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ON GENERAL CONTINUOUS-PARAMETER SUM RULES

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Abstract: A class of general finite-energy sum rules not necessarily derived from analyticity and crossing property of the amplitude is discussed. An example is given of a sum rule which has over the continuous-moment sum rules the advantages of a more equilibrated participation of low-energy data and a better separation of two nearby Regge-pole contributions.

1. INTRODUCTION

During the last two years finite-energy sum rules (FESR) [1] and continuous-moment sum rules (CMSR) [2, 3] have been used to obtain Regge parameters from low- and medium-energy data. These sum rules (SR) may be derived using a trick based on analyticity, crossing symmetry and high-energy Regge behaviour of the amplitude [3] and are generally called right-signature SR.

Some modifications [4-6] and generalizations [7] of the previous techniques have been recently proposed together with SR which cannot be obtained from the above assumptions [8, 9], so that they can be called wrong-signature SR, and where wrong-signature fixed poles may contribute [10].

In practice, all these SR may be viewed as a fit to Regge poles of the scattering amplitude multiplied by a weight function over a finite range of energy. Their approximate experimental validity is a test of "duality" (in the global sense).

Our aim in this paper is to point out the fundamental reasons for which a general class of continuous parameter SR is more appropriate to be fitted with Regge poles than the simple high-energy amplitudes are. Thus certain criteria will be provided for choosing a convenient weight function and to tell when a SR is really useful for the prediction of Regge parameters.

2. CONTINUOUS-PARAMETER MELLIN SUM RULES

Let us begin deriving a general right-signature SR in the traditional way.

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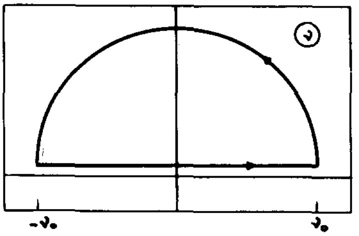

Fig. 1. Path of integration for right-signature SR.

We consider for simplicity a crossing symmetric amplitude for $t = 0$, i.e.

$$F^{(+)}(\nu) = F^{(+)}(-\nu)^* , \tag{1}$$

such that for high-energy we may write the Regge expansion

$$F^{(+)}(\nu) = F_R^{(+)}(\nu) = \sum_\alpha \beta_\alpha F_\alpha^{(+)}(\nu) + F_B(\nu) ,$$

$$F_\alpha^{(+)}(\nu) = (-\mathrm{cotg}\,\tfrac{1}{2}\pi\alpha + i)\,\nu^\alpha , \tag{2}$$

where the background term $F_B(\nu)$ corresponds to the integral along the axis $-1+ib$ of the J-plane, so that problems related to fixed poles, which appear when it is pushed more to the left, do not arise here; we consider Regge poles with intercepts $-1 < \alpha \leqslant 1$ and all singularities to the left of -1 are reflected in the background term.

One may obtain a continuous parameter family of SR by writing

$$\oint g(a,\nu)\,F^{(+)}(\nu)\,\mathrm{d}\nu = 0 , \tag{3}$$

where $g(a,\nu)$ is a function of the continuous parameter a and is regular in the upper ν half plane, and the path of integration is taken as in fig. 1. Assuming that for $\nu = \nu_0$ the expression (2) is valid, we obtain from eq. (3)

$$\int_0^{\nu_0} \mathrm{Re}\,\{g_s(a,\nu)[F^{(+)}(\nu) - F_R^{(+)}(\nu)]\}\,\mathrm{d}\nu = 0 , \tag{4}$$

$$\int_0^{\nu_0} \mathrm{Im}\,\{g_a(a,\nu)[F^{(+)}(\nu) - F_R^{(+)}(\nu)]\}\,\mathrm{d}\nu = 0 , \tag{5}$$

where we have introduced the general separation

$$g(a,\nu) = g_s(a,\nu) + g_a(a,\nu) , \tag{6}$$

which satisfies on the real axis

$$g_s(a,\nu) = g_s(a,-\nu)^*, \qquad g_a(a,\nu) = -g_a(a,-\nu)^*. \tag{7}$$

If we now restrict ourselves to $g(a,\nu)$ which are regular functions of ν, we may write

$$g_s(a,\nu) = \sum_{n=0}^{\infty} c_n(a)(i\nu)^n, \qquad g_a(a,\nu) = i \sum_{n=0}^{\infty} d_n(a)(i\nu)^n, \tag{8}$$

with c_n and d_n real functions of a, so that it is clear that SR (4) and (5) take the same form and we are allowed to retain only the former. With restriction (8) we do not include in the present treatment the CMSR although these, with the modification given by ref. [4], may be considered as a superposition of FESR.

We may argue that expansion (2), better if considered in Khuri representation and leaving aside eventual cuts, is probably an exact expression for all values of ν, provided we could describe the complicated structure of background, so that SR (4) would be a trivial identity independent of g_s. However, since one cannot hope to describe background in detail, SR (4) may be useful depending on the choice of g_s; indeed, if the contribution of background to SR (4) is simpler and smaller than the contribution to the amplitude, we may expect to extract from (4) information about Regge poles, even the non-leading ones, which usually cannot be determined from high-energy fits ‡.

Starting the analysis of (4) with restriction (8) by considering the contribution of one Regge pole we have

$$\int_0^{\nu_0} \mathrm{Re}\,[g_s(a,\nu)\, F_\alpha^{(+)}(\nu)]\,\mathrm{d}\nu$$

$$= -\frac{\nu_0^\alpha}{\sin\frac{1}{2}\pi\alpha}\left[\cos\tfrac{1}{2}\pi\alpha \sum_{n=0}^{\infty} (-)^n c_{2n} \frac{\nu_0^{2n+1}}{\alpha+2n+1} + \sin\tfrac{1}{2}\pi\alpha \sum_{n=0}^{\infty} (-)^n c_{2n+1} \frac{\nu_0^{2n+2}}{\alpha+2n+2}\right]$$

$$= -\frac{\nu_0^\alpha}{\sin\frac{1}{2}\pi\alpha} R(a,\alpha), \tag{9}$$

where $\nu_0^\alpha/\sin\frac{1}{2}\pi\alpha$ is the modulus of the contribution of the pole to the amplitude, so that $R(a,\alpha)$ may be considered as reflecting the effect introduced by the SR. We may note that $R(a,\alpha)$ converges in the limit $\alpha \to -1$, being

$$\lim_{\epsilon\to 0} R(a,-1+\epsilon) = \tfrac{1}{2}\pi c_0 \nu_0 - \sum_{n=0}^{\infty} (-)^n c_{2n+1} \frac{\nu_0^{2n+2}}{2n+1}, \tag{10}$$

‡ In usual fits of SR (4) one neglects F_B or replaces it by an easily parametrized term F_B' (e.g. a pole with $J = -1$) hoping that the unaccounted for part of the background vanishes by integration over the semicircle of fig. 1.

which is tantamount to saying that the wrong-signature fixed pole at -1 does not contribute to the SR.

Now we must take into account the background integral which we write as

$$F_{\mathrm{B}}(\nu) = \lim_{\epsilon \to 0} \frac{1}{\nu^{1-\epsilon}} \int_{-\infty}^{\infty} B(b)\, \nu^{ib}\, \mathrm{d}b = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} F_{\mathrm{B}}(b, \nu)\, \mathrm{d}b\,, \tag{11}$$

so that the contribution to the SR of each part $F_{\mathrm{B}}(b, \nu)$ is

$$\int_0^{\nu_0} \mathrm{Re}\,[g_{\mathrm{s}}(a, \nu)\, F_{\mathrm{B}}(b, \nu)]\, \mathrm{d}\nu$$

$$= \frac{\nu_0^{\epsilon}}{\epsilon^2 + b^2} \mathrm{Re}\left\{(\epsilon - ib) \exp\,[ib \ln \nu_0]\, B(b) \sum_n c_n \frac{\epsilon + ib}{n + \epsilon + ib}\, (i\nu_0)^n\right\}, \tag{12}$$

we see that in general for large values of b the contribution $B(b)$ is damped by a factor $1/b$ in the SR. It is clear that we can say nothing about the contribution of small values of b (unless we knew the explicit form of $B(b)$); to obtain a good fit of the SR with Regge poles, i.e. for the validity of the "duality" concept, it will be necessary to choose a weight function g_{s} which does not emphasize the low-b part. Anyhow, because of the damping of large-b contributions, we obtain in general a simplification in the description of background.

One may readily verify by the same procedure that the above properties are also shared by CMSR, where

$$g_{\mathrm{s}}(\delta, \nu) = \frac{1}{\nu_0^{\delta+1}}\, \nu^{\delta} \exp\,[-i\tfrac{1}{2}\pi\delta]\,, \tag{13}$$

so that a one-pole contribution is

$$\frac{1}{\nu_0^{\delta+1}} \int_0^{\nu_0} \nu^{\delta}\, \mathrm{Re}\left\{\exp\,[-i\tfrac{1}{2}\pi\delta]\, F_{\alpha}^{(+)}(\nu)\right\} \mathrm{d}\nu$$

$$= -\frac{\nu_0^{\alpha}}{\sin \frac{1}{2}\pi\alpha} \frac{\sin \frac{1}{2}\pi(\alpha + \delta + 1)}{\alpha + \delta + 1} = -\frac{\nu_0^{\alpha}}{\sin \frac{1}{2}\pi\alpha} R(\delta, \alpha)\,. \tag{14}$$

At this stage one may wonder whether these properties depend on analyticity and crossing which were used in the derivation of SR (4). The answer is that we may think of more general SR; indeed, the damping of background for large b is simply a property of the power expansion of the weight function; as for the convergence of the SR at $\alpha = -1$, it is sufficient that the term independent of ν (equivalent to c_0) appears only in the function which multiplies $\cos \frac{1}{2}\pi\alpha$ in the SR (see eq. (9)).

For example we may multiply the integrand of eq. (4) by a real function

$$h(a,\nu) = \sum_{n=0}^{\infty} e_n(a)\,\nu^n , \tag{15}$$

and the SR

$$\int_0^{\nu_0} h(a,\nu)\,\mathrm{Re}\,\{g_S(a,\nu)[F^{(+)}(\nu) - F_R^{(+)}(\nu)]\}\,d\nu = 0 , \tag{16}$$

satisfies the previous requirements, though it cannot be derived using the trick based on analyticity and crossing.

With this in mind, the general SR of type (16) may be considered as Mellin transforms and we will call them Mellin Sum Rules (MSR). What we have to ask from MSR, apart from the damping of background and the convergence at $\alpha = -1$ already assured, is:

(i) that contributions of two nearby poles can be separated as clearly as possible;

(ii) that for decreasing α, the contributions of Regge poles do not increase (in the sense of an average over the continuous parameter) with respect to the corresponding ones in the scattering amplitude at energy $\approx \nu_0$;

(iii) that the low-b background contribution is small.

To analyze these aspects, which are partly interlocked, it is better to consider examples.

3. MSR WITH OSCILLATING WEIGHT

Besides the requirements shown above, one should choose as g_S a function with a Mellin transform reasonably simple. One example is to take $h = 1$ and for g_S an oscillating function with frequency $a = c/\nu_0$

$$g_S(c,\nu) = \frac{c}{\nu_0}\exp\left[i\frac{c}{\nu_0}\nu\right] , \tag{17}$$

so that the one-pole contribution to MSR turns out to be [11]

$$\frac{c}{\nu_0}\int_0^{\nu_0} \mathrm{Re}\,\{\exp\left[i\frac{c}{\nu_0}\nu\right]F_\alpha^{(+)}(\nu)\}\,d\nu$$

$$= -\frac{\nu_0^\alpha}{\sin\frac{1}{2}\pi\alpha}\frac{1}{c^\alpha}\,\mathrm{Im}\,[\gamma(\alpha+1,\, ic)] = -\frac{\nu_0^\alpha}{\sin\frac{1}{2}\pi\alpha} R(c,\alpha) , \tag{18}$$

where γ is the incomplete gamma function. The power expansion of eq. (18) is of the type (9) and (10) with coefficients

$$c_n(a) = a^{n+1}/n!. \tag{19}$$

If we represent graphically the function $R(c,\alpha)$ of eq. (18) for $-1 \leq \alpha \leq 1$

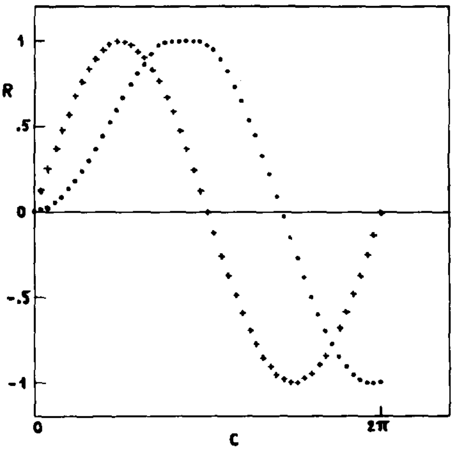

Fig. 2. One-pole contribution $R(c,\alpha)$ to SR with oscillating weight eq. (18). +++ $\alpha = 0$; ··· $\alpha = 1$.

(see as example the cases $\alpha = 0$ and 1 in fig. 2), it is seen that the maxima are approximately equal for all values of α, satisfying (ii). Besides, the curves are not only displaced but show also a shape which changes with α. This is a point in favour of the present SR over CMSR, where only a similar displacement appears, the shape remaining the same for different α's, since the sum of two of these curves will be less easily confounded with the one of an intermediate α. To prove this we have added for example two functions $R(c,\alpha)$ of the type (18) for $\alpha = 0.5$ and $\alpha = 1.0$ which could be fitted with the single pole contribution 1.9 $R(c, 0.7)$. Performing the same calculation with two CMSR, a fit with 2.0 $R(\delta, 0.7)$ was obtained which was about 10% better than in the previous case; similar results appear in the comparison of fits of mixtures of two poles with any α. This shows that SR (18) provides a better separation of two poles contributions than CMSR. It is clear that one may look for further SR which give a still better separation. A second advantage of SR (18) over CMSR is that in the former the weight function allows an equilibrated contribution of all low- and medium-energy data for all values of the continuous parameter; in CMSR, on the contrary, for most of the range of the continuous parameter (usually $0 \leq \delta \leq 5$) the region of energy near ν_0 has the predominant weight.

A common characteristic of both SR (18) and CMSR is the displacement of curves for different α's; this is one of the most important features which make it easier to fit those SR than the amplitude itself, and is a direct consequence of the interference of real and imaginary part of the amplitude in the SR. Those SR, as FESR for integer moment, which do not share this property do not satisfy (i) and have little advantage over a direct high-energy fit of the amplitude in the prediction of Regge parameters. To put this in evidence we may compare SR (18) with, for example

$$a \int_0^{\nu_0} \sin a\nu \, \mathrm{Im}\, F_\alpha^{(+)}(\nu)\, \mathrm{d}\nu = \nu_0^\alpha R'(a, \alpha), \tag{20}$$

where only the imaginary part contributes. While in eq. (18) $R(c, \alpha)$ for $\alpha = 0$ and $\alpha = 1$, for example, shows maxima of the same size ≈ 1 but displaced about $\frac{1}{2}\pi$, in eq. (20) maxima of $R'(a, \alpha)$ are practically not displaced and their values are ≈ 2 and ≈ 1 respectively, so that also condition (ii) is not satisfied here. Then the interference of real and imaginary part of the amplitude is a necessary condition to satisfy (i), and g_s of eq. (17) is chosen to emphasize it.

The condition (iii) is the most difficult to predict; it is clearly attained if g_s gives weight only to high-energy data but then the SR would be of little use. One may hope that (iii) is satisfied if g_s is sufficiently smooth so that the wiggles due to resonances are not emphasized. Referring to SR (18) the frequency a cannot be too large; for $\nu_0 \approx 4$ GeV a reasonable range for c seems to be $0 \leqslant c \leqslant 2\pi$ because (see sect. 4) a smooth integral over an interval of ~ 2 GeV is likely to show duality between resonances and Regge poles (or equivalently Regge background will not play an important role).

An application of SR (18) to πN scattering is in progress.

4. GAUSSIAN SR

As a second example in which only the imaginary part of the amplitude contributes, let us consider $h = 1$ and

$$g_s(\nu_\Delta, \Delta; \nu) = -\frac{i}{\Delta\sqrt{\pi}} \left\{ \exp\left[-\left(\frac{\nu - \nu_\Delta}{\Delta}\right)^2\right] - \exp\left[-\left(\frac{\nu + \nu_\Delta}{\Delta}\right)^2\right] \right\}. \tag{21}$$

The SR turns out to be [12]

$$\frac{1}{\Delta\sqrt{\pi}} \int_0^\infty \{\exp[-(x - x_\Delta)^2] - \exp[-(x + x_\Delta)^2]\} \, \mathrm{Im}\, F_\alpha^{(+)}(\nu)\, \mathrm{d}\nu$$

$$= \frac{\Delta^\alpha}{2^{\alpha-1}} \frac{\Gamma(\alpha+1)}{\Gamma(\frac{\alpha+1}{2})} x_\Delta \exp[-x_\Delta^2]\, M(\tfrac{1}{2}\alpha + 1, \tfrac{3}{2}, x_\Delta^2), \tag{22}$$

where $x = \nu/\Delta$ and M is the confluent hypergeometric function. In this case we have taken $\nu_0 \to \infty$ because if ν_0 is not too high the contribution of very high-energy imaginary parts is negligible. With this SR one would hope to relate Regge parameters to a low-energy region restricted to the neighbourhood of ν_Δ. However, as we have discussed above, we cannot expect much more than a simple fit with Regge poles of the region near ν_Δ because of the lack of interference between real and imaginary parts. The damping of Regge background, common to every MSR, has to be greater the greater is Δ, for ν_Δ fixed; it is obvious that the higher is ν_Δ the smaller may be Δ

Table 1
Gaussian SR for symmetric πN.

Δ [GeV]	$S(\Delta)$ [GeV^{-1}]	$S_P(\Delta)$	$S_{P'}(\Delta)$	$S_R(\Delta)$	$S_B(\Delta)$
0.745	144.5	93.1 (*A*) 87.7 (*B*)	56.4 (*A*) 59.2 (*B*)	91.3	88.0
0.368	68.5	46.0 (*A*) 43.3 (*B*)	42.7 (*A*) 38.8 (*B*)	72.6	21.7
0.166	59.9	20.9 (*A*) 19.6 (*B*)	31.0 (*A*) 24.0 (*B*)	92.8	2.9

$S(\Delta)$: experimental contribution.
$S_P(\Delta)$ and $S_{P'}(\Delta)$: P and P' contributions respectively, according to sets of parameters *A* and *B*.
$S_R(\Delta)$: resonances contribution with narrow-width approximation.
$S_B(\Delta)$: non-resonating background contribution (see eq. (24)).

to attain the same conditions of duality, since Regge background becomes less and less important.

As an example we have fixed $x_\Delta = 2$ and evaluated SR (22) for πN symmetric amplitude with $\nu_\Delta = 1.490$, 0.736 and 0.332 GeV, values which coincide with some of the most important resonances. Evaluating the right-hand side of eq. (22) we see that the dependence on α is very similar to ν_Δ^α, in agreement with what was said above. The contributions of experimental data to the SR are reported in table 1 under $S(\Delta)$. For comparison we report the contributions of P and P' poles under $S_P(\Delta)$ and $S_{P'}(\Delta)$ using two sets of parameters [13] *. Set *A* corresponds to those obtained from CMSR with the constraint of reproducing the high-energy data of Foley et al. [14]

$$\alpha_P = 1\,, \quad \beta_P = 111\ \text{GeV}^{-1}\,, \quad \alpha_{P'} = 0.41\,, \quad \beta_{P'} = 87\ \text{GeV}^{-1}\,.$$

Set *B* corresponds to the prediction of CMSR without further restrictions

$$\alpha_P = 1\,, \quad \beta_P = 104.5\ \text{GeV}^{-1}\,, \quad \alpha_{P'} = 0.60\,, \quad \beta_{P'} = 84\ \text{GeV}^{-1}\,.$$

We see that with set *B* we obtain a better agreement for the values of Δ where we may expect it; with this set we have a discrepancy with the experimental value of $\sim$ 2% for $\Delta = 0.745$ GeV, of $\sim$ 20% for $\Delta = 0.368$ (both in excess) and $\sim$ 27% for $\Delta = 0.166$ (in defect; this is due to having centered the gaussian on the N_{33} peak). Now, as said in ref. [13], the set *B* predicts too large cross sections for energies > 6 GeV; this fact gives support to the hypothesis of a more complex parametrization for P and P', for example a cut [15].

* In the present paper we normalize Regge parameters by

$$\sigma_- + \sigma_+ = \sum_\alpha \beta_\alpha \nu^{\alpha-1}$$

To analyze better the duality resonances - Regge poles we have also evaluated SR (22) in the narrow-width approximation, the contribution of resonances being

$$\mathrm{Im}\, F^{(+)}(\nu) = \frac{1}{3M} \sum_{l\pm} K_{l\pm} (R^{\mathrm{A}}_{l\pm} + \nu_{l\pm} R^{\mathrm{B}}_{l\pm})\, \Gamma_{l\pm}\, \delta(\nu - \nu_{l\pm})\,, \tag{23}$$

where $K_{l\pm}$ = 1 and 2 for resonances of type N and Δ respectively, and $R^{\mathrm{A}}_{l\pm}$ and $R^{\mathrm{B}}_{l\pm}$ are defined as in ref. [16].

The results, reported in table 1 under $S_{\mathrm{R}}(\Delta)$, show that for $\Delta = 0.745$ resonances are clearly in defect (due to the presence of a large non-resonating background), whereas for $\Delta = 0.166$ they are in excess (due to the exaggerated contribution of N_{33} given by the narrow-width approximation). We have then added the contribution of a background ($S_{\mathrm{B}}(\Delta)$ in table 1) of the type

$$\begin{aligned} \mathrm{Im}\, F^{(+)}_{\mathrm{B}}(\nu) &= c^{(+)}(\nu^2 - \mu^2)\,, \qquad \nu \leq 2.3 \text{ GeV}\,, \\ &= \beta_{\mathrm{P}}\, \nu\,, \qquad \nu > 2.3 \text{ GeV}\,, \end{aligned} \tag{24}$$

taking a parabolic behaviour for low-energy (as in ref. [16]) and equating it to the Pomeranchon for high-energy.

We see that $S_{\mathrm{R}} + S_{\mathrm{B}}$ show errors in excess, with respect to the experimental values S, of 25%, 35% and 60% for $\Delta = 0.745$, 0.368 and 0.166 respectively, probably due to the use of narrow-width approximation instead of, say, Breit-Wigner expressions.

We may finally devote some attention to wrong-signature SR which are not even of type (16). In order to compare with eq. (22) we may take as weight function the symmetric combination of gaussians; calculating the one-pole contribution for the same values of Δ and ν_Δ we see that it practically does not differ from the right-hand side of eq. (22), except for $\alpha \to -1$ where it diverges; evaluating this new SR with experimental data we obtain agreement with SR (22) within 1%; this shows that in both SR the contribution of Regge background for $J = -1$ is negligible. The same happens if we evaluate with experimental data the Schwarz SR for the πN symmetric amplitude

$$\int_0^{\nu_0} \mathrm{Im}\, F^{(+)}(\nu)\, \mathrm{d}\nu = \sum_\alpha \beta_\alpha \frac{\nu_0^{\alpha+1}}{\alpha+1}\,. \tag{25}$$

The set of Regge parameters B is perfectly in agreement with the left-hand side of eq. (25) for ν_0 = 2 GeV showing again (as in ref. [13]) that background is not important in this SR or, in other words, that one should not worry too much about wrong-signature fixed poles contribution *.

* This shows the experimental validity of global duality independently of crossing properties and analyticity.

5. CONCLUSIONS

Summarizing, we may write a great variety of continuous-parameter sum rules for different choices of the weight function $g(a, \nu)$. Assuming g to be regular in ν, all these SR damp the contribution of Regge background, at least for large imaginary part of J. Apart from this fact, these SR are really useful to fit Regge parameters if g induces an interference of real and imaginary parts of the amplitude and if it does not give more weight to Regge contributions with low α (including $\alpha = -1$) than they have in the amplitude in the neighbourhood of the cutoff energy ν_0. Furthermore, g has to be sufficiently smooth, otherwise the SR would be strongly dependent on the fine details of the amplitude, i.e. on the Regge background. Provided all these features are satisfied, g has to be chosen with the criterion of giving the best separation of two poles with similar α; in this regard we have introduced an example of a SR (eq. (18)) which gives a better separation than CMSR, apart from a more balanced contribution of data of different regions of energy spectrum.

Finally we must stress that SR with these properties may be "derived" by means of a trick based on analyticity, crossing and Regge-pole behaviour for ν_0; but to state that general FESR are a consequence of such fundamental properties is certainly misleading since the hypothesis that the amplitude tends to a Regge-pole behaviour for large energies does not imply that the contribution of Regge background to the SR is either simple or small. So, analyticity and crossing are neither necessary nor sufficient conditions; we may think of SR (see eq. (16)) which have all the correct properties though they cannot be derived from analyticity and crossing; and on the contrary we may "derive" SR from these assumptions (see eq. (22)) which can be of little use to predict Regge parameters.

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