COMISION NACIONAL DE ENERGIA ATOMICA

NEUTRON FLUX WITHIN CYLINDRICAL AIR GAPS

Por

S. P. Levy de Bollini y C. G. Bollini

•

BUENOS AIRES

1957

J. Nuclear Energy I, 1957, Vol. 5, pp. 33 to 40. Pergamon Press Ltd., London

NEUTRON FLUX WITHIN CYLINDRICAL AIR GAPS

S. P. LEVY DE BOLLINI and C. G. BOLLINI
Comisión Nacional de la Energía Atómica, Buenos Aires, Argentina

(*Received* 30 *July* 1956; *in revised form* 18 *September* 1956)

Abstract—A method, which allows the calculation of the neutron-flux distribution at the interior of air gaps of any geometry, is applied. From the values of the vectorial flux at the surface, and from the form of this surface, the neutron flux at any interior point of the gap can be evaluated.

The diffusion theory gives the values of the flux and its normal derivative at any point of the surface. Those values are all we need for the determination of the vectorial flux (in this approximation). In the case of two concentric infinite cylinders (the interior one being a black rod), we arrived at a remarkable result: the flux shows a maximum inside the gap. We explain the physical significance of this effect.

1. INTRODUCTION

IN this work we shall consider the neutron-flux distribution in a gap which is supposed to be filled with non-absorbent, non-dispersing substances (air). Though the method described here is in principle quite general, we have paid special attention to the case corresponding to the flux between two concentric infinite cylinders, not only as an example but to consider some properties particular to the case, for instance the maximum of the flux present in the gap, near the external wall.

2. GENERALITIES

From now on we mean by interface all surface dividing any material from air (which we suppose without absortion or dispersion).

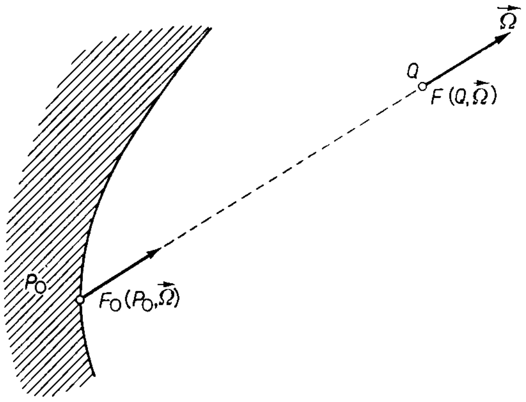

FIG. 1.

We begin by supposing that the function $F_0(P,\mathbf{\Omega})$ is known. It gives the neutron flux at each point of the interface and in every direction from P to the air. We shall not determine the flux F_0, which is obtained by analysing the behaviour of the neutron flux in the dispersing substance.

It is remarkable that, knowing only $F_0(P,\mathbf{\Omega})$, we may obtain the vectorial flux at any point of the air. If we want to know the flux $F(Q,\mathbf{\Omega})$ at any point Q, all we have to do is draw a parallel to $-\mathbf{\Omega}$ (see Fig. 1). This line will meet the boundary surface at a certain point $P_0(Q,\mathbf{\Omega})$ and then:

$$F(Q,\mathbf{\Omega}) = F_0(P_0,\mathbf{\Omega})$$

This is so because under the supposed conditions the directional flux is conserved.

To determine the integral flux $\Phi(Q)$

$$\Phi(Q) = \int F(Q,\mathbf{\Omega}) \,.\, d\Omega$$

it is sufficient to vary $\mathbf{\Omega}$ and to find the corresponding points $P_0(Q,\mathbf{\Omega})$. Then we have:

$$\Phi(Q) = \int F_0(P_0(Q,\mathbf{\Omega}),\mathbf{\Omega}) \,.\, d\Omega \tag{1}$$

Therefore, once we know $F_0(P,\mathbf{\Omega})$ we are able to find the vectorial or the integrated flux at the interior of the gap.

3. THE DIFFUSION THEORY APPROXIMATION

The diffusion theory does not give the vectorial neutron flux, but the integrated flux Φ_0 and the normal derivative $\partial/\partial n\Phi_0$ at every point of the interface. However, that is all we need for the determination of F_0 in the approximation of the diffusion theory. The procedure is similar to that used, for example, by GLASSTONE and EDLUND (1952) (pages 92ff.), for estimating the current density. The only difference is that there is no integration over the angular variables. We shall only give here the result because the deduction does not offer any difficulty.

$$F_0(P,\mathbf{\Omega}) = \frac{1}{4\pi}\left[\Phi_0(P) + \frac{1}{\Sigma_s}\cdot\frac{\partial\Phi_0}{\partial\mu}(P)\right] \tag{2}$$

where $\partial/\partial\mu$ is the derivative in a direction opposite to $\mathbf{\Omega}$.

Of course, the directional derivative can be divided in two tangential and one normal derivatives:

$$\frac{\partial}{\partial\mu} = -\mathbf{\Omega}\,.\,\mathbf{t}_1\frac{\partial}{\partial t_1} - \mathbf{\Omega}\,.\,\mathbf{t}_2\frac{\partial}{\partial t_2} - \mathbf{\Omega}\,.\,\mathbf{n}\frac{\partial}{\partial n} \tag{3}$$

As we have already said, $F_0(P,\mathbf{\Omega})$ can be obtained from the values of Φ and $\partial/\partial n\ \Phi$ at the interface by means of (2) and (3).

4. FLUX BETWEEN TWO CONCENTRIC INFINITE CYLINDERS

At the point Q of the air channel we establish a co-ordinate system (Fig. 2), in which θ is the angle formed by $\mathbf{\Omega}$ with a direction parallel to the axis of the cylinder, while φ is the angle which the projection of $\mathbf{\Omega}$ over a plane perpendicular to the axis, forms with a direction perpendicular to that axis and to the radial direction which passes through Q. In these conditions and by geometrical considerations:

$$\cos\theta' = \sin\theta\sqrt{1 - \frac{r^2}{R^2}\cos^2\varphi} \tag{4}$$

where θ' is the angle formed by $-\mathbf{\Omega}$ with the normal $\mathbf{n}$ to the interface. In (4), if the interface corresponds to the internal cylinder, we must take R_1 for R; R_2 is used if it corresponds to the external cylinder.

From now on we shall suppose that the internal cylinder is a black rod, so that the flux in the gap originates only from the external surface. This is the most important case; e.g. the cylinder is a control rod or an uranium rod, which is supposed to absorb all the thermal neutrons impinging on it.

The tangential derivatives of the flux are zero, and (3) gives

$$\frac{\partial}{\partial \mu}\Phi = \cos\theta' \frac{\partial}{\partial n}\Phi$$

and from (2)

$$F_0(\mathbf{\Omega}) = \frac{1}{4\pi}\left[\Phi_0 + \frac{\cos\theta'}{\Sigma_s}\frac{\partial\Phi_0}{\partial n}\right] \tag{5}$$

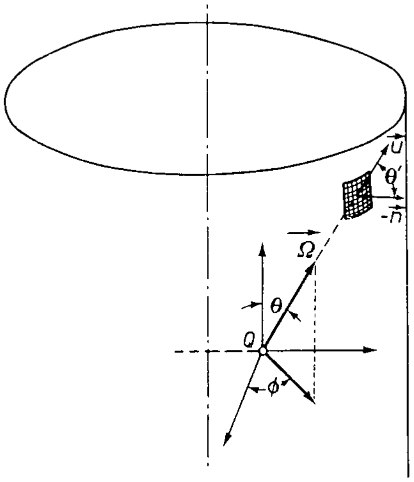

FIG. 2.

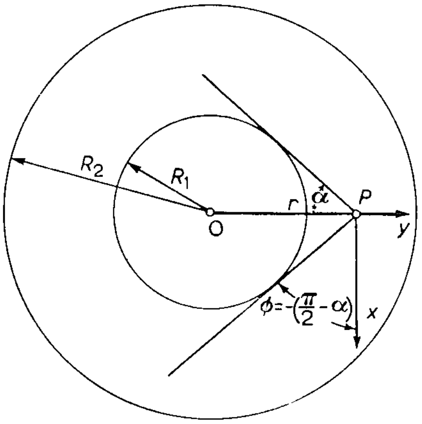

FIG. 3.

In order to find the flux $\Phi(r)$, we would have to integrate $F_0(\mathbf{\Omega})$ over all the directions of the straight lines which starting from P intersect only with the exterior interface. These directions are those for which the angle φ (Fig. 3) is between $-\left(\frac{\pi}{2}-\alpha\right)$ and $\left(\frac{3\pi}{2}-\alpha\right)$, with $\sin\alpha = R_1/r$. Owing to the symmetry of the integrand, it is sufficient to take $-\left(\frac{\pi}{2}-\alpha\right) \leq \varphi \leq \frac{\pi}{2}$.

$$\Phi(r) = 2\frac{1}{4\pi}\int_0^{\pi}\int_{-(\frac{\pi}{2}-\alpha)}^{\pi/2}\left[\Phi_0 + \frac{\sin\theta}{\Sigma_s}\sqrt{1-\frac{r^2}{{R_2}^2}\cos^2\varphi}\,\frac{\partial\Phi_0}{\partial n}\right]\sin\theta\,d\theta\,d\varphi$$

$$\Phi(r) = \left(1-\frac{\alpha}{\pi}\right)\Phi_0 + \frac{1}{4\Sigma_s}\frac{\partial\Phi_0}{\partial n}\int_{-(\frac{\pi}{2}-\alpha)}^{\pi/2}\sqrt{1-\frac{r^2}{{R_2}^2}\cos^2\varphi}\;d\varphi$$

$$= \left(1-\frac{\alpha}{\pi}\right)\Phi_0 + \frac{1}{4\Sigma_s}\frac{\partial\Phi_0}{\partial n}[2E(x) - E(x,\alpha)]$$

$$\Phi(r) = \left(1-\frac{\alpha}{\pi}\right)\Phi_0 + \frac{1}{4\Sigma_s}\frac{\partial\Phi_0}{\partial n}E(x,\pi-\alpha) \tag{6}$$

where $E(x)$ and $E(x,y)$ are the complete and incomplete elliptic functions of the second kind respectively.

5. MATCHING OF THE VALUES AT THE BOUNDARY SURFACE

(a) *By continuity of current*

The current on the exterior part of the interface is:

$$J(R_2) = \frac{1}{2\pi}\int_0^{\pi}\int_0^{\frac{\pi}{2}-\alpha_0}\left[\Phi_0 + \frac{\sin\theta \,.\, \sin\varphi}{\Sigma_s}\frac{\partial\Phi_0}{\partial n}\right]\sin^2\theta\sin\varphi\, d\theta\, d\varphi$$

$$J(R_2) = \frac{1}{4}\left(1 - \frac{R_1}{R_2}\right)\Phi_0 + \frac{1}{6\Sigma_s}\left(1 - \frac{2\alpha_0}{\pi} - \frac{\sin 2\alpha_0}{\pi}\right)\frac{\partial\Phi_0}{\partial n} \tag{7}$$

being $\sin\alpha_0 = R_1/R_2$.

On the other hand, the current which leaves the channel passing through the exterior interface is given by the formula

$$J'(R_2) = \frac{1}{4}\Phi_0 - \frac{1}{6\Sigma_s}\frac{\partial\Phi_0}{\partial n} \tag{8}$$

From the continuity of the current follows that (7) is equal to (8). Therefore we obtain

$$\frac{1}{6\Sigma_s}\frac{\partial\Phi_0}{\partial n} = \frac{1}{4}\frac{R_1}{R_2}\cdot\frac{1}{\left(2 - \frac{2\alpha_0}{\pi} - \frac{\sin 2\alpha_0}{\pi}\right)}\Phi_0 \tag{9}$$

Using (9), (6) can be written

$$\Phi(r) = \Phi_0\left[1 - \frac{\alpha}{\pi} + \frac{3}{8}\frac{R_1}{R_2}\cdot\frac{E(x, \pi - \alpha)}{\left(2 - \frac{2\alpha_0}{\pi} - \frac{\sin 2\alpha_0}{\pi}\right)}\right] \tag{10}$$

This formula gives the flux distribution at the interior of the gap, when the continuity of current is imposed.

From (10), a relation between the fluxes at the internal and external walls (Φ_i and Φ_0), may be deduced. On account of the assumptions made before, the emergent current from the internal cylinder must be zero:

$$\frac{1}{4}\Phi_i - \frac{1}{6\Sigma_{s_i}}\cdot\frac{\partial\Phi_i}{\partial n_i} = 0 \tag{11}$$

Consequently, the entrance current must be:

$$J'(R_1) = \frac{1}{4}\Phi_i + \frac{1}{6\Sigma_{s_i}}\cdot\frac{\partial\Phi_i}{\partial n_i} = \frac{1}{2}\Phi_i \tag{12}$$

Now, estimating by integration the current entering the channel:

$$J(R_1) = \frac{1}{4\pi}\int_0^{\pi}\int_0^{\pi}\left[\Phi_0 + \frac{1}{\Sigma_s}\frac{\partial\Phi_0}{\partial n}\sin\theta\sqrt{1 - \frac{R_1^2}{R_2^2}\,.\cos^2\varphi}\right]\sin^2\theta\sin\varphi\, d\theta\, d\varphi$$

$$J(R_1) = \frac{1}{4}\Phi_0 + \frac{1}{6\Sigma_s}\frac{\partial\Phi_0}{\partial n}\cdot\frac{R_2}{R_1}\cdot\left(\frac{2\alpha_0}{\pi} + \frac{\sin 2\alpha_0}{\pi}\right)$$

and on account of (9)

$$J(R_1) = \frac{1}{2}\frac{1}{\left(2 - \frac{2\alpha_0}{\pi} - \frac{\sin 2\alpha_0}{\pi}\right)}\Phi_0$$

Equating this expression to (11) we find:

$$\frac{\Phi_0}{\Phi_i} = 2 - \frac{2\alpha_0}{\pi} - \frac{\sin 2\alpha_0}{\pi} \tag{13}$$

This expression coincides with that given by NEWMARCH (1955). Besides, if we put

$$\frac{\Phi_0 - \Phi_i}{\Phi_i} = 1 - \frac{2\alpha_0}{\pi} - \frac{\sin 2\alpha_0}{\pi}$$

considering (12), we arrive at the formula

$$\Phi_0 - \Phi_i = 2J_i'\left(1 - \frac{2\alpha_0}{\pi} - \frac{\sin 2\alpha_0}{\pi}\right)$$

which coincides with the result obtained by MERCIER (as given in a report of M. BUGAARDT-SACLAY).

(b) *By flux continuity*

Formula (10), obtained by continuity of current, *does not account* for the requirements of the continuity of the flux. In fact,

$$\Phi(R_2) \neq \Phi_0 \qquad \text{and also} \qquad \Phi(R_1) \neq \Phi_i$$

The explanation is quite simple; (10) cannot be valid at the interface, not even in the approximation of the diffusion theory. It could only be applicable at the *interior* of a material, or at the most at the interface between two materials whose Σ_s are not too different. But this is not our case, for we suppose that one medium is air. By the error introduced in (8) the continuity of the flux is destroyed, as reflected in (10).

From the above considerations, it is apparent that it is more reasonable to ask for the continuity of the flux than for the continuity of the current (which was obtained equating (8) to (7)).

The continuity of flux at $r = R_2$ implies $\Phi_0 = \Phi(R_2)$, where $\Phi(R_2)$ is obtained from (6) with $\alpha = \alpha_0$ and $x = \pi/2$.

$$\Phi_0 = \Phi(R_2) = \left(1 - \frac{\alpha_0}{\pi}\right)\Phi_0 + \frac{1}{4\Sigma_s}\frac{\partial\Phi_0}{\partial n}\left[2E\left(\frac{\pi}{2}\right) - E\left(\frac{\pi}{2},\alpha_0\right)\right]$$

Taking into account that; $E(\pi/2) = 1$, $E(\pi/2, \alpha_0) = \sin\alpha_0$, it follows:

$$\frac{1}{4\Sigma_s}\cdot\frac{\partial\Phi_0}{\partial n} = \frac{\alpha_0/\pi}{2 - \sin\alpha_0}\Phi_0 \tag{14}$$

and substituting in (6),

$$\Phi(r) = \Phi_0\left[1 - \frac{\alpha}{\pi} + \frac{\alpha_0}{\pi}\frac{E(x, \pi - \alpha)}{2 - \sin\alpha_0}\right] \tag{15}$$

As in (a), we can deduce a relation between Φ_i and Φ_0.

By continuity of flux at R_1:

$$\Phi_i = \Phi(R_1) = \Phi_0 \left[1 - \frac{1}{2} + \frac{\alpha_0}{\pi} \frac{E(\alpha_0, \pi/2)}{2 - \sin \alpha_0}\right] ,$$

i.e.

$$\frac{\Phi_i}{\Phi_0} = \frac{1}{2} + \frac{\alpha_0}{\pi} \frac{E(\alpha_0)}{2 - \sin \alpha_0} \tag{16}$$

Only for the extreme cases ($\alpha_0 = 0$ and $\alpha_0 = \pi/2$), (16) becomes equal to (13).

It is remarkable that (15) and (16) are valid whichever be the coefficient chosen for the angular part of (5). Indeed, if we make the supposition that F_0 may be written as

$$F_0(\mathbf{\Omega}) = \frac{1}{4\pi} (\Phi_0 + k \cos \theta') \tag{17}$$

without taking into account the form of k, (14) would again be valid with $\frac{1}{\Sigma_s} \frac{\partial \Phi_0}{\partial n} = k$,

$$k/4 = \frac{\alpha_0/\pi}{2 - \sin \alpha_0} \Phi_0 \tag{14'}$$

and (15) and (16) would remain unchanged.

If desired, k could be chosen so that besides the continuity of the flux, the current also would be continuous.

6. GRAPHICAL REPRESENTATION OF THE RADIAL VARIATION OF THE FLUX

Fig. 4 shows the variation of $\Phi(r)$ of (15) as a function of r and for different ratios R_2/R_1. In all this cases Φ_i has been taken as unity of flux, and all the distances $R_2 - R_1$ have been reduced to the same value. The values given by (12) are represented by the dotted lines.

It is seen that the curves have the following general characteristics: As we get away from the external wall, the flux grows at first slowly, then reaches a maximum, and afterwards, as it approaches the internal wall, it rapidly decreases. For R_2/R_1 tending to infinity the flux tends to be constant, except near the internal wall, where it steeply decreases and approaches one-half of the value corresponding to the external wall. The dotted lines approach the solid ones for R_2/R_1 tending to infinity, while the difference between them is a maximum for this ratio approaching 1. In the extreme case $R_2/R_1 = 1$, (15) gives the correct result $\Phi(r) = 1$ (O curve), while (10) gives $\Phi(r) = 7/8$ (O' curve).

7. EXPLANATION OF THE SHAPE OF THE CURVES

Formula (15), which gives $\Phi(r)$, may be divided in two parts:

$$\Phi(r) = \Phi_1(r) + \Phi_2(r)$$

$$\Phi_1(r) = \Phi_0 \left[1 + \frac{2\alpha_0}{\pi} \frac{E(x)}{2 - \sin \alpha_0}\right] \tag{18}$$

$$\Phi_2(r) = \Phi_0 \left[\frac{\alpha}{\pi} + \frac{\alpha_0}{\pi} \frac{E(x,\alpha)}{2 - \sin \alpha_0}\right] \tag{19}$$

The neutron flux which would be observed at r if the central rod were transparent to neutrons, corresponds to $\Phi_1(r)$ (assuming fixed values at the boundary). $\Phi_2(r)$ accounts for the intercepting effect of the rod. This effect is smaller near R_2, but it quickly increases as r approaches R_1. This produces a pronounced lowering of the flux at the vicinity of R_1. On the other hand, for a greater R_2/R_1, the intercepting effect is less sensible and only is appreciable for a radius a little larger than R_1. In such a case the angle α_0 has a very small value and the functions are reduced to:

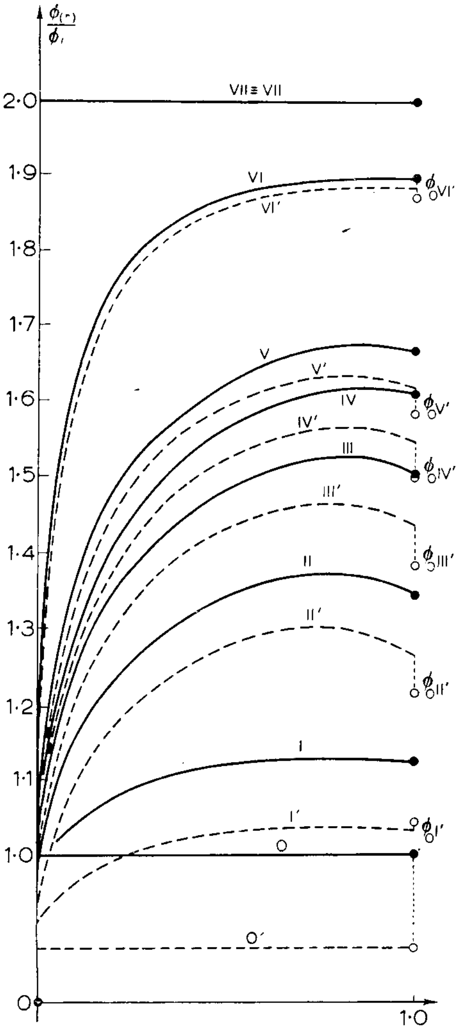

FIG. 4.—Variation of $\Phi(r)$ (formula 15), as a function of r for different ratios R_2/R_1. Φ_i has been taken unity. Dotted lines represent the values given by (10). Φ_0' denotes the flux at the exterior wall (from (13)). The corresponding ratio R_2/R_1 for the different curves are: O and $O' \to 1$; I and I′ → 1,1; II and II′ → 1,5; III and III′ → 2; IV and IV′ → 2,5; V and V′ → 3; VI and VI′ → 10; VII and VII′ → ∞.

$$\Phi_1 \simeq \Phi_0; \qquad \Phi_2 \simeq \alpha/\pi \, \Phi_0$$

Therefore, it is seen that the total flux must be nearly constant except close to R_1. Furthermore, we have $\alpha = \pi/2$ and $\Phi_2 = \Phi_0/2$ for $r = R_1$. This signifies that on the internal wall the central rod annihilates half of the flux coming from the external wall. This explains why the limit curve for $R_2/R_1 = \infty$ is constant and equal to Φ_0, except for $r = R_1$, where its value is equal to $\Phi_0/2$.

Formula (18) exhibits the influence of two terms. One of them is constant and is derived from the angle-independent part in (5) or in (17). The other one is the angular part in those formulae. It points out that the flux is higher in a direction normal to the interface.

The considerations made above imply that neutrons enter the channel preferably in a forward direction. Therefore, $\Phi_1(r)$ must be a maximum at the cylinder axis, as it is there where the maximum contribution from perpendicular directions to the interface are received. At larger distances from the axis, central contributions are lost and the lateral surfaces are seen under larger angles, decreasing in that way the values of $\Phi_1(r)$.

The position of the maximum (of $\Phi(r)$) may be found analytically, as is shown in the Appendix.

8. CONCLUSIONS

The distribution of the neutron flux at a gap depends essentially on the values at the boundary and on the geometry of the problem.

We have estimated the flux distribution between two concentric cylinders with the boundary flux given by the diffusion-theory approximation (supposing that the internal cylinder is perfectly absorbing). The resultant flux is not linear, but has a maximum inside the duct. Finally, we should like to point out that nearly all the developments made here are valid independently of the energy of the neutrons. They need not necessarily be thermal, not even monoenergetic. This is true only if the boundary values have the supposed form.

APPENDIX

The maximum of $\Phi(r)$ (formula 15), may be found in the usual way. Putting the first derivative equal to zero, we obtain:

$$F(x, \pi - y) - E(x, \pi - y) = \left(\frac{2 - \sin\alpha_0}{\alpha_0} + \cos\alpha_0\right)\tan y \qquad \text{(A)}$$

where $F(x,y)$ is the incomplete elliptic function of the first kind (FLUGGE, 1954); $\sin x = r/R_2$; $\sin y = R_1/r$; $\sin\alpha_0 = R_1/R_2$.

If we vary r from $r = R_1$ to $r = R_2$, the $\tan y$ varies correspondingly from ∞ to $R_1/\sqrt{R_2^2 - R_1^2}$. On the other hand, the left-hand side of (A) goes from $F(x_0) - E(x_0)$ up to $F(\pi/2, \pi - x_0) - E(\pi/2, \pi - x_0)$. Consequently, (A) can always be solved, except for the case that the coefficient of $\tan y$ be zero, which in our case is physically impossible.

REFERENCES

FLUGGE W. (1954) *Four-Place Tables of Trascendental Functions.*

GLASSTONE S. and EDLUND M. (1952) *The Elements of Nuclear Reactor Theory.*

NEWMARCH D. A. (1955) A modification to the diffusion theory of the thermal fine structure in a reactor. *J. Nucl. Energy.* **2,** No. 1, 52.