THERMALLY ORTHOTROPIC, DOUBLY CONNECTED PLATES

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INTRODUCTION

Solutions of the diffusion equation in doubly connected domains are of interest in several fields of applied science. In general, approximate analytical or numerical techniques must be used: conformal mapping, variational methods, point-matching, finite differences, finite elements, etc.(Finlayson, 1972).

It appears at this point that a significant number of investigations has been performed on the determination of steady-state solutions of the diffusion equation in the case of complex, doubly connected regions (Hockney, 1964, Lewis, 1968, Balcerzak and Raynor, 1961 and Laura and Susemihl, 1973).

On the other hand, the study of unsteady-state solutions has not been pursued to a significant extent.

The present investigation constitutes an extension of the study previously performed by Laura and Ercoli, 1972, since the approach developed there is used in the present paper for the analysis of thermally orthotropic materials.

STATEMENT OF THE PROBLEM AND ITS APPROXIMATE SOLUTION

It is assumed that the heat conduction coefficients k_x and k_y do not vary with temperature. The problem is then governed, in the case of an unsteady, two dimensional phenomenon, by the partial differential equation:

$$k_x \frac{\partial^2 T}{\partial x^2} + k_y \frac{\partial^2 T}{\partial y^2} = c_p . \rho . \frac{\partial T}{\partial t} \qquad (1)$$

Let the initial and boundary conditions be:

$$T(x,y,o) = T_o \tag{2a}$$

$$T[L_i(x,y) = o,t] = 0 \tag{2b}$$

where $L_i(x,y)=0$ is the functional relation which defines each boundary of the domain (i=1,2).

Making

$$T(x,y,t) = T_1(x,y)\tau_1(t) \tag{3}$$

and substituting in (1) one obtains, following the classical procedure of separation of variables:

$$\frac{1}{T_1}\left(k_x \frac{\partial^2 T_1}{\partial x^2} + k_y \frac{\partial^2 T_1}{\partial y^2}\right) = c_p.\rho.\frac{\tau_1'}{\tau_1} = -\beta^2 \tag{4}$$

The function $\tau_1(t)$ is simply:

$$\tau_1 = A\, e^{\frac{-\beta^2}{c_p.\rho} t} \tag{5}$$

where A is an arbitrary constant.

On the other hand, the function $T_1(x,y)$ is a solution of the partial differential equation:

$$k_x \frac{\partial^2 T_1}{\partial x^2} + k_y \frac{\partial^2 T_1}{\partial y^2} + \beta^2 T_1 = 0 \tag{6}$$

which may be expressed as:

$$\nabla_0^2\, T_1 + \beta^2 T_1 = 0 \tag{7}$$

where ∇_0^2 is the orthotropic Laplacian operator ($\nabla_0^2 = k_x \frac{\partial^2}{\partial x^2} + k_y \frac{\partial^2}{\partial y^2}$) and equation (7) will be referred to as the "orthotropic Helmholtz equation".

It is shown in calculus of variations that the solution of the partial differential equation (6) is equivalent to the minimization of the functional:

$$J|T_1| = \iint_D \left[k_x\left(\frac{\partial T_1}{\partial x}\right)^2 + k_y\left(\frac{\partial T_1}{\partial y}\right)^2 - \beta^2 T_1^2\right] dx\, dy \tag{8}$$

subject to the boundary condition:

$$T_1\, [L_i(x,y) = 0] = 0 \quad ; \quad i=1,2 \tag{9}$$

Let:

$$z = x + iy = f(\zeta) \; ; \qquad \zeta = \xi + i\eta \tag{10}$$

be the functional relation which transforms the given, complicated shape in the z-plane, onto an annulus in the ζ-plane (Figure 1).

Admittedly finding $z= f(\zeta)$ is not an easy task, but it is known for many simply and doubly connected shapes of practical significance (Laura and Ercoli, 1972).

Substituting (10) in equation (8), one obtains:

$$J|T_1| = k_x \iint_C \left[2\ \mathrm{Re}\left[\left(\frac{\partial T_1}{\partial \zeta}\right)^2 \frac{1}{f'^2(\zeta)}\right] + 2\ \frac{\partial T_1}{\partial \zeta} \cdot \frac{\partial T_1}{\partial \zeta}\ \frac{1}{\|f'(\zeta)\|}\right]$$
$$\|f'(\zeta)\|\ d\xi d\eta - k_y \iint_C \left[2\ \mathrm{Re}\left[\left(\frac{\partial T_1}{\partial \zeta}\ \frac{1}{f'^2(\zeta)}\right]-2\frac{\partial \phi}{\partial \zeta}\ \frac{\partial \phi}{\partial \zeta}\ \frac{1}{\|f'\zeta\|}\right]$$
$$\|f'(\zeta)\|\ d\xi.d\eta-\beta^2 \iint_C T_1^2\ \|f'(\zeta)\|\ d\xi.d\eta \tag{11}$$

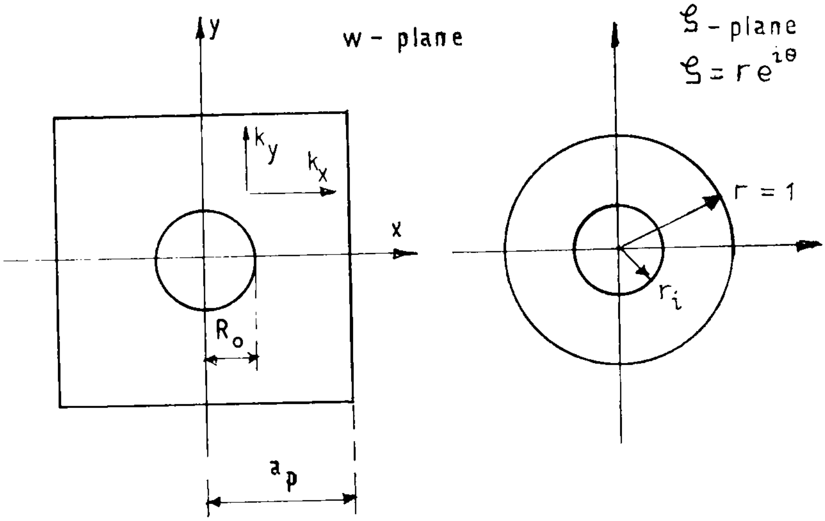

Figure 1. Thermally orthotropic system under study.

The simplest approximation which satisfies the transformed boundary condition (Figure 1):

$$T_1(\zeta,\bar{\zeta}) = 0 \quad \text{for } |\zeta|= 1,r_i \tag{12}$$

is probably the expression:

$$T_1 \simeq T_{1a}(\zeta,\bar{\zeta}) = (\sqrt{\zeta\bar{\zeta}} - r_i)\left[A_0(1-\sqrt{\zeta\bar{\zeta}}) + A_1(1-\zeta\bar{\zeta})\right] \tag{13}$$

Substituting (13) in (11) and using the minimization condition:

$$\frac{\partial J|T_{1a}|}{\partial A_n} = 0 \quad (n=1,2\ldots N) \tag{14}$$

one obtains a linear system of equations in the A_n's.

From the non-triviality condition one obtains a secular determinant whose roots are the desired eigenvalues β_n^2's.

It should be clear at this point that expression (13) makes the algorithmic procedure quite simple from the point of view of calculating the separation constants. When expressing the temperature field in its final form, it is considerably more expedient to use the Fourier-Bessel expansion, which has the following advantages:

a) it is the exact solution in the case of a circular, isotropic, annular plate, and
b) the coefficients are obtained by means of a well-known orthogonality relation.

Accordingly, as a first order approximation, one has in the ζ-plane:

$$T_{1a} = \sum_m B_{om}\left[J_o(\eta_{om}r)\, Y_o(\eta_{om}) - J_o(\eta_{om})\, Y_o(\eta_{om}r)\right] \tag{15}$$

where:

$$B_{om} = \pi \frac{J_o(\eta_{om}\, r_i)}{J_o(\eta_{om} r_i) + J_o(\eta_{om})} \tag{16a}$$

and the η_{om}'s are the roots of the transcendental equation:

$$J_o(\eta_{om} r_i)\, Y_o(\eta_{om}) - J_o(\eta_{om})\, Y_o(\eta_{om} r_i) = 0 \tag{16b}$$

It is important to point out that it has already been shown that the accuracy of the approach presented in this section is very good in the case of isotropic configurations (Laura and Ercoli, 1972).

The unsteady temperature field is then described by the functional relation:

$$T(\zeta,\bar{\zeta},t) \simeq \sum_m B_{om}\left[J_o(\eta_{om}r)Y_o(\eta_{om}) - J_o(\eta_{om})Y_o(\eta_{om}r)\right] e^{\frac{-\beta_{om}^2\, t}{c_p\cdot\rho}} \tag{17}$$

It is important to point out that finding the mapping function which transforms an arbitrary, doubly connected domain onto a circular annulus is a very difficult task. However, in the case of domains of regular polygonal shape with a concentric circular perforation the finding of the approximate mapping function is quite easy if $R_o/a_p \ll 1$. In the case of simply connected regular polygonal shapes, one has:

$$z = a_p . A_s \int_0^{\zeta} \frac{d\zeta}{(1+\zeta^s)^{2/s}} \tag{18}$$

where a_p is the apothem; $\underline{s}$ is the order of the polygon and A_s is a parameter which has been tabulated in the open literature.

If one expresses (18) in series form it is quite simple to shown that (18) transforms, approximately, the circular hole in the z-plane onto a circular hole of radius r_i in the ζ-plane where:

$$r_i \simeq \frac{R_o}{a_p . A_s} \tag{19}$$

THE FINITE ELEMENT SOLUTION

A finite element code was developed in order to tackle any geometry since the methodology explained in the previous section is rather restricted in the sense that the mapping function must be known in advance.

Triangular elements are used and a linear variation of the temperature field is assumed inside the element.

NUMERICAL RESULTS

In the present investigation numerical results are obtained for square and pentagonal shapes with concentric circular holes, such that $R_o/a_p = 0.10$. The orthotropy parameter ratio k_y/k_x has been taken equal to 1.20.

Figures 2 through 5 show comparisons of temperature values as a function of $\tau = k_x . t/c_p . \rho . a_p^2$, determined:

a) analytically (using the first two terms of (17)), and
b) by means of the finite element algorithm for $k_y/k_x = 1.20$.

The agreement is quite good specially considering the simplicity of relation (17) and the small number of terms taken when using the analytical approach.

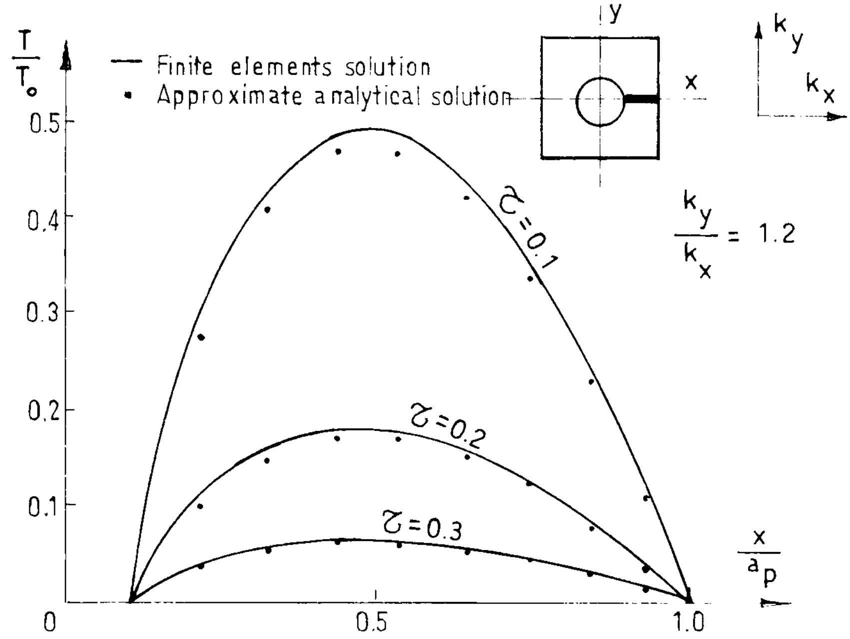

Figure 2 - Temperature variation along the apothem (Outer square boundary).

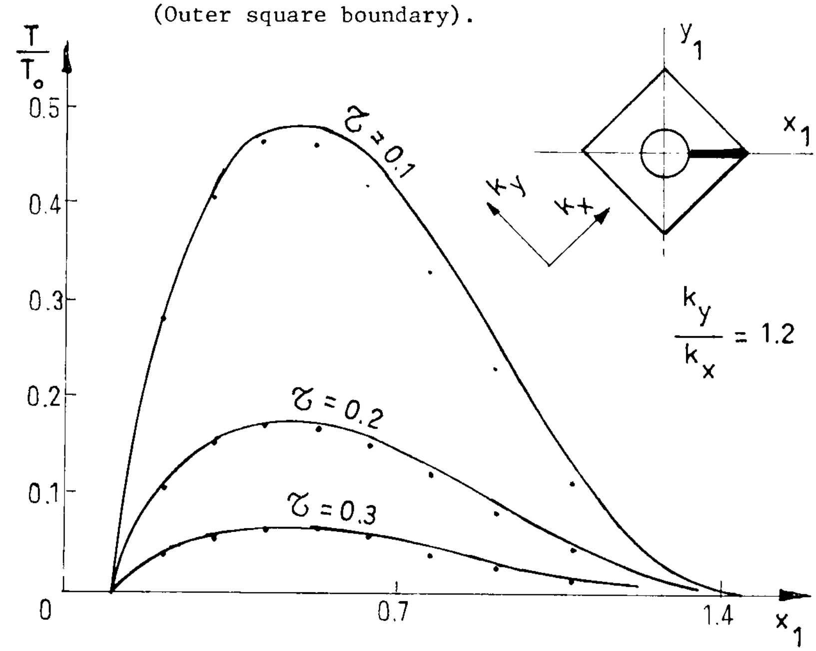

Figure 3 - Temperature variation along the diagonal.

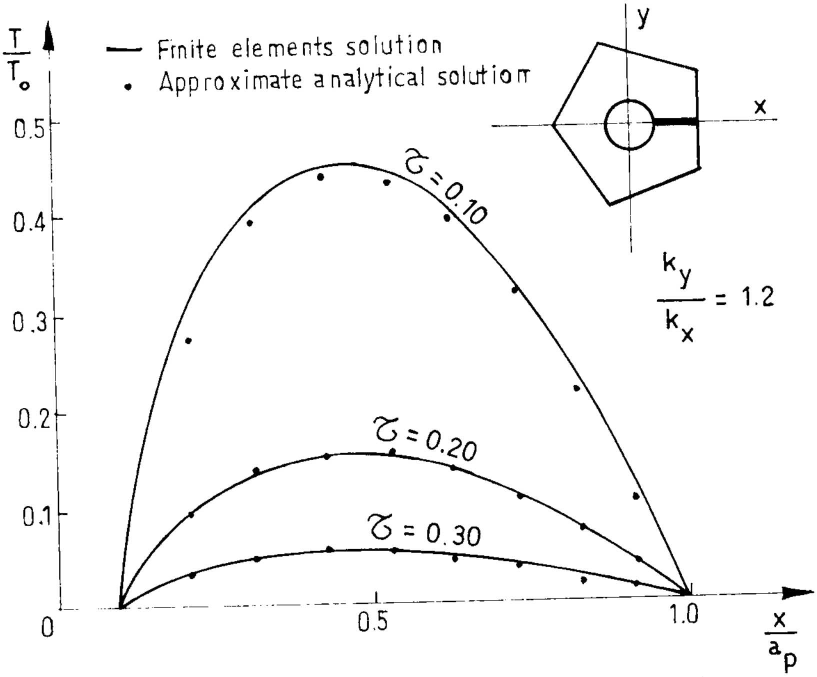

Figure 4 - Temperature variation along the apothem (Outer pentagonal boundary).

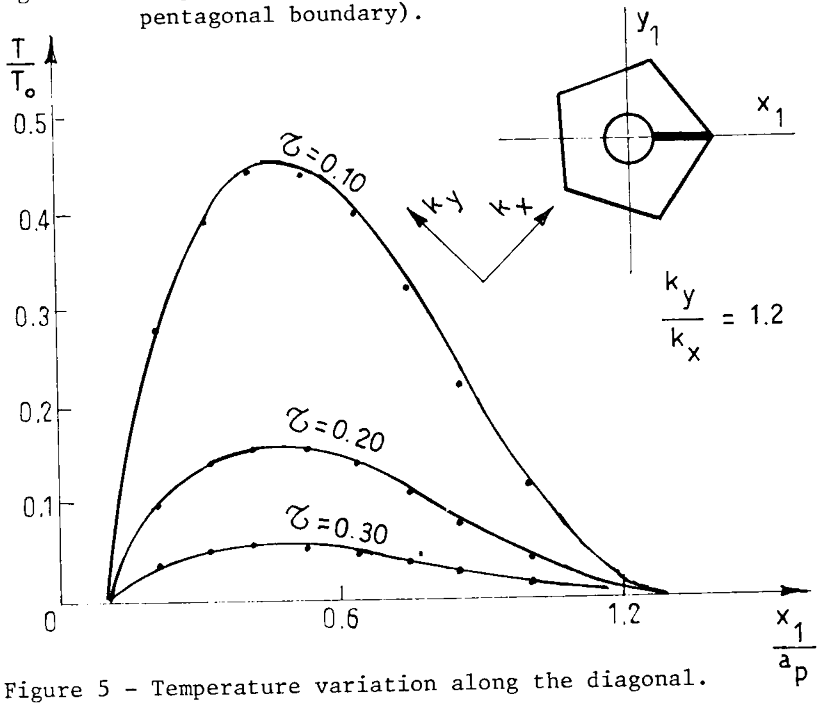

Figure 5 - Temperature variation along the diagonal.

For larger values of the parameter k_y/k_x the agreement is not as good as in the case $k_y/k_x = 1.20$; but it is still acceptable from the point of view of many engineering applications, specially if rather large values of τ are considered.

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