



Finite-Velocity Diffusion in Random Media

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Abstract

We investigated a diffusion-like equation with a bounded speed of signal propagation (the so called telegrapher's equation) in a random media. We discuss some properties of the mean-value solution in a well-defined perturbation theory. The frequency-dependent effective-velocity of propagation is studied in the long and short time regime. We show that due to the wave-like character of telegrapher's equation the effective-velocity is a complex dispersive function in time. Exact results and asymptotic perturbative long-time behaviors (for an exponential space-correlated binary disorder) are presented, showing their agreement and corroborating the goodness of the effective medium approximation in continuous system.

Keywords Telegrapher's equation · Random media · Effective Medium Approximation · Exact solutions

Mathematics Subject Classification 82C44 · 82D30 · 60G60 · 60H25 · 58J65

1 Introduction

The simplest generalization of a diffusion process, but which is also characterized by a bounded propagation speed is the so-called telegrapher's equation. Telegrapher's equation was originally introduced by Lord Kelvin when he was studying dissipation of electromagnetic fields in waveguides. On the other hand, a pioneer work on telegrapher's equation, from a discrete point of view, was introduced by Golsdtein [1], from which many generalization in the context of persistent random walk were later implemented [2]. In a different context, a comprehensive set of references on heat waves and its connection with telegrapher's equation can be seen in [3,4], also open questions on thermal waves' velocity can be read in [5,6].

Telegrapher's equation, in one dimension, read:

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$$\left[\partial_t^2 + \frac{1}{T} \partial_t - v^2 \partial_x^2 \right] P(x, t) = 0 \quad (1)$$

where T is a characteristic time and v is a finite propagation speed. The solution of this equation shows at short times ($t \ll T$) a wave-like feature, while at long-times the diffusion behavior is restored. Equation (1) shears many applications in diffusion at finite speed [7], Electronics and engineers boundary conditions [8], Relativistic Brownian motion [9], Persistent random walk [10,11], Anisotropic scattering [12], and finite-velocity diffusion on non-regular lattice [13], to mention a few of them (see also Appendix A).

Finite-velocity diffusion models in heterogeneous media is an important problem which, to the best of our knowledge, has not been addressed due to the inherent difficulty in working with continuous disordered systems. Discrete models of diffusion in disordered media have successfully been studied using the Effective Medium Approximations (EMA) [14], while the theory of transport in (lattice) random media using Terwiel cumulants [15] has shown to be suitable to go beyond EMA [16]. In fact, the utility of Terwiel's cumulants have also been used to solve different situations in calculating the mean-first passage-time in a lattice with weak and strong disorder [17,18], as well as in (lattice) diffusion problems in finite random media with a dynamic boundary condition [19].

In an important contribution [20], authors introduce Terwiel's cumulants to work out correlated-disorder in a one-dimensional continuous diffusion model. In the present work we propose to study a one-dimensional finite-velocity diffusion model in an heterogeneous medium. Then, we will show that due to the wave character (at short times) of telegrapher's equation, there exist an effective-velocity which is a complex function of time due to the disorder. Only asymptotically at very long-time the effective-velocity goes to a constant, if the disorder is weak. Also the dispersive character of the solution of telegrapher's equation, in a disordered media, will be studied using EMA. The disorder character in the telegrapher equation will be associated to a non-homogeneous finite-velocity of propagation $v(x)$. We will replace, in the non-local in time Fick's law [8], the constant speed v by a space-correlated random field, see Appendix A.

In addition the exact result for the exponential space-correlated binary disorder is presented. Then EMA (for this binary disorder) is compared against exact results, showing agreement and proving the accuracy of the critical exponents predicted by EMA in continuous systems.

2 Telegrapher's Equation in Random Media

We can consider a finite-velocity diffusion process in an infinite 1D disordered medium by generalizing Fick's law, then telegrapher's equation in a random media looks like (see Appendix A for its derivation):

$$\left[\partial_t^2 + \frac{1}{T} \partial_t - v^2 \partial_x^2 \right] P(x, t) = \partial_x \xi(x) \partial_x P(x, t) \equiv \Theta_\xi P(x, t). \quad (2)$$

Here $P(x, t)$ is going to represent a probability and $v^2 + \xi(x)$ must be positive, so $\xi(x)$ is an arbitrary space-dependent correlated random variable bounded from below $\xi(x) > -v^2$.

Interestingly the heterogeneous diffusion term in (2) has the form of Hänggi-Klimontovich's interpretation. Nevertheless it should be noted that equation (2) does not come from a multiplicative Langevin-like equation with Gaussian white noise, which is

when Itô versus Stratonovich dilemma appears [21,22]. Equation (2) arises from the use of a generalized Fick's law and the conservation of particles, see (A2) and (A3).

First, we introduce projector operators over the disorder:

$$\mathcal{P}[\bullet] \equiv \int \cdots \int \mathcal{P}(\xi(x_1), \dots, \xi(x_n)) [\bullet] \prod_{j=1}^n d\xi_j, \quad \forall n = \text{integer} \quad (3)$$

$$\mathcal{Q}[\bullet] \equiv (\mathbf{1} - \mathcal{P})[\bullet], \quad (4)$$

where $\mathcal{P}(\xi(x_1), \dots, \xi(x_n))$ is an arbitrary n -points joint probability associated to the set of random variables $\{\xi_1, \dots, \xi_n\}$. Note that projector $\mathcal{P}[\bullet]$ operates to the right of any functional of the random field $\xi(x)$.

Then, we can write an exact evolution equation for the mean-value $\mathcal{P}P \equiv \langle P(x, t) \rangle$ in the form (see Appendix B for its derivation) [16,17,22]:

$$\Lambda \mathcal{P}P(x, s) = \left(s + \frac{1}{T}\right) \delta(x) + \mathcal{P}\Theta_\xi \sum_{n=0}^{\infty} [\Lambda^{-1} \mathcal{Q}\Theta_\xi]^n \mathcal{P}P(x, s), \quad (5)$$

where Λ is the differential operator:

$$\Lambda = \left[s^2 + \frac{s}{T} - v^2 \partial_x^2\right],$$

and Λ^{-1} is its integral representation in terms of the kernel:

$$g(x, s) = \frac{\exp\left(-\left|\frac{x}{v}\right| \sqrt{s(s+1/T)}\right)}{2\sqrt{v^2 s(s+1/T)}}. \quad (6)$$

It is simple to note that $g(x, s)$ is the inverse Fourier transform $\mathcal{F}_k^{-1} \left[\left(s^2 + \frac{s}{T} + v^2 k^2\right)^{-1} \right]$. The generalized telegrapher equation (5) is equivalent to the expression found for conventional diffusion in a disordered continuous medium [20].

Taking the Fourier transform of (5) and integrating by parts, the exact solution for the mean-value $\mathcal{P}P(x, s)$ can be written in the form (with $P(x, t=0) = \delta(x)$, $\partial_t P(x, t)|_{t=0} = 0$, see Appendix B):

$$\mathcal{P}P(k, s) \equiv \langle P(k, s) \rangle = \frac{(s+1/T)}{s(s+1/T) + (v^2 + \gamma(k, s))k^2}, \quad (7)$$

where $\gamma(k, s)$ takes care of the average over the disorder in the finite-velocity propagation process

$$\begin{aligned} \gamma(k, s) = & \sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} dy_1 e^{iky_1} \cdots \int_{-\infty}^{+\infty} dy_n e^{iky_n} \langle \xi(0) \xi(y_1) \cdots \xi(y_n) \rangle_T \\ & \times \left[\partial_{y_1}^2 g(0 - y_1, s) \right] \cdots \left[\partial_{y_n}^2 g(y_{n-1} - y_n, s) \right]. \end{aligned} \quad (8)$$

Expressions (7) and (8) point out that the heterogeneous media introduces a new dispersive structure in $\langle P(k, s) \rangle$, through the function $\gamma(k, s)$. This function can be studied analyzing all contributions from Terwiel's cumulants: $\langle \xi(0) \xi(y_1) \cdots \xi(y_n) \rangle_T$. Note that because $\xi(x)$ may be correlated to any other random variable ξ at the continuous site x' , Terwiel's cumulants in (8) have contributions all along the domain of integration.

2.1 Short-Times Analysis

A general analysis of (8) at high-frequencies is straightforward to perform using the structure of $g(x, s)$. From (6) we see that any contribution in the series (8) is a well-ordered perturbation when $s \rightarrow \infty$. In fact the n -th term has a multiplicative factor of order $(2\sqrt{v^2 s(s+1/T)})^{-n}$. So we only need to know a n -th Terziel cumulant to perform any integration.

In Sect. 3, after introducing a particular statistical disorder, we will present an exact result and do some calculations in the short and long time regimes. In that occasion, some remarks about the “Markovian” space-correlated binary disorder will be pointed out facing the possibility to find an exact solution.

2.2 Long-Time Analysis

In general from (6) and the fact that:

$$\partial_x^2 g(x, s) = \frac{1}{v^2} [s(s+1/T)g(x, s) - \delta(x)]. \quad (9)$$

it is possible to see, from (8), that even the calculation of $\gamma(k, s=0)$ involves an infinite number of terms. Thus a rearrangement of (8) becomes necessary. Following previous experience [16,17,20] we can reorganize these multiple integrals in such a way to arrive to a simpler expression for $\gamma(k, s)$, that is:

$$\begin{aligned} \gamma(k, s) = & \langle \psi(0) \rangle + \sum_{n=1}^{\infty} \left(\frac{s(s+1/T)}{v^2} \right)^n \int_{-\infty}^{+\infty} dy_1 \cdots \int_{-\infty}^{+\infty} dy_n e^{-iky_n} \langle \psi(0) \psi(y_1) \cdots \psi(y_n) \rangle_T \\ & \times g(0-y_1, s) \cdots g(y_{n-1}-y_n, s). \end{aligned} \quad (10)$$

In this way we have partially taken into account infinite contributions coming from the sum (8) (see Appendix C for its derivation). Here $\psi(x)$ is a random operator acting to the right on any functional of the field $\xi(x)$; that is, if $f = f(\xi(x))$ we get:

$$\begin{aligned} \psi(x)[f] &= \sum_{p=0}^{\infty} \left(\xi(x) (\mathbf{1} - \mathcal{P}) \left(\frac{-1}{v^2} \right) \right)^p \xi(x)[f] \\ &= \left[1 + \frac{1}{v^2} \xi(x) (\mathbf{1} - \mathcal{P}) \right]^{-1} \xi(x)[f]. \end{aligned} \quad (11)$$

We remark that even when we can chose $\langle \xi(x) \rangle = 0$ this will not be the case for the random operator $\psi(x)$. In fact, using (C6) and (C9) it can be seen that $\langle \psi(x) \rangle = \langle \xi(x) / (v^2 + \xi(x)) \rangle / \langle 1 / (v^2 + \xi(x)) \rangle$. Therefore, expression (10) is not useful to calculate the low-frequency behavior of $\gamma(k, s \sim 0)$, because of the non-vanishing average $\langle \psi(x) \rangle \neq 0$. Indeed, frequency-dependent corrections to $\gamma(k, s=0)$ come from an infinite of terms, a detailed review of the intervening Terziel’s diagrams can be seen in [16]. A second rearrangement of the series (10) is necessary to have a useful perturbation theory for low frequency. This new perturbation will be done around an effective-velocity.

The next step in calculating the function $\gamma(k, s)$ (in its asymptotic regime) consists in introducing an EMA. In this way we can simplify many diagrams representing Terziel’s cumulants in series (10). To deal with this issue we introduce an effective-velocity in the random telegrapher’s equation. By adding and subtracting an unknown quantity Γ in (2) we get:

$$\left[\partial_t^2 + \frac{1}{T} \partial_t - (v^2 + \Gamma) \partial_x^2 \right] P(x, t) = \partial_x (\xi(x) - \Gamma) \partial_x P(x, t). \quad (12)$$

Now, we can follow the same steps as before introducing the transformation $v^2 \rightarrow (v^2 + \Gamma) \equiv \tilde{v}^2$ and $\xi(x) \rightarrow (\xi(x) - \Gamma) \equiv \tilde{\xi}(x)$ and so defining a new random operator $\psi(x) \rightarrow \tilde{\psi}(x)$ where $v^2 \rightarrow \tilde{v}^2$, $\xi \rightarrow \tilde{\xi}$, and $g(x, s) \rightarrow \tilde{g}(x, s)$. In this manner by demanding that $\langle \tilde{\psi}(x) \rangle = 0$ we reduce a large amount of Terwiel's diagrams appearing in (10), when the position of the field $\xi(x)$ is far enough from any other at position x' . This condition allows to solve the value of Γ and to calculate the asymptotic behavior of $\tilde{\gamma}(k, s \sim 0)$ at any order in the small parameter s .

2.3 The Effective Medium Approximation

As previously mentioned, from (12) and introducing the same steps as before an expression for the perturbative calculation of $\tilde{\gamma}(k, s \rightarrow 0)$ will be reached; that is, the dispersive effective-velocity in the random medium. In this way the mean-value solution of telegrapher's equation will be (see Appendix C.2):

$$\langle P(k, s) \rangle = \left[s + \frac{(\tilde{v}^2 + \tilde{\gamma}(k, s)) k^2}{s + 1/T} \right]^{-1}, \quad (13)$$

where:

$$\begin{aligned} \tilde{\gamma}(k, s) = & \langle \tilde{\psi}(0) \rangle + \sum_{n=1}^{\infty} \left(\frac{s(s + 1/T)}{\tilde{v}^2} \right)^n \int_{-\infty}^{+\infty} dy_1 \dots \int_{-\infty}^{+\infty} dy_n e^{-iky_n} \langle \tilde{\psi}(0) \tilde{\psi}(y_1) \dots \tilde{\psi}(y_n) \rangle_T \\ & \times \tilde{g}(0 - y_1, s) \dots \tilde{g}(y_{n-1} - y_n, s), \end{aligned} \quad (14)$$

with $\tilde{g}(y, s)$, read from (6), as

$$\tilde{g}(x, s) = \frac{\exp\left(-\left|\frac{x}{\tilde{v}}\right| \sqrt{s(s + 1/T)}\right)}{2\sqrt{\tilde{v}^2 s(s + 1/T)}}, \quad (15)$$

and from (11):

$$\tilde{\psi}(x) = \left[1 + \frac{\tilde{\xi}(x)}{\tilde{v}^2} (\mathbf{1} - \mathcal{P}) \right]^{-1} \tilde{\xi}(x). \quad (16)$$

Now expression (14) is appropriate to obtain the low-frequency behavior of $\tilde{\gamma}(k, s)$. The order of each term $\tilde{\gamma}_n(k, s)$ can be estimated noting that:

$$\lim_{s \rightarrow 0} \frac{s(s + 1/T)}{\tilde{v}^2} \tilde{g}(x, s \sim 0) = \frac{\sqrt{s/T}}{2|\tilde{v}|^3} - \frac{|x|s}{2\tilde{v}^4 T} + \left(\frac{\sqrt{T}}{|\tilde{v}|^3} + \frac{x^2}{T^{3/2}|\tilde{v}|^5} \right) \frac{s^{3/2}}{4} + \mathcal{O}(s^2) \quad (17)$$

Introducing (17) in (14) the first dominant correction is:

$$\tilde{\gamma}_1(k, s \sim 0) \simeq \int_{-\infty}^{+\infty} dy_1 e^{-iky_1} \langle \tilde{\psi}(0) \tilde{\psi}(y_1) \rangle_T \frac{\sqrt{\tilde{v}^2 s/T}}{2\tilde{v}^4} + \mathcal{O}(s) \quad (18)$$

In order to continuous with this calculation it is necessary to specify the statistics of the random field $\xi(x)$, to be able to evaluate the Terwiel cumulant appearing in (18). In the same manner it is possible to see that $\tilde{\gamma}_2(k, s \sim 0) \sim \mathcal{O}(s)$.

Here it is important to note that we can write the random operator $\tilde{\psi}(x)$ in a simpler form to be handled for future calculations

$$\tilde{\psi}(x)[\bullet] = \left\{ \tilde{M}(x) + \frac{\tilde{M}(x)/\tilde{v}^2}{\left\{1 - \langle \tilde{M}(x) \rangle / \tilde{v}^2\right\}} \mathcal{P} \tilde{M}(x) \right\} [\bullet], \quad (19)$$

where $\tilde{M}(x) = \tilde{\xi}(x) / \left[1 + \tilde{\xi}(x) / \tilde{v}^2\right]$ is a random number, with $\tilde{v}^2 = (v^2 + \Gamma)$ and $\tilde{\xi}(x) = (\xi(x) - \Gamma)$, see (C12).

Imposing the condition $\langle \tilde{\psi}(x) \rangle = 0$ (for all x due to the stationary property of the random field $\xi(x)$) allows a precise value for Γ , and so we get (see Appendix C.2):

$$\Gamma = \left\langle \frac{\xi(x)}{v^2 + \xi(x)} \right\rangle / \left\langle \frac{1}{v^2 + \xi(x)} \right\rangle. \quad (20)$$

This general expression gives the effective-velocity in the static regime (frequency limit $s \rightarrow 0$). We can apply this formula to any statistic of the random filed $\xi(x)$ provided that $\xi(x) > -v^2$. The particular situation when $\tilde{v}^2 = v^2 + \Gamma = 0$ corresponds to a strong disorder case which will not be treated systematically here. In the next section we will present a binary correlated disorder that also allows to see the strong disorder transition to $\tilde{v}^2 \rightarrow 0$.

3 Space-Correlated Binary Disorder

In this section we are going to apply our previous results to a particular class of correlated disorder, this model can also be called dichotomic space-disorder. In this case the random filed $\xi(x)$ assumes two possible values $\pm\Delta$ which will be taken equally probable, thus the 2-points correlation function has an exponential space translational structure:

$$\langle \xi(x_1) \xi(x_2) \rangle = \Delta^2 \exp(-|x_1 - x_2|/\lambda), \quad (21)$$

here λ is the space-correlation length and Δ measures the intensity of the disorder (in units of square velocity). Higher order correlation functions $\langle \xi(x_1) \dots \xi(x_j) \rangle$ can easily be calculated because the 2-points exponential structure (21) emulates a space “Markov” disorder [22,23]. In order to keep the positivity of $v^2 + \xi(x)$ for any realization of the disorder the amplitude of the random field $\xi(x)$ must be bounded to $\Delta < v^2$, the case $\Delta = v^2$ would correspond to strong disorder.

Other interesting models of space-correlated disorder can also be used with the present approach. For example a space intermittent binary disorder (non-Markovian) can be worked out knowing all the corresponding n -points correlation functions [24].

3.1 Exact Result (Exponential-Correlated Space Disorder)

We have commented in Sect. 2.1 that the high frequency analysis of $\gamma(k, s)$ can be done using (8). In addition, it is simple to prove [22] that for an exponential-correlated *symmetric* binary disorder, any Terwiel’s cumulant of order $m \geq 3$ is zero. Therefore, using (8) for dichotomic space-disorder and applying (9) we get the exact result:

$$\begin{aligned}
\gamma_E(k, s) &= \int_{-\infty}^{+\infty} dy_1 e^{iky_1} \langle \xi(0) \xi(y_1) \rangle_T \left[\partial_{y_1}^2 g(0 - y_1, s) \right] \\
&= \frac{s(s+1/T) \Delta^2}{2v^2 \sqrt{v^2 s(s+1/T)}} \int_{-\infty}^{+\infty} dy e^{iky} \exp(-|y|/\lambda) \exp\left(-|y| \sqrt{\frac{s(s+1/T)}{v^2}}\right) - \frac{\Delta^2}{v^2} \\
&= \frac{\Delta^2 \sqrt{s(s+1/T)/v^2}}{v^2} \frac{\left(\frac{1}{\lambda} + \sqrt{\frac{s(s+1/T)}{v^2}}\right)}{\left(\left(\frac{1}{\lambda} + \sqrt{\frac{s(s+1/T)}{v^2}}\right)^2 + k^2\right)} - \frac{\Delta^2}{v^2}, \quad (22)
\end{aligned}$$

the last line has been calculated doing the integral by Fourier transform. We noted that this expression is only exact for the “Markovian” symmetric dichotomic disorder. Formula (22) gives the exact dispersive character of the effective-velocity. Then from (7) and (22) we can write the exact mean-value telegrapher’s solution in the form:

$$\begin{aligned}
\langle P(k, s) \rangle &= \left(s + \frac{v^2 + \gamma_E(k, s)}{(s+1/T)} k^2 \right)^{-1} \\
&\equiv (s + \mathcal{D}_E(k, s) k^2)^{-1}. \quad (23)
\end{aligned}$$

This resembles the diffusion propagator. Then, we can call $\mathcal{D}_E(k, s)$ the effective dispersive diffusive function, which can be study in the limit $s \gg 1/T$ (wave regime), and in the limit $s \ll 1/T$ (diffusive regime). In fact it is possible to see from function $\mathcal{D}_E(k=0, s)$ that in the wave regime if the traveled distance $(v/s) \gg \lambda$, the effective-velocity is modified by the disorder. While in the opposite case $(v/s) \ll \lambda$, there is not a net effect from the action of the disorder, see inset in Fig. 1. On the other hand, in the limit $s \ll 1/T$ if the diffusive distance $\sqrt{T v^2/s} \gg \lambda$, there are some effects from the disorder; while in the opposite case $\sqrt{T v^2/s} \ll \lambda$, there is not a net effect from the disorder.

It can be seen from (23) and using the generalized Einstein relation [22], that function $\mathcal{D}_E(k=0, s)$ corresponds to the Laplace representation of the stationary 2-times velocity correlation function associated to the mean-value telegrapher’s solution.

3.1.1 Long-Times

We can also use our formula to calculate the exact long-time behavior of the total effective-velocity. Therefore, in the limit $s \rightarrow 0$, from (22) we get

$$v^2 + \gamma_E(k, s \rightarrow 0)|_{k=0} \simeq v^2 - \frac{\Delta^2}{v^2} + \frac{\lambda \Delta^2}{|v|^3} \sqrt{\frac{s}{T}}. \quad (24)$$

In the next section we will show that this behavior can be recovered by using the EMA, a theory that is fundamental for the calculus of the mean-value over the disorder when there is not possibilities to get any exact result.

3.1.2 Strong-Disorder

In the particular case when the disorder produces a zero *static* total effective-velocity, it is called strong disorder. This situation can be achieved with the present “Markovian” binary model when the intensity of the random field $\xi(x)$ takes random values $\pm \Delta = \pm v^2$. From

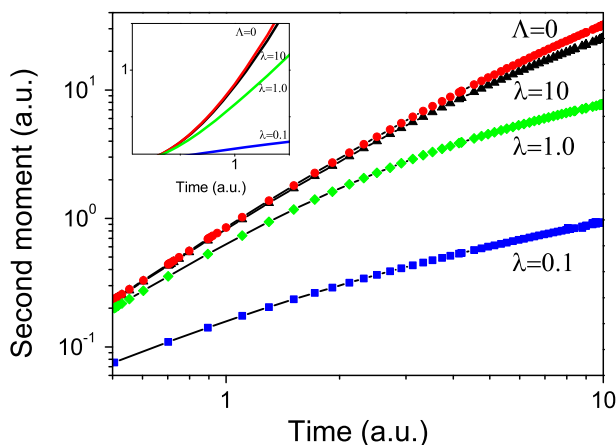


Fig. 1 Log-log time evolution of the second moment $\langle x(t)^2 \rangle$ of telegrapher's probability in presence of strong binary disorder, for different values of the correlation length λ . The intensity of the noise is $\Delta = 1$ and telegrapher's parameters are: $T = 2$ and $v = 1$. As a reference the ordered case ($\Delta = 0$) is also included. On-line color figure: red circles $\Delta = 0$, black triangles $\lambda = 10$, green rhomboid $\lambda = 1$, blue square $\lambda = 0.1$. Inset corresponds to short-time regime, in linear scale, the competition between wave-motion and strong disorder (as a function of λ) can be seen

current result (22) it is possible to analyze this situation. Using the asymptotic regime (24) and introducing the value $\Delta = v^2$ we get:

$$v^2 + \gamma_E(k, s \rightarrow 0)|_{k=0} \simeq \lambda \sqrt{\frac{sv^2}{T}}, \quad (25)$$

this is a genuine anomalous behavior for the total effective velocity. In addition from function $\mathcal{D}_E(k, s)$, evaluated at $\Delta = v^2$, we can study the exact behavior of telegrapher's solution in presence of a strong disorder model. In Fig. 1 we have plotted the temporal behavior of $\langle x(t)^2 \rangle$ from the solution of mean-value telegrapher's equation. This second moment can easily be computed from (23) after calculating the inverse Laplace transform:

$$\langle x(t)^2 \rangle = \mathcal{L}_s^{-1} \left[\frac{2\mathcal{D}_E(k=0, s)}{s^2} \right],$$

as a matter of reference we also plotted, in the same figure, the case with zero disorder $\Delta = 0$. Thus, we can compare different behaviors of the second moment as a function of the correlation length λ , for this particular case of strong disorder.

On the other hand, using the Tauberian theorem [22] it is possible to see, from (23), that at short-times there is a ballistic limit $\langle x(t \rightarrow 0)^2 \rangle \propto t^2$, and if the disorder is weak at long-times it is diffusive $\langle x(t \rightarrow \infty)^2 \rangle \propto t$. While if the disorder is strong, in the short time regime there is an anomalous ballistic behavior, see inset in Fig. 1,

$$\langle x(t \sim 0)^2 \rangle \simeq v^2 t^2 \left(1 - \frac{|v|}{\lambda} t + \dots \right),$$

and in the long time limit the behavior is subdiffusive

$$\langle x(t \rightarrow \infty)^2 \rangle \propto t^{1/2}.$$

3.1.3 The Pure Diffusive Regime

From telegrapher's equation (1) if we take $T \rightarrow 0$ and $v^2 \rightarrow \infty$ such that $v^2 T \rightarrow D$ (diffusion constant) this equation turns to be the conventional diffusion one. The same limits can be taken considering an heterogeneous diffusion coefficient $D(x) \rightarrow D + \xi(x) > 0$. Then, in this case the *exact* mean-value solution is:

$$\langle P(k, s) \rangle = \frac{1}{s + (D + D_E(k, s)) k^2}, \quad (26)$$

where $D_E(k, s)$ can be read from (22) in the same limits. Note that in this case the binary random field $\xi(x)$ take values $\xi(x) = \pm \Delta$ given in the dimension of the diffusion coefficient D . So in the proper dimensions, we get from (22) a formula for the dispersive effective diffusion coefficient:

$$D_E(k, s) = \left(\sqrt{s/D} \frac{(\frac{1}{\lambda} + \sqrt{s/D})}{((\frac{1}{\lambda} + \sqrt{s/D})^2 + k^2)} - 1 \right) \frac{\Delta^2}{D}, \quad (27)$$

Formula (26) and (27) give the exact mean-value diffusion solution under the action of a (Markov) space-correlated dichotomic disorder. From (26) it is possible to see, also, the agreement with the asymptotic results presented in [20]; as well as the value of the static effective diffusion coefficient: $(D^2 - \Delta^2)/D$.

3.1.4 The Pure Wave Regime

The wave equation can be obtained from telegrapher's equation by taking $T \rightarrow \infty$, therefore we can also analyze this case from our exact solution (22). Thus $\langle P(k, s) \rangle$, the solution of the wave equation under the action of a (Markov) space-correlated dichotomic disorder follows from (23) as:

$$\langle P(k, s) \rangle = (s + \mathcal{D}_E(k, s) k^2)^{-1}, \quad \partial_t P(k, t)|_{t=0} = 0 \quad (28)$$

with:

$$\mathcal{D}_E(k, s) = \frac{1}{s} \left(v^2 + \frac{\Delta^2}{v^2} \left(\frac{s}{|v|} \frac{1/\lambda + s/|v|}{(1/\lambda + s/|v|)^2 + k^2} - 1 \right) \right)^{-1}.$$

This formula is valid for any value of the parameter v, λ and $\Delta \leq v^2$. In particular, the case $\Delta = v^2$ corresponds to strong binary disorder.

4 EMA for Space-Correlated Binary Disorder

4.1 The Static Effective-Velocity Γ

Because Eq. (20) demands only an average at the same site x , the calculation of Γ can easily be done using the stationary 1-point probability:

$$P(\xi) = (\delta_{\xi, \Delta} + \delta_{\xi, -\Delta})/2. \quad (29)$$

Then, from (20) we get $\Gamma = -\frac{\Delta^2}{v^2}$ and so the total effective-velocity at very long-times turns to be

$$\tilde{v}^2 = (v^2 + \Gamma) = \frac{v^4 - \Delta^2}{v^2}, \quad (30)$$

and does not depend on the correlation λ , this is so due to the structure of the 1-point probability $P(\xi)$.

The limit when the amplitude of the random field $\xi(x)$ goes to $\Delta \rightarrow v^2$ corresponds to a strong disorder case (the static total effective-velocity is zero). This result coincide with our exact calculation (24).

4.2 Corrections to the Long-Time Behavior $\tilde{\gamma}(k, s \sim 0)$

Generally in order to calculate the Laplace-dependent contributions of the effective-velocity $\tilde{\gamma}(k, s)$, higher order Terziel's cumulants must be considered; that is, $\tilde{\gamma}(k, s \sim 0) = \sum_{n=1}^{\infty} \tilde{\gamma}_n(k, s \sim 0)$. This can be done order-by-order in power of s from (14).

In the particular case of binary disorder, by using expression (19) we finally arrive to the alternative expression

$$\tilde{\psi}(x) [\bullet] = \frac{(v^4 - \Delta^2)}{v^4} \left[\xi(x) + \frac{1}{v^2} \xi(x) \mathcal{P}\xi(x) \right] [\bullet], \quad (31)$$

which is much simpler to be used (see Eq. (D3) in Appendix D for its derivation). The first correction to the zero frequency $\tilde{\gamma}(k, s = 0)$ demands the calculation of Terziel's cumulant: $\langle \tilde{\psi}(0) \tilde{\psi}(y_1) \rangle_T$, this object can be calculated from (31) as:

$$\begin{aligned} \langle \tilde{\psi}(0) \tilde{\psi}(y_1) \rangle_T &= \left(\frac{v^4 - \Delta^2}{v^4} \right)^2 \mathcal{P} \left[\xi(0) + \frac{1}{v^2} \xi(0) \mathcal{P}\xi(0) \right] \mathcal{Q} \left[\xi(y_1) + \frac{1}{v^2} \xi(y_1) \mathcal{P}\xi(y_1) \right] \\ &= \left(\frac{v^4 - \Delta^2}{v^4} \right)^2 \Delta^2 \exp(-|y_1|/\lambda), \end{aligned} \quad (32)$$

in the second line we have explicitly used (3) and (21).

In order to study the effective-velocity we keep the Fourier dependence in $\tilde{\gamma}_1(k, s)$ and carried on the evaluation of the integral, then by introducing $\tilde{g}(0 - y_1, s)$, from (15), we get:

$$\begin{aligned} \tilde{\gamma}_1(k, s) &= \left(\frac{s(s + 1/T)}{\tilde{v}^2} \right) \int_{-\infty}^{+\infty} dy_1 e^{-iky_1} \langle \tilde{\psi}(0) \tilde{\psi}(y_1) \rangle_T \tilde{g}(0 - y_1, s), \\ &= \frac{\sqrt{\tilde{v}^2 s(s + 1/T)} \Delta^2}{v^4} \left[\frac{\left(\frac{1}{\lambda} + \sqrt{s(s + 1/T)/\tilde{v}^2} \right)}{\left(\frac{1}{\lambda} + \sqrt{s(s + 1/T)/\tilde{v}^2} \right)^2 + k^2} \right] \end{aligned} \quad (33)$$

Therefore, from this expression, up to order $\mathcal{O}(s^{1/2})$, the total effective-velocity gets:

$$\tilde{v}^2 + \tilde{\gamma}_1(k, s)|_{k=0} = \frac{v^4 - \Delta^2}{v^2} + \sqrt{1 - \frac{\Delta^2}{v^4} \frac{\Delta^2 \lambda}{|v|^3}} \sqrt{s(s + 1/T)} + \mathcal{O}(s). \quad (34)$$

This dominant correction agrees with the exact result given in (24), we also noted that even when the exponent is the same, there is some difference with the prefactor. As it is known EMA predicts time-exponents while the factors are not correct. This is something that has

been found in lattice disordered models [16], while in continuous systems it is harder to work out.

Higher order corrections can also be calculated in a similar way, let me show here the next one:

$$\begin{aligned}\tilde{\gamma}_2(k, s) &= \left(\frac{s(s+1/T)}{\tilde{v}^2} \right)^2 \int_{-\infty}^{+\infty} dy_1 \int_{-\infty}^{+\infty} dy_2 e^{-iky_2} \left\langle \tilde{\psi}(0) \tilde{\psi}(y_1) \tilde{\psi}(y_2) \right\rangle_T \tilde{g}(0-y_1, s) \\ &\quad \tilde{g}(y_1-y_2, s) \\ &= \frac{s(s+1/T)\Delta^4}{v^8} \left(\frac{\left(\frac{1}{\lambda} + \sqrt{s(s+1/T)/\tilde{v}^2} \right)}{\left(\frac{1}{\lambda} + \sqrt{s(s+1/T)/\tilde{v}^2} \right)^2 + k^2} \right)^2,\end{aligned}\quad (35)$$

which of course has a dominant contribution in agreement with the exact result (22).

5 Discussions

We have presented a perturbation theory to study telegrapher's equation in a random media. The method is based on the application of projector operators techniques that lead to a Terwiel cumulant expansion for the effective-velocity. This series has been presented in two different forms: the first expansion is appropriate for a perturbation in the high-frequency limit $s \rightarrow \infty$ (short-times), this is so because it is a perturbation around the ordered value of the velocity of propagation v , see (8). While the second expansion has been done around an EMA which allows for calculation of the "static" effective-velocity of propagation Γ , and it is suitable for a perturbation in the low-frequency limit $s \rightarrow 0$ (long-times). That is, the EMA allows also to get corrections to the effective-velocity at low-frequencies $\tilde{\gamma}(k=0, s \sim 0)$, see (14).

The present approach is valid for a general random field $\xi(x)$ with the physical constraint imposed by considering positive square velocity for any realization of the disorder $v^2 + \xi(x) > 0$. This restriction imposed a bounded random field and, hence, prevent the use of a Gaussian space-correlated field. In view of our results this fact is essential to obtain a positive and real square velocity. We have applied our approach considering an exponential space-correlated binary disorder to emulate the random media. Different random fields can be used into the present theory. For example, an interesting model of disorder in the context of plasma physics, is when the random field $\xi(x)$ shows space intermittence as in non-Markovian binary models [24].

In the particular case of a Markovian space-correlated symmetric binary disorder the exact solution has been found, which was used to test the goodness of the EMA in continuous systems. We have shown that EMA gives the correct asymptotic exponent, while the multiplicative factor is underestimated. Higher order corrections with n -th Terwiel's cumulants have also been presented to show the accuracy of EMA for the calculation of the mean-value solution $\langle P(k, s) \rangle$, see (33) and (35). In addition the exact result (22), for weak and strong binary disorder, can be mapped to a pure diffusion equation, then we have also compared our result against the perturbative ones presented in Ref. [20]. Exact result (23) can also be mapped to study a pure undulatory regime, giving insight into the important problem of wave propagation in random media [23], see (28).

For the present binary disorder and due to the wave-like character of telegrapher's equation, the exact frequency-dependent effective-velocity $\gamma_E(k, s)$ shows a complex structure, see (22). In fact, the time-dependent velocity correlation function has a non-monotonic behavior which can be seen from the inverse Laplace transform of the function $\mathcal{D}_E(k=0, s)$, see

(23). We have shown that due to the correlated space disorder the wave-packet moves with a reduced total effective-velocity, this reduction in the velocity of propagation depends on the value of the correlation length λ and the intensity of the disorder Δ . At very long-times when the diffusion regime dominates the solution, if the disorder is weak the effective-velocity $\tilde{\gamma}(k, s)$ reduces to a constant value Γ that depends only on the intensity of the disorder (30). In general the time-structure of this relaxation depends on the correlation length λ , the intensity Δ and parameters of telegrapher's equation: T and v . The exact mean square displacement (second-moment) of the mean-value telegrapher's solution shows the signature of the mentioned time-dependent complex structure in the effective velocity, see Fig. 1. For strong binary disorder we have shown that in the short time regime the behavior is anomalous ballistic, while in the long-time limit it is sub-diffusive (see Sect. 3.1.2).

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Appendix A: Telegrapher's Equation on Random Media

The simplest kind of absorptive (electromagnetic) wave model is telegrapher's equation (1), there T^{-1} is a measure of the wave attenuation and v is a characteristic velocity, both parameters given in terms of Maxwell's equations and Ohm's law, that is:

$$T = \varepsilon/4\pi\sigma \quad \text{and} \quad v^2 = c^2/\mu\varepsilon,$$

here ε is the dielectric and μ the permeability constants, c the light velocity, while σ is the conductivity (responsible of Joule effects in a waveguides).

In the context of an electronic circuit with inductance L , resistance R , capacity C and leakage conductance per unit length G , the telegrapher parameters T and v can be identified as [25]:

$$T = (G/C - R/L)^{-1} \quad \text{and} \quad v = (LC)^{-1/2}.$$

In the context of a diffusion-like process we noted that telegrapher's equation (1) can alternative be transformed to an integral equation with a time-memory kernel:

$$\partial_t P(x, t) = v^2 \int_0^t e^{-(t-\tau)/T} \partial_x^2 P(x, \tau) d\tau \quad \text{with} \quad \partial_t P(x, t)|_{t=0} = 0, \quad (\text{A1})$$

and conditions for positive and normalization of solutions. In this form T and v can also be related with parameters of a persistent lattice random walk [8, 11, 12]. The last non-local in time equation (A1) can also be considered a generalized master equation [26]. Therefore, the continuous-time random-walk (CTRW) theory, with internal states [27–29], allows successfully to be used to consider further generalizations of telegrapher's equation with fractional calculus in the context of fractal-time and/or fractal-space persistent random-walk dynamics [2, 22].

Interestingly, telegrapher's equation (1) can also be introduced from a phenomenological point of view considering a non-local in time Fick's law [8]:

$$J(x, t) = -v^2 \int_0^t e^{-(t-\tau)/T} \partial_x P(x, \tau) d\tau. \quad (\text{A2})$$

Combining this equation with the conservation law: $\partial_t P(x, t) + \partial_x J(x, t) = 0$, the telegrapher equation (1) is recovered.

In addition, if in the non-local Fick's law (A2) we consider a random velocity field (as possible effects from the heterogeneous media) we can replace $v^2 \rightarrow v^2 + \xi(x)$ in Eq. (A2). Then, noting that:

$$\partial_t J(x, t) = - (v^2 + \xi(x)) \partial_x P(x, t) - J(x, t) / T \quad (\text{A3})$$

and using the conservation law, the telegrapher equation (2) (in a random media) is obtained.

Appendix B: Projector Operator Techniques Revisited

Projector operator techniques has a long history in physics, for a comprehensive review see [23]. Consider the Laplace transform of Eq. (2) and initial conditions: $P(x, t = 0) = \delta(x)$, $\partial_t P(x, t)|_{t=0} = 0$. After some algebra we get:

$$\left(s^2 + \frac{s}{T} - v^2 \partial_x^2\right) P(x, s) = \left(s + \frac{1}{T}\right) \delta(x) + \partial_x \xi(x) \partial_x P(x, s). \quad (\text{B1})$$

If we apply the projectors operator $\mathcal{P}$ and $\mathcal{Q} = (\mathbf{1} - \mathcal{P})$ to Eq. (B1) we get:

$$\Lambda \mathcal{P} P(x, s) = \left(s + \frac{1}{T}\right) \delta(x) + \mathcal{P} \Theta_\xi (\mathcal{P} + \mathcal{Q}) P(x, s) \quad (\text{B2})$$

$$\Lambda \mathcal{Q} P(x, s) = \mathcal{Q} \Theta_\xi (\mathcal{P} + \mathcal{Q}) P(x, s), \quad (\text{B3})$$

where we have defined the differential operator

$$\Lambda = \left(s^2 + \frac{s}{T} - v^2 \partial_x^2\right), \quad (\text{B4})$$

and

$$\Theta_\xi = \partial_x \xi(x) \partial_x.$$

Solving $\mathcal{Q} P(x, t)$ in terms of $\mathcal{P} P(x, t)$ and replacing this expression in (B2) we finally get a formal result in terms of the integral convolutional operator Λ^{-1} with kernel:

$$g(x, s) = \exp\left(-\left|\frac{x}{v}\right| \sqrt{s(s + 1/T)}\right) / 2\sqrt{v^2 s(s + 1/T)}. \quad (\text{B5})$$

In fact, the Green function of operator (B4) is $g(k, s) = (s^2 + \frac{s}{T} + v^2 k^2)^{-1}$, thus in real space we get:

$$g(x, s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx} \frac{dk}{(s^2 + \frac{s}{T} + v^2 k^2)},$$

this integral can be done by residues and the result is (B5). Using (B5) and the formal expression for $\mathcal{P} P(x, t)$, gives, after some algebra, the exact equation:

$$\Lambda \mathcal{P} P(x, s) = \left(s + \frac{1}{T}\right) \delta(x) + \mathcal{P} \Theta_\xi \sum_{n=0}^{\infty} [\Lambda^{-1} \mathcal{Q} \Theta_\xi]^n \mathcal{P} P(x, s), \quad (\text{B6})$$

Sum terms in Eq. (B6) take care of the average over the disorder and it is represented in its integral form as:

$$\mathcal{P}\partial_x \sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} dx_1 \dots \int_{-\infty}^{+\infty} dx_n \langle \xi(x) \xi(x_1) \dots \xi(x_n) \rangle_T \quad (B7)$$

$$\times \left[\partial_{x_1}^2 g(x - x_1, s) \right] \dots \left[\partial_{x_n}^2 g(x_{n-1} - x_n, s) \right] \partial_{x_n} \langle P(x_n, s) \rangle,$$

where the object $\langle \xi(x) \xi(x_1) \dots \xi(x_n) \rangle_T$ is a Terwiel cumulant of order $n + 1$, that is:

$$\langle \xi(x) \xi(x_1) \dots \xi(x_n) \rangle_T \equiv \mathcal{P}\xi(x) \mathcal{Q}\xi(x_1) \mathcal{Q} \dots \mathcal{Q}\xi(x_n). \quad (B8)$$

Using the space-convolution structure of (B6) we can straightforwardly take the Fourier transform, from which we get the explicit result:

$$\mathcal{P}P(k, s) = \frac{(s + 1/T)}{s(s + 1/T) + (v^2 + \gamma(k, s))k^2}, \quad P(k, t = 0) = 1, \quad \partial_t P(k, t)|_{t=0} = 0, \quad (B9)$$

where the function $\gamma(k, s)$ is given by:

$$\gamma(k, s) = \sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} dy_1 e^{iky_1} \dots \int_{-\infty}^{+\infty} dy_n e^{iky_n} \langle \xi(0) \xi(y_1) \dots \xi(y_n) \rangle_T \quad (B10)$$

$$\times \left[\partial_{y_1}^2 g(0 - y_1, s) \right] \dots \left[\partial_{y_n}^2 g(y_{n-1} - y_n, s) \right],$$

this is just (8). To prove (B10) note that the first term in the sum of (B6) explicitly read:

$$\mathcal{P}\Theta_{\xi} \langle P(x, s) \rangle = \partial_x \langle \xi(x) \rangle \partial_x \langle P(x, s) \rangle, \quad (B11)$$

which cancels if the mean-value of $\xi(x)$ is zero. The second term read:

$$\mathcal{P}\Theta_{\xi} \Lambda^{-1} \mathcal{Q}\Theta_{\xi} \langle P(x, s) \rangle = \mathcal{P}\Theta_{\xi x} \int_{-\infty}^{+\infty} dx_1 g(x - x_1, s) \mathcal{Q}\Theta_{\xi_1} \langle P(x_1, s) \rangle$$

$$= \mathcal{P}\Theta_{\xi x} \int_{-\infty}^{+\infty} dx_1 g(x - x_1, s) \mathcal{Q}\partial_{x_1} \xi(x_1) \partial_{x_1} \langle P(x_1, s) \rangle, \quad (B12)$$

here the last integral can be integrated by parts to get:

$$\mathcal{P}\Theta_{\xi} \Lambda^{-1} \mathcal{Q}\Theta_{\xi} \langle P(x, s) \rangle = -\mathcal{P}\Theta_{\xi x} \int_{-\infty}^{+\infty} dx_1 \left[\partial_{x_1} g(x - x_1, s) \right] \mathcal{Q}\xi(x_1) \partial_{x_1} \langle P(x_1, s) \rangle$$

$$= -\mathcal{P}\partial_x \xi(x) \int_{-\infty}^{+\infty} dx_1 \mathcal{Q}\xi(x_1) \left[\partial_x \partial_{x_1} g(x - x_1, s) \right] \partial_{x_1} \langle P(x_1, s) \rangle$$

$$= -\partial_x \int_{-\infty}^{+\infty} dx_1 \mathcal{P}\xi(x) \mathcal{Q}\xi(x_1) \left[\partial_x \partial_{x_1} g(x - x_1, s) \right] \partial_{x_1} \langle P(x_1, s) \rangle$$

$$= \partial_x \int_{-\infty}^{+\infty} dx_1 \langle \xi(x) \xi(x_1) \rangle_T \left[\partial_{x_1}^2 g(x - x_1, s) \right] \partial_{x_1} \langle P(x_1, s) \rangle. \quad (B13)$$

If we now introduce the Fourier transform in (B13) and use the stationary behavior of Terwiel's cumulants we get:

$$\begin{aligned}
\mathcal{F}_k & \text{ [B13]} \\
& \equiv \int_{-\infty}^{+\infty} dx e^{ikx} \partial_x \int_{-\infty}^{+\infty} dx_1 \langle \xi(x) \xi(x_1) \rangle_T \left[\partial_{x_1}^2 g(x - x_1, s) \right] \partial_{x_1} \langle P(x_1, s) \rangle \\
& = (-ik) \int_{-\infty}^{+\infty} dx e^{ik(x-x_1+x_1)} \int_{-\infty}^{+\infty} dx_1 \langle \xi(x) \xi(x_1) \rangle_T \left[\partial_{x_1}^2 g(x - x_1, s) \right] \partial_{x_1} \langle P(x_1, s) \rangle \\
& = (-ik) \int_{-\infty}^{+\infty} dx_1 e^{ikx_1} \left[\int_{-\infty}^{+\infty} dx e^{ik(x-x_1)} \langle \xi(x) \xi(x_1) \rangle_T \left[\partial_{x_1}^2 g(x - x_1, s) \right] \right] \partial_{x_1} \langle P(x_1, s) \rangle \\
& = -k^2 \left[\int_{-\infty}^{+\infty} dy_1 e^{iky_1} \langle \xi(0) \xi(y_1) \rangle_T \left[\partial_{y_1}^2 g(0 - y_1, s) \right] \right] \langle P(k, s) \rangle. \tag{B14}
\end{aligned}$$

Similar calculus can be done to any term appearing in (B6), therefore leading to (B9) and (B10).

Appendix C: Resuming Techniques on Telegrapher's Equation

Terwiel's cumulant contributions of neighbors sites in a discrete random systems, were presented in the pioneer work [16]. In this Appendix we briefly introduce these ideas for the (continuous) telegrapher equation.

Consider the infinite sum of integrals (B10); there, expressions like $[\partial_x^2 g(x, s)]$ appear n -times. Using (9) we can write (8) in the form:

$$\begin{aligned}
\gamma(k, s) & \\
& = \mathcal{P} \sum_{p=0}^{\infty} \int_{-\infty}^{+\infty} dy_1 \dots \int_{-\infty}^{+\infty} dy_p e^{ik(y_1 + \dots + y_p)} \xi(0) \mathcal{Q}_{F_{0,y_1}} \xi(y_1) \mathcal{Q} \dots \xi(y_{p-1}) \mathcal{Q}_{F_{y_{p-1},y_p}} \xi(y_p), \tag{C1}
\end{aligned}$$

where $F_{x,y} \equiv [s(s + 1/T)g(x - y, s) - \delta(x - y)]/v^2$.

One important difference with the resummation techniques in continuous models is that the “diagonal” part of $F_{x,y}$ has two contributions because $g(0, s) \neq 0$. So following previous experience we introduce a first possible resummation consisting in integrating all $\delta(x)$ contributions coming from $F_{x,y}$, and they are of the form:

$$\begin{aligned}
& \mathcal{P} \left(\xi(0) - \xi(0) \mathcal{Q} \frac{1}{v^2} \xi(0) + \xi(0) \mathcal{Q} \frac{1}{v^2} \xi(0) \mathcal{Q} \frac{1}{v^2} \xi(0) - \dots \right) \\
& = \mathcal{P} \sum_{p=0}^{\infty} \left(-\xi(0) \mathcal{Q} \frac{1}{v^2} \right)^p \xi(0) \equiv \mathcal{P} \psi(0),
\end{aligned}$$

here the random operator $\psi(x)$ at site $x = 0$, defined in (11), has been used. Integral (C1) can therefore be reordered leading to:

$$\begin{aligned}
\gamma(k, s) & = \mathcal{P} \sum_{n=0}^{\infty} \left(\frac{s(s + 1/T)}{v^2} \right)^n \int_{-\infty}^{+\infty} dy_1 \\
& \dots \int_{-\infty}^{+\infty} dy_n e^{-iky_n} \psi(0) \mathcal{Q} g(0 - y_1, s) \psi(y_1) \mathcal{Q} g(y_1 - y_2, s) \dots \\
& \times \dots \psi(y_{n-2}) \mathcal{Q} g(y_{n-2} - y_{n-1}, s) \psi(y_{n-1}) \mathcal{Q} g(y_{n-1} - y_n, s) \psi(y_n), \tag{C2}
\end{aligned}$$

which is (10).

In terms of the continuous field operator $\psi(x)$, a difference in this integral-resummation is noted with respect to the one coming from a discrete model [16]. There the same discrete-resummation is restricted to terms in which a given ψ_{n_j} at lattice site n_j does not coincide with its nearest neighbors $\psi_{n_{j\pm 1}}$ at sites $n_{j\pm 1}$. As we mention before this effect comes from the non-diagonal behavior of the continuous function $F_{x,y}$. In addition to this, in discrete models a random field ξ_{n_j} is assumed to be statistical independent from any other ξ_{n_i} if $n_i \neq n_j$. As mentioned this is not the case in continuous space-correlated models.

1. On the (Continuous) Random Operator $\psi(x)$

In order to study the small s behavior of $\gamma(k, s)$ it is convenient to rewrite the random operator $\psi(x)$ into a more friendly way to be used with any statistics of the random field $\xi(x)$. From (11) we can write:

$$\left[1 + \frac{1}{v^2} \xi(x) (\mathbf{1} - \mathcal{P})\right] \psi(x) [\bullet] = \xi(x) [\bullet], \quad (\text{C3})$$

working out the LHS of (C3) we get:

$$\left(1 + \frac{1}{v^2} \xi(x)\right) \psi(x) [\bullet] = \xi(x) [\bullet] + \frac{1}{v^2} \xi(x) \mathcal{P} \psi(x) [\bullet], \quad (\text{C4})$$

dividing by $\left(1 + \frac{1}{v^2} \xi(x)\right)$ we get:

$$\begin{aligned} \psi(x) [\bullet] &= \left(1 + \frac{1}{v^2} \xi(x)\right)^{-1} \left\{ \xi(x) + \frac{1}{v^2} \xi(x) \mathcal{P} \psi(x) \right\} [\bullet] \\ &\equiv \left\{ M(x) + \frac{M(x)}{v^2} \mathcal{P} \psi(x) \right\} [\bullet], \end{aligned} \quad (\text{C5})$$

where $M(x)$ is a random number written in terms of the field $\xi(x)$:

$$M(x) = \left(\frac{\xi(x)}{1 + \xi(x)/v^2} \right). \quad (\text{C6})$$

Now, we can solve $\mathcal{P} \psi(x)$ by applying $\mathcal{P}$ to (C5)

$$\begin{aligned} \mathcal{P} \psi(x) [\bullet] &= \mathcal{P} \left\{ M(x) + \frac{M(x)}{v^2} \mathcal{P} \psi(x) \right\} [\bullet] \\ &= \left\{ \mathcal{P} M(x) + \mathcal{P} \left(\frac{M(x)}{v^2} \right) \mathcal{P} \psi(x) \right\} [\bullet], \end{aligned} \quad (\text{C7})$$

then we get:

$$\left\{ 1 - \frac{\langle M(x) \rangle}{v^2} \right\} \mathcal{P} \psi(x) [\bullet] = \mathcal{P} M(x) [\bullet], \quad (\text{C8})$$

from which,

$$\mathcal{P} \psi(x) [\bullet] = \left\{ 1 - \langle M(x) \rangle / v^2 \right\}^{-1} \mathcal{P} M(x) [\bullet]. \quad (\text{C9})$$

Introducing (C9) into (C5) we arrive to a simpler expression for the random operator $\psi(x)$:

$$\psi(x) [\bullet] = M(x) \left\{ 1 + \frac{1/v^2}{\left\{ 1 - \langle M(x) \rangle / v^2 \right\}} \mathcal{P} M(x) \right\} [\bullet]. \quad (\text{C10})$$

Long time behavior of $\gamma(k, s)$ could be studied from Terwiel's cumulants contributions in the low-frequency regime, that is, in the small Laplace parameter s from $\gamma(k, s) = \sum_{n=1}^{\infty} \gamma_n(k, s)$, with:

$$\gamma_n(k, s) = \left(\frac{s(s+1/T)}{v^2} \right)^n \int_{-\infty}^{+\infty} dy_1 \dots \int_{-\infty}^{+\infty} dy_n e^{-iky_n} \langle \psi(0) \psi(y_1) \dots \psi(y_n) \rangle_T \quad (\text{C11})$$

$$\times g(0 - y_1, s) \dots g(y_{n-1} - y_n, s).$$

Nevertheless, a crucial point in this formula is the fact that $\langle \psi(x) \rangle \neq 0$. Thus to any order $\mathcal{O}(s^v)$ there are infinite diagrammatic contributions, see [16]. Thus a better arrangement must be done to consider a suitable perturbation to calculate $\gamma_n(k, s \sim 0)$. In fact the new reordering, around some EMA leads to a well-defined perturbation theory in the small s .

2. The EMA for Telegrapher's Equation on Random Media

As we mention before the EMA consists in defining a new random operator $\tilde{\psi}(x)$. In this way introducing the changes $(v^2 + \Gamma) \equiv \tilde{v}^2$, $(\xi(x) - \Gamma) \equiv \tilde{\xi}(x)$ and $g(x, s) \rightarrow \tilde{g}(x, s)$ we can write $\tilde{\psi}$, from (C10), in the form:

$$\tilde{\psi}(x) [\bullet] = \tilde{M}(x) \left\{ 1 + \frac{1/\tilde{v}^2}{1 - \langle \tilde{M}(x) \rangle / \tilde{v}^2} \mathcal{P} \tilde{M}(x) \right\} [\bullet], \quad (\text{C12})$$

where,

$$\tilde{M}(x) = \left(\frac{(\xi(x) - \Gamma)}{1 + (\xi(x) - \Gamma)/(v^2 + \Gamma)} \right). \quad (\text{C13})$$

Note that from (C12) we can immediately calculate $\langle \psi(x) \rangle$ and therefore the self-consistent equation $\mathcal{P} \tilde{\psi}(x) [1] = 0$ implies $\langle \tilde{M}(x) \rangle = \langle (\xi(x) - \Gamma) (v^2 + \Gamma) / (v^2 + \xi(x)) \rangle = 0$, which gives (20).

Then expression for the effective-velocity $\tilde{\gamma}(k, s) = \sum_{n=1}^{\infty} \tilde{\gamma}_n(k, s)$ can be read from (C11) as:

$$\tilde{\gamma}_n(k, s) = \left(\frac{s(s+1/T)}{\tilde{v}^2} \right)^n \int_{-\infty}^{+\infty} dy_1 \dots \int_{-\infty}^{+\infty} dy_n e^{-iky_n} \langle \tilde{\psi}(0) \tilde{\psi}(y_1) \dots \tilde{\psi}(y_n) \rangle_T \quad (\text{C14})$$

$$\times \tilde{g}(0 - y_1, s) \dots \tilde{g}(y_{n-1} - y_n, s),$$

which is (14). From this formula higher orders terms of $\mathcal{O}(s^v)$ can be calculated considering contributions from $n \geq 1$.

Appendix D: Space-Correlated Binary Disorder

Equation (C14) is a general expression for the calculation of the dispersive function $\tilde{\gamma}(k, s)$ in orders of s^v , and for any statistics of the random field $\xi(x)$. An important model of disorder is the binary one, which allows for considering different options for modeling its n -points correlation functions. In general we have to deal with the random operator $\tilde{\psi}(x)$ given in (C12). Nevertheless, in the particular case of an equiprobable binary disorder this random operator adopts a simpler structure. Using the stationary probability of the symmetric

binary disorder: $P(\xi) = (\delta_{\xi, \Delta} + \delta_{\xi, -\Delta})/2$, the expression of $\tilde{\psi}(x) [\bullet]$ can be simplified as follows. Introducing $\tilde{M}(x)$ from (C13) (noting that $\langle \tilde{M}(x) \rangle = 0$ and $\Gamma = -\Delta^2/v^2$) in (C12) we get:

$$\begin{aligned} \tilde{\psi}(x) [\bullet] &= \tilde{M}(x) \left\{ 1 + \frac{1}{(v^2 + \Gamma)} \mathcal{P} \tilde{M}(x) \right\} [\bullet] \\ &= \frac{(v^4 - \Delta^2)}{v^2} \left(\frac{v^2 \xi(x) + \Delta^2}{v^2 + \xi(x) v^2} \right) \left(1 + \mathcal{P} \left(\frac{v^2 \xi(x) + \Delta^2}{v^2 + \xi(x) v^2} \right) \right) [\bullet]. \end{aligned} \quad (D1)$$

Using binary disorder we also get that $\mathcal{P} \left(\frac{v^2 \xi(x) + \Delta^2}{v^2 + \xi(x) v^2} \right) [f] = \frac{1}{v^2} \mathcal{P} \xi(x) [f]$, then introducing this expression in (D1) and taking the mean value, we obtain after some algebra:

$$\mathcal{P} \tilde{\psi}(x) [\bullet] = \frac{(v^4 - \Delta^2)}{v^2} \left(\frac{1}{v^2} \right) \mathcal{P} \left(\xi(x) + \frac{1}{v^2} \xi(x) \mathcal{P} \xi(x) \right) [\bullet], \quad (D2)$$

from which we can write a simpler operational expression:

$$\tilde{\psi}(x) [\bullet] = \frac{(v^4 - \Delta^2)}{v^4} \left[\xi(x) + \frac{1}{v^2} \xi(x) \mathcal{P} \xi(x) \right] [\bullet]. \quad (D3)$$

With this expression for $\tilde{\psi}(x) [\bullet]$, any n -points Terwiel's cumulant associated with the symmetric binary disorder can straightforwardly be calculated, so we can proceed to study the effective-velocity $\gamma(k, s)$ order-by-order in s from (C14).

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