g}$ to the gauge fields h through

$$\overset{M}{g}_{\mu\nu} = h_\mu^i h_\nu^j \gamma_{ij} \tag{4.32}$$

where γ_{ij} is the Minkowskian metric, without the necessity of additional conditions and within a coherent formulation.

4.7. *Theory Based on the Group T(4) × GL(4)*

Briefly, it is almost strictly a generalization of the preceding theory to the case in which $\overset{M}{\Gamma}$ may also be nonmetric. The principal bundle is denoted $P[T(4) \times GL(4)]$.

4.8. *Evaluation of the Theories*

This evaluation has been made by inquiring for each theory whether or not the aspects selected in Table I possess an adequate physical-geometrical meaning.

TABLE I. Evaluation of the Theories by Inquiring for Each One, Whether or Not the Aspects Selected in the Table Possess an Adequate Physical-Geometrical Meaning.

Group on which the theory is based	$T(4)$	$O(1,3)$	$GL(4)$	$IO(1,3)$	$IGL(4)$	$T(4) \times O(1,3)$	$T(4) \times GL(4)$
Connection coefficients as gauge potentials	yes	yes	yes	yes	yes	yes	yes
Transformation law for connection coefficients as the cross section changes	yes	yes	yes	no	no	yes	yes
Covariant differentiation field	yes	yes	yes	no	no	yes	yes
Associated bundles; fields	no	yes	yes	yes	yes	yes	yes
Metric of the base M, related or not to some gauge field	yes	no	no	no*	no*	yes	yes
Curvature of connection	no	yes	yes	yes	yes	yes	yes
Variational principle (2.33)	no	yes	yes	no	no	yes	yes
Gauge invariance of the Lagrangian (2.33)	yes	yes	yes	no	no	yes	yes
Observations	–	–	$\overset{M}{\Gamma}$ is not completely defined by the variational principle, except in the case that $\overset{M}{\Gamma}$ is metric	*For the general case	*For the general case	–	$\overset{M}{\Gamma}$ is not completely defined by the variational principle, except in the case that $\overset{M}{\Gamma}$ is metric

After observing Table I and from the standpoint of this approach, the "best" theories are neatly those based on the direct product groups. In second place we would include the Lorentz group and $GL(4)$ group theories.

The Poincaré group theory is perhaps the one that aroused more interest. Nevertheless we observe that from this point of view it presents more difficulties than others. On the contrary our "best" theories do not seem, to our knowledge, to have been studied–at least not explicitly.

An additional consideration for this evaluation will be commented on at the end of Section 5.

§(5): *Relations between Theories: Homomorphisms*

The scope of this approach is not complete without making some reference to the existent relations among the theories involved. As was previously mentioned, this can be done within the structure of a category, the objects being the proper theories, and the morphisms being the existent natural homomorphisms between the underlying principal fiber bundles, over the same base M, the space-time.

This category is too broad. We distinguish in it two important subcategories with the same objects but less rich in morphisms. One is that whose morphisms are injective immersions ([11], embeddings [16]). The other, on the contrary, has morphisms that are surjective submersions [11].

We wish to study the class of connections induced from one bundle to another by some natural homomorphisms linking them. If the homomorphism is a surjective submersion the induced class coincides with the class of all connections of the second bundle. Nothing new is added. Otherwise for embeddings, the induced class will in general be a strict subclass whose characterization may be of interest. We shall then only consider the first class, the embeddings, shown in Figure 1.

The arrows indicate the nonempty sets hom. The broken lines correspond to homomorphisms needing a strong hypothesis over M, that we will comment on further on.

Let us now discuss the sets hom and the induced connections subclasses.

For any bundle P the sets hom(P, P) are only constituted by the identity on P.

The set hom$(M \times T(4), P[T(4) \times O(1,3)])$ does not contain in general natural homomorphisms. Under the strong hypothesis that M admits global cross sections, a class of homomorphisms may be introduced. Let $q(x), x \in M$ be a global orthonormal cross section of $L(M)$. For each q on such conditions a differentiable mapping f_q is defined in the following way: to each tetrad t of the first bundle, on $x \in M$, there corresponds the pair $(t, q(x)) \in P[T(4) \times O(1,3)]$. The group homomorphism φ assings to each $\lambda \in T(4)$ the pair $(\lambda, e) \in$

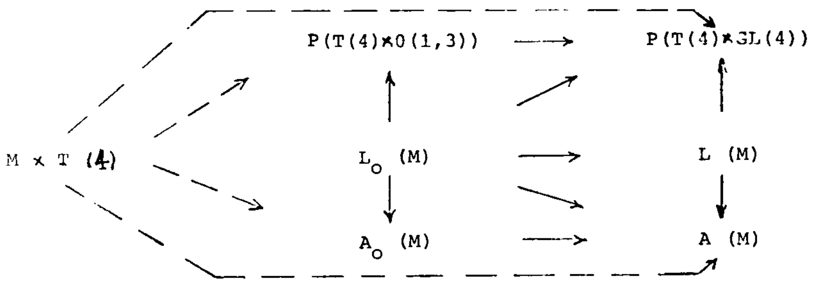

Fig. 1. Diagram of the embeddings considered.

$T(4) \times O(1,3)$ [where e is the identity of $O(1,3)$]. One easily checks through (3.1) that the pair (f_q, φ) is a fiber bundles homomorphism. Let hom be constituted of such elements. For the earlier chosen fundamental fields the transfer coefficients of Section 3 are $W_{b_1}^{b_2} = \delta_{b_1}^{b_2}$ if $b_1, b_2 = 1,2,3,4$; vanishing for $b_2 > 4$. If σ is a cross section of the first bundle the connection forms so induced on the cross section $f_q(\sigma)$ of the second bundle are, from (3.6)

$$\overset{2}{A}{}^{\alpha}_{[f_q(\sigma)]} = \overset{1}{A}{}^{\alpha}_{(\sigma)}, \quad \overset{2}{A}{}^{pq}_{[f_q(\sigma)]} = 0, \quad \alpha, p, q = 1,2,3,4 \tag{5.1}$$

For any other section $\rho(x) = f_q(\sigma)a(x)$ the transformation formulas (4.27) allow one to obtain the correspondent 1-forms $A_{(\rho)}$.

The induced class leads, according to (4.28), to particular theories for which the space-time M possesses vanishing curvature but not vanishing torsion. For hom$(M \times T(4), P[T(4) \times GL(4)])$, hom$(M \times T(4), A_0(M))$ and hom$(M \times T(4), A(M))$ essentially the same conclusion holds.

hom$(L_0(M), L(M))$ only contains the canonical homomorphism. As is well known [15] the induced connections are the metric ones. The transfer coefficients for this case

$$\begin{aligned} W_{b_1}^{b_2} &= \delta_i^{\beta}\gamma_{i'\beta'} - \delta_{i'}^{\beta}\gamma_{i\beta'} \\ b_1 &= (i, i'), \qquad b_2 = \binom{\beta}{\beta'} \end{aligned} \tag{5.2}$$

permit one to verify that assertion.

Similar conclusions are valid for hom$(P[T(4) \times O(1,3)], P[T(4) \times GL(4)])$ if this set is constituted as before, only by the canonical homomorphism.

Now let $\tau(x)$ be a global differential mapping from M to $\mathbb{R}^4$. Let f_τ: $L_0(M) \to P[T(4) \times O(1,3)]$ and φ: $O(1,3) \to T(4) \times O(1,3)$ be the functions such that

$$f_\tau: q \longrightarrow (\tau(\pi(q)), q) \in P[T(4) \times O(1,3)], \qquad q \in L_0(M)$$

$$\varphi: \Lambda \longrightarrow (0, \Lambda) \in T(4) \times O(1,3), \qquad \Lambda \in O(1,3)$$

For every τ, the pairs (f_τ, φ) will be the elements of $\hom(L_0(M), P[T(4) \times O(1,3)])$. The transfer coefficients are

$$W_{b_1}^{b_2} = \begin{cases} 0 & \text{if } b_2 \leqslant 4 \\ \delta_{b_1}^{(b_2-4)} & \text{if } b_2 = 5, 10 \end{cases} \tag{5.3}$$

If τ is the constant mapping, (f_τ, φ) induces the class of connections already chosen in Section 4.6 through equations (4.28). The constancy of τ is a necessary and sufficient condition for that. For $\hom(L_0(M), P[T(4) \times GL(4)])$ the situation is the same. The induced connections are metric. Another similar case is $\hom(L(M), P[T(4) \times GL(4)])$.

Let us now examine $\hom(L_0(M), A_0(M))$. The only element (f, φ) of that set is such that

$$f\colon q \longrightarrow (0, q) \text{ homogeneous affine orthonormal frame } q \in L_0$$

$$\varphi\colon \Lambda \longrightarrow (0, \Lambda) \in IO(1,3), \qquad \Lambda \in O(1,3)$$

The transfer coefficients are also given by (5.3), and the induced class is that of the generalized affine connections with vanishing tensor field of type $\binom{1}{1}$. Similar conclusions hold for $\hom(L_0(M), A(M))$ and for $\hom(L(M), A(M))$.

Finally, $\hom(A_0(M), A(M))$ only contains the natural homomorphism. The induced connections $\overset{M}{\Gamma}$ are of course metrics. The $\binom{1}{1}$ field is not modified. The rest of the sets hom are empty.

We observe a formal difficulty inherent to the theories of Sections 4.2, 4.4, and 4.6 based essentially on the Lorentz group. Expression (2.30) [but not (2.33)] of the variational principle does not have a clear meaning at all. Such bundles depend on one of the magnitudes to be varied: the metric $\overset{M}{g}$. This difficulty can be avoided by simply replacing the theories mentioned by those of Sections 4.3, 4.5, and 4.7 involving $GL(4)$, but restricting their connection to the subclasses respectively induced by the previous bundles. These are the "metric" connections.

§(6): *Concluding Remarks*

This approach is an attempt to present a unified overview of certain gauge theories of the gravitational field, with the aim of obtaining a better understanding and a greater synthesis of them and their existent interrelations. Such an approach in no way constitutes itself as a conclusive formalism for evaluating every gauge theory of gravitation. On the contrary, it is only a possible point of view among many others.

From Table I and the last part of Section 5, the theory based on the group $T(4) \times GL(4)$ appears neatly to be the most advantageous. In second place comes that of the bundle $L(M)$. Both involve the induced "metric" connections. This is the most important conclusion of the paper.

The first theory has the advantage over the second of associating the metric of the space-time with a gauge field through equations (4.28) and (4.32).

"Matter" fields have not been considered in this approach. Nevertheless the formalism is specially suitable for such purposes through the associated fiber bundles. This will be accomplished in subsequent stages. In particular the two selected theories accomodate that extension well.

The Lagrangian here proposed, constructed with the "adapted connection," is more flexible and general than the usual one for the Christoffel-Cartan connection of P. It does not exclude, for instance, important theories with torsion such as the Einstein-Cartan theory. The approach also leads to theories that expand that of Mansouri and Chang [9].

The "adapted connections" further involves very natural geometrical assumptions.

Interesting extensions of this work would consist of applying the formalism to theories based on the group $SL(2, C)$ ([51] and references there quoted), and to conformal and sympletic approaches of gravity.

Summarizing this approach has revealed from its standpoint certain misstatements of known gauge theories of the gravitational field; it has also contributed to putting in evidence the role played by the direct product groups and has allowed us to propose a theory based on one such group, which corrects the mentioned deficiencies and possesses the positive attributes of the preceding theories.

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