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SEMICOHERENT STATES AND THE GROUP ISp(2, *R*)

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The properties of two structures arising from a particular subset of the general coherent states of ISp(2, R) are studied. At the quantum level, these states support a Hilbert space of analytic functions in two variables which generalizes the Bargmann–Segal space. At a classical level they generate a symplectic manifold on which a Hamiltonian flow is obtained through dequantization via the time-dependent variational principle. This flow provides an approximate description of the coupled motion of the centre of a wave packet and its covariance matrix in phase space.

The coherent states associated with the Weyl group in one dimension span the state space of a quantum system in the analytic version due to Segal[1]) and Bargmann[2]). Dequantization by application of the time-dependent variational principle yields a two-dimensional phase space with a symplectic structure[3]). We choose here the larger group ISp(2, R) of inhomogeneous symplectic transformations, determine its coherent states and apply them to the dynamics of one-dimensional systems. The detailed proofs, calculations for specific systems, other applications and generalizations are left for a subsequent publication.

Definition 1

The Lie algebra of ISp(2, R) $\simeq$ ISU(1, 1) has the basis A_+, A_-, K_0, K_+, K_- with the commutators

$$[A_-, A_+] = 1,$$

$$[K_0, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = -2K_0,$$

$$[K_+, A_-] = -A_+, \quad [K_-, A_+] = A_-, \quad [K_+, A_+] = [K_-, A_-] = 0,$$

$$[K_0, A_\pm] = \pm\tfrac{1}{2}A_\pm.$$

The adjoint properties leading to unitary representations are $(A_\pm)^+ = A_\mp$, $(K_\pm)^+ = K_\mp$, $K_0^+ = K_0$. By use of the Weyl algebra of creation and annihilation operators A_+, A_- we get the following proposition.

Proposition 1

The operators A_+, A_-, M_0, M_+, M_- with $M_0 = \frac{1}{4}(A_+A_- + A_-A_+)$, $M_\pm = \frac{1}{2}A_\pm A_\pm$

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have the same commutators and adjoint properties as A_+, A_-, K_0, K_+, K_-. This allows us to split the Lie algebra of definition 1 according to the next proposition.

Proposition 2

Introduce the decomposition $L_0 = K_0 - M_0$, $L_\pm = K_\pm - M_\pm$. Then the algebra A_+, A_-, M_0, M_+, M_- has the commutators described in proposition 1 and moreover

$$[L_0, L_\pm] = \pm L_\pm, [L_+, L_-] = -2L_0, [L_0, A_\pm] = 0, [L_+, A_\pm] = 0, [L_-, A_\pm] = 0.$$

We turn to the irreducible representations and consider extremal states.

Definition 2

An extremal state $|0, q\rangle$ of a representation for ISp(2, R) obeys

$$A_-|0, q\rangle = 0, K_-|0, q\rangle = 0, M_0|0, q\rangle = |0, q\rangle\tfrac{1}{4}, K_0|0, q\rangle = |0, q\rangle q, q > 0.$$

The corresponding representation is of positive discrete type.

Definition 3

The coherent states associated with this representation are defined in terms of two complex numbers α, β, $1 - \beta\bar{\beta} > 0$ as

$$|\alpha, \beta\rangle = \exp[\bar{\alpha}A_+ + \bar{\beta}K_+]|0, q\rangle.$$

With the help of the operators L_0, $L_\pm$, the coherent state could be decomposed into a product. In what follows we are interested only in the case where L_0, $L_\pm$ act trivially on $|0, q\rangle$ so that the operators K reduce to the operators M. In the one-dimensional case considered here this implies $q = \frac{1}{4}$. However, we write formulae for general q because they have applications for systems with more degrees of freedom.

Definition 4

The semicoherent states associated with the Weyl group are given by

$$|\alpha, \beta\rangle = \exp[\bar{\alpha}A_+ + \bar{\beta}M_+]|0, q\rangle.$$

We now give without proofs the basic properties of these states.

Proposition 3

Let (g, c) be an element of the group ISp(2, R) where

$$g = \begin{pmatrix} \lambda & \mu \\ \bar{\mu} & \bar{\lambda} \end{pmatrix}, \quad \lambda\bar{\lambda} - \mu\bar{\mu} = 1$$

belongs to Sp(2, R) in its complex form[2]) and where

$$\gamma' = (c + \bar{c})/\sqrt{2}, \ \gamma'' = -\mathrm{i}(c - \bar{c})/\sqrt{2}$$

are the position and momentum shifts for the Weyl operators. Let $U(g, c)$ be its unitary representation. Then the semicoherent states transform as

$$\langle\alpha\beta|U(g, c) = \langle\tilde{\alpha}\tilde{\beta}|m(\alpha, \beta, g, c),$$

where

$$\tilde{\alpha} = (-\bar{\mu}\beta + \lambda)^{-1}(\alpha - c + \beta\bar{c}), \quad \tilde{\beta} = (-\bar{\mu}\beta + \lambda)^{-1}(\bar{\lambda}\beta - \mu),$$

$$m(\alpha, \beta, g, c) = (-\bar{\mu}\beta + \lambda)^{-1/2}\exp \Omega,$$

$$\Omega = -\tfrac{1}{2}c\bar{c} + \tfrac{1}{2}\bar{c}^2\beta + \bar{c}\alpha + \tfrac{1}{2}(\alpha - c + \beta\bar{c})^2(-\bar{\mu}\beta + \lambda)^{-1}\bar{\mu}.$$

We can now introduce a Hilbert space of analytic functions in the variables α, β.

Proposition 4

The semicoherent states $|\alpha, \beta\rangle$ span a Hilbert space of analytic functions $f(\alpha, \beta)$ with the properties that for $4q - 3 \geqslant 0$ there exist

a) *Reproducing kernel*

$$\langle\alpha'\beta' \mid \alpha\beta\rangle = (1 - \beta'\bar{\beta})^{-2q}\exp\{\tfrac{1}{2}(1 - \beta'\bar{\beta})^{-1}(\bar{\alpha}^2\beta' + \alpha'^2\bar{\beta} + 2\alpha'\bar{\alpha})\}.$$

b) *Inner product*

$$\langle f, g\rangle = \int \overline{f(\alpha, \beta)}g(\alpha, \beta)\,\mathrm{d}\mu(\alpha, \beta),$$

$$\mathrm{d}\mu(\alpha, \beta) = \pi^{-2}(2q - \tfrac{3}{2})(1 - \beta\bar{\beta})^{-3}\langle\alpha\beta \mid \alpha\beta\rangle^{-1}\,\mathrm{d}\alpha\,\mathrm{d}\bar{\alpha}\,\mathrm{d}\beta\,\mathrm{d}\bar{\beta}$$

and the integration is over the whole α plane and for $\beta\bar{\beta} < 1$.

c) *Generators*

$$A_+ = \alpha + \beta(\partial/\partial\alpha), \quad A_- = \partial/\partial\alpha,$$

$$M_0 = \tfrac{1}{2}\alpha(\partial/\partial\alpha) + \beta(\partial/\partial\beta) + q, \quad M_- = \partial/\partial\beta,$$

$$M_+ = \tfrac{1}{2}\alpha^2 + 2q\beta + \alpha\beta(\partial/\partial\alpha) + \beta^2(\partial/\partial\beta).$$

The Hilbert space of proposition 4 provides a framework where the full quantum dynamics of a one-dimensional system can be analyzed.

Next we turn to the symplectic structure associated with these states. We follow closely the methods of ref. 3.

Definition 5

The dequantization of a quantum operator F through semicoherent states is the mapping that assigns to F a function $\mathscr{F}$ of two complex variables through

$$F \to \mathscr{F}(\alpha, \beta, \bar{\alpha}, \bar{\beta}) = \langle\alpha\beta|F|\alpha\beta\rangle/\langle\alpha\beta \mid \alpha\beta\rangle.$$

Of particular significance are the dequantized generators. They can be obtained by differentiation of the reproducing kernel. We obtain

Proposition 5

The dequantized generators are

$$\mathscr{A}_+ = \omega(\alpha + \beta\bar{\alpha}), \quad \mathscr{A}_- = \bar{\mathscr{A}}_+,$$

$$\mathscr{K}_+ = 2q\omega\beta + \tfrac{1}{2}\mathscr{A}_+^2, \quad \mathscr{K}_- = \bar{\mathscr{K}}_+, \quad \mathscr{K}_0 = q\omega(1 + \beta\bar{\beta}) + \tfrac{1}{2}\mathscr{A}_+\mathscr{A}_-,$$

where $\omega = (1 - \beta\bar{\beta})^{-1}$.

The manifold M spanned by the real and imaginary parts of α and β subject to the restriction $1 - \beta\bar{\beta} > 0$ has a natural symplectic structure which we construct following ref. 3.

Definition 6

The symplectic bracket on M is given by

$$\{\mathscr{F}, \mathscr{G}\} = (\partial\mathscr{F}/\partial\bar{\alpha}, \partial\mathscr{F}/\partial\bar{\beta})[C^{-1}]\begin{pmatrix}\partial\mathscr{G}/\partial\alpha \\ \partial\mathscr{G}/\partial\beta\end{pmatrix} - (\mathscr{F} \leftrightarrow \mathscr{G}),$$

where the 2×2 matrix C^{-1} is

$$C^{-1} = (2q\omega^2)^{-1}\begin{pmatrix}2q\omega + \mathscr{A}_+\mathscr{A}_- & -\mathscr{A}_- \\ -\mathscr{A}_+ & 1\end{pmatrix}.$$

The matrix C can be obtained from the second derivatives of the reproducing kernel. The bracket is antisymmetric, non-singular and it satisfies the Jacobi identity and the derivative property. Moreover, we have the following proposition.

Proposition 6

The dequantized generators of proposition 5 with the bracket of definition 6 provide a symplectic realization of the Lie algebra of ISp(2, R).

Proposition 7

The equations of motion resulting from the application of the time-dependent variation principle when the wave function is restricted to be a semicoherent state at all times are

$$i\dot{\bar{\alpha}} = \{\bar{\alpha}, \mathscr{H}\}, \quad i\dot{\bar{\beta}} = \{\bar{\beta}, \mathscr{H}\},$$

where $\mathscr{H}$ is the dequantized Hamiltonian. The full quantum dynamics is therefore projected to a Hamiltonian flow on the symplectic manifold M. The time evolution of any function on M is given by $i\dot{\mathscr{F}} = \{\mathscr{F}, \mathscr{H}\}$. To make clear the meaning of these equations we rewrite them directly in terms of quantities of physical interest. We define

$$\mathscr{Q} = (\mathscr{A}_+ + \mathscr{A}_-)/\sqrt{2}; \quad \mathscr{P} = -i(\mathscr{A}_+ - \mathscr{A}_-)/\sqrt{2},$$

$$\chi = \langle\alpha\beta|(Q-\mathscr{Q})^2|\alpha\beta\rangle/\langle\alpha\beta \mid \alpha\beta\rangle = \tfrac{1}{2}\omega(1+\beta)(1+\bar{\beta}),$$

$$\phi = \langle\alpha\beta|(P-\mathscr{P})^2|\alpha\beta\rangle/\langle\alpha\beta \mid \alpha\beta\rangle = \tfrac{1}{2}\omega(1-\beta)(1-\bar{\beta}),$$

$$\sigma = \tfrac{1}{2}\langle\alpha\beta|(P-\mathscr{P})(Q-\mathscr{Q}) + (Q-\mathscr{Q})(P-\mathscr{P})|\alpha\beta\rangle/\langle\alpha\beta \mid \alpha\beta\rangle = \tfrac{1}{2}i\omega(\beta-\bar{\beta}).$$

χ, ϕ, σ are the elements of the covariance matrix of P and Q in phase space.

Proposition 8

If the dequantized Hamiltonian can be expressed explicitly as $\mathscr{H}(\mathscr{P}, \mathscr{Q}, \chi, \phi, \sigma)$ the equations of motion can be written in terms of the above quantities as

$$\begin{pmatrix}\dot{\mathscr{P}}\\ \dot{\mathscr{Q}}\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\partial\mathscr{H}/\partial\mathscr{P}\\ \partial\mathscr{H}/\partial\mathscr{Q}\end{pmatrix},$$

$$\begin{pmatrix}\dot{\chi}\\ \dot{\phi}\\ \dot{\sigma}\end{pmatrix} = 2\begin{pmatrix}0 & 2\sigma & \chi\\ -2\sigma & 0 & -\phi\\ -\chi & \phi & 0\end{pmatrix}\begin{pmatrix}\partial\mathscr{H}/\partial\chi\\ \partial\mathscr{H}/\partial\phi\\ \partial\mathscr{H}/\partial\sigma\end{pmatrix}.$$

These equations imply, for any Hamiltonian, that the quantity $\chi\phi - \sigma^2$, which is the symplectic Casimir invariant, is conserved. For one-dimensional systems its value is 1/4. When this constraint is explicitly taken into account we recover the equations of proposition 6 in terms of independent parameters α, $\bar{\alpha}$, β, $\bar{\beta}$. The energy is conserved, in accordance with a general feature of the time-dependent variational principle.

In proposition 8 we obtained a set of coupled equations for the evolution of the centroid and of the dispersion of a wave packet. These equations yield the exact solution for the quantum problem when the quantum Hamiltonian is at most quadratic in the position and momentum operators. This encompasses the attractive and repulsive oscillator and the free particle. For more general quantum Hamiltonians they yield approximate solutions which represent the best possible description in terms of a wave packet characterized only by its centroid and its dispersions, taken in the sense of the time-dependent variational principle.

The semicoherent states introduced in this paper support two structures. On the quantum level, they provide a Hilbert space of analytic functions in two variables. In this Hilbert space the full quantum dynamics can be studied. At the dequantized level the semicoherent states determine a symplectic manifold that supports a Hamiltonian flow. This flow represents an approximation to the quantum dynamics. We stress that the dequantization is not a classical limit (in the sense of $\hbar \rightarrow 0$). Specific quantum effects like the spreading of wave packets and the tunnelling through a potential barrier can be obtained from the dequantized equations of motion. A detailed investigation of these effects as well as numerical solutions of both the quantum and dequantized solutions is being performed in a number of examples and will be reported elsewhere. Other applications including large amplitude collective motion in many-particle systems and generalizations to more degrees of freedom are under study.

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References

1) I.E. Segal, Lectures in Applied Mathematics, Vol. 2, Proc. Summer Seminar in Appl. Math., Boulder, Colorado, 1960 (Am. Math. Soc., Providence, Rhode Island, 1963).
2) V. Bargmann, Commun. Pure Appl. Math. **14** (1961) 187.
3) P. Kramer and M. Saraceno, Lecture Notes Phys. **140** (1981) 1.