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# On Spinors and Gravity (*).

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**Summary.** — A consistent gauge theory of gravitation based on the group $SL_{2,C}$ and the two-component spinor formalism is presented. Under this unified fibre bundle approach, matter may be coupled to the gravitational field leading finally to a linear and/or quadratic Lagrangian on such fields.

## 1. – Introduction.

Two different aspects converge in favour of a gauge theory of gravitation whose basic group is $SL_{2,C}$. The first one concerns the great recent importance attributed to gauge theories as unifying theories for different fields; the second one takes into account the relevance of spinor formulations of general relativity.

$SL_{2,C}$ gauge theory of gravitation has been intensively studied by Carmeli and co-workers (see, for example, [1-8]), recasting the Newman-Pen-

(*) To speed up publication, the author of this paper has agreed to not receive the proofs for correction.
(**) Comisión Nacional de Energía Atómica.
(1) M. Carmeli: *J. Math. Phys.* (*N. Y.*), **11**, 2728 (1970).
(2) M. Carmeli and S. I. Fickler: *Phys. Rev. D*, **5**, 290 (1972).
(3) M. Carmeli: *Ann. Phys.* (*N. Y.*), **71**, 603 (1972).
(4) M. Carmeli: *Nucl. Phys. B*, **38**, 621 (1972).
(5) M. Carmeli: *Phys. Rev. D*, **14**, 1727 (1976).
(6) M. Carmeli and S. Malin: *Ann. Phys.* (*N. Y.*), **103**, 208 (1977).
(7) M. Carmeli and M. Kaye: *Ann. Phys.* (*N. Y.*), **113**, 177 (1978).
(8) M. Kaye: *Nuovo Cimento B*, **43**, 293 (1978).

rose (9) spinor formalism of general relativity into a Yang-Mills–type theory. Recently KAYE (8) has reviewed these developments using the fibre bundle language.

The present « real manifold » approach, in certain aspects similar to that of Kaye (8), is also an attempt to contribute to a mathematically better founded $SL_{2,C}$ spinor theory of the gravitational field, essentially under the general point of view presented in a previous work (10). For « complex manifold » approaches see, for example, ref. (11).

Section **2** is a compendium of well-known algebraic properties of spinor and related subjects, to be used later supporting the notion of spinor manifold. This is discussed in sect. **3** together with other topics connected to the spinorial structure of space-time. Section **4** concerns the definition of the forms of total dimension on the principal fibre bundle of spin frames, leading to the final Lagrangian in interaction with matter.

## 2. – Spinors.

Let $\mathscr{M}$ be the Minkovski space with pseudometric $\gamma$ (*) and $\mathscr{F}$ the real four-dimensional space of complex Hermitian $2\times2$ matrices. These two spaces are certainly isomorphic (for present developments see also (12-15)).

Let us consider the usual representation of the Lorentz group $O_{1,3}$ for $\mathscr{M}$ and for $\mathscr{F}$ the representation

$$\Lambda \to \alpha_\Lambda\,, \qquad \Lambda \in SL_{2,C}\,, \tag{1}$$

such that for every $A \in \mathscr{F}$

$$\alpha_\Lambda A \stackrel{\text{def}}{=} \Lambda A \bar{\Lambda}^{\mathrm{T}}\,, \tag{2}$$

$\bar{\Lambda}$: complex conjugate of $\Lambda$; $\Lambda^{\mathrm{T}}$: tranpose of $\Lambda$.

It is immediately checked that $\alpha_\Lambda$ conserves the Hermiticity and the determinant of $A$.

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(9) E. NEWMANN and R. PENROSE: *J. Math. Phys.* (*N. Y.*), **3**, 566 (1962).
(10) F. G. BASOMBRÍO: *A comparative review of certain gauge theories of the gravitational field*, *Gen. Rel. Grav.* (in press).
(11) M. J. HAYASHI: *Phys. Rev. D*, **18**, 3523 (1978).
(*) $\gamma$ is the diagonal form $(1, -1, -1, -1)$.
(12) T. KAHAN: *Théorie des groupes en physique classique et quantique*, Tome I (Paris, 1960).
(13) E. CARTAN: *Leçons sur la théorie des spineurs* (Paris, 1938).
(14) S. KÄSTNER: *Vectoren, Tensoren, Spinoren* (Berlin, 1964).
(15) W. L. BADE and H. JEHLE: *Rev. Mod. Phys.*, **25**, 714 (1953).

We define on $\mathscr{F}$ the real bilinear form

(3) $$E(A, B) \stackrel{\text{def}}{=} \mathrm{Tr}(\varepsilon^{\mathrm{T}} A \bar{\varepsilon} B^{\mathrm{T}}), \qquad A, B \in \mathscr{F}.$$

One easily verifies the relation

(4) $$E(A, A) = 2 \det(A)$$

if $\varepsilon$ is taken as the usual unimodular skew-symmetric matrix

(5) $$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \exp[i\theta], \qquad \theta \in \mathbf{R}.$$

Let $\hat{\sigma}$ be an isomorphism $\hat{\sigma}: \mathscr{F} \to \mathscr{M}$ such that

(6) $$\gamma(\hat{\sigma}A, \hat{\sigma}B) = E(A, B).$$

Such isomorphisms, which certainly exist, will be called isometries.

In such conditions an homomorphism $\mathscr{H}_{\hat{\sigma}}: SL_{2,C} \to O_{0\,1,3}$, the proper Lorentz group, is established:

(7) $$\mathscr{H}_{\hat{\sigma}} A \stackrel{\text{def}}{=} \hat{\sigma} \circ \alpha_A \circ \hat{\sigma}^{-1}.$$

It is a well-known fact that for each proper Lorentz matrix $L$ there correspond exactly two unimodular matrices $A$.

As a consequence of (6) and (7) we also have

(8) $$\hat{\sigma}(\alpha_A A) = (\mathscr{H}_{\hat{\sigma}} A)\, \hat{\sigma}A, \qquad \forall A \in \mathscr{F}.$$

If $\hat{\sigma}$ and $\tilde{\sigma}$ are two isometries, then

(9) $$\tilde{\sigma} = L \circ \hat{\sigma}$$

necessary holds for them, for some Lorentz transformation $L$, and also

(10) $$\tilde{\mathscr{H}} A \equiv \mathscr{H}_{\tilde{\sigma}} A = L(\mathscr{H}_{\hat{\sigma}} A) L^{-1}.$$

Let now $\mathscr{S}$ be the two-component complex spinor space with the usual representation of $SL_{2,C}$ (*). $\bar{\mathscr{S}}$ stands for the same space but the complex conjugate representation. Such spaces will be, respectively, endowed with a skew-symmetric bilinear form $\varepsilon \in \mathscr{S}^* \otimes \mathscr{S}^*$ (* means dual space) and with the complex conjugate form $\bar{\varepsilon} \in \bar{\mathscr{S}}^* \otimes \bar{\mathscr{S}}^*$. $\varepsilon$ is given by (5) and satisfies $\varepsilon(\phi, \psi) = \varepsilon(A\phi, A\psi)$, $\forall A \in SL_{2,C}$, $\forall \phi, \psi \in S$ (1).

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(*) Notice that the spinor space is identified with $\mathbf{C}^2$ and in this way the action of $SL_{2,C}$ is defined.

We shall make the following identification:

$$\mathscr{F} = (\mathscr{S} \otimes \overline{\mathscr{S}})_{\mathscr{H}} \text{: Hermitian } 2\times 2 \text{ tensors}. \tag{11}$$

It is then clear that

$$E = \varepsilon \otimes \bar{\varepsilon}. \tag{12}$$

The isometries $\hat{\sigma}$ may be regarded as elements of the space $\mathscr{M} \otimes \mathscr{S}^* \otimes \overline{\mathscr{S}}^*$.

For natural basis of the just-mentioned vector spaces the previously defined elements may be written in components as $\gamma_{kl}$, $\hat{\sigma}^k{}_{A\dot{B}}$, $\varepsilon_{AB}$, $\bar{\varepsilon}_{\dot{A}\dot{B}}$, etc., where $A, B = 1, 2$, $k = 0, 1, 2, 3$, a dot over a capital letter is the usual notation for spinor indices which transforms according to the complex conjugate representation. In this notation, we mention the property

$$(\hat{\sigma}^{-1})_k{}^{C\dot{D}} = \hat{\sigma}_k{}^{C\dot{D}} \overset{\text{def}}{=} \gamma_{kl}\varepsilon^{CE}\bar{\varepsilon}^{\dot{D}\dot{F}}\hat{\sigma}^l{}_{E\dot{F}}. \tag{13}$$

A concrete isometry $\hat{\sigma}$ may be realized in the following way, by taking (*)

$$\hat{\sigma}_k \overset{\text{def}}{=} \frac{1}{\sqrt{2}}\sigma_k, \tag{14}$$

where the $\sigma_k$ are the identity matrix $\mathbf{1}$ and the three Pauli matrices $\sigma_\alpha$, $\alpha = 1, 2, 3$:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{15}$$

From (13) we get (*)

$$\begin{cases} \hat{\sigma}_0 = \dfrac{1}{\sqrt{2}}\sigma_0, & \hat{\sigma}^1 = \dfrac{1}{\sqrt{2}}\sigma_1, \\ \hat{\sigma}^2 = -\dfrac{1}{\sqrt{2}}\sigma_2, & \hat{\sigma}^3 = \dfrac{1}{\sqrt{2}}\sigma_3. \end{cases} \tag{16}$$

To $\hat{\sigma}$ there corresponds the homomorphism $\mathscr{H}_{\hat{\sigma}} = \mathscr{H}$ according to expression (7).

Any other isometry $\tilde{\sigma}$ is related to $\hat{\sigma}$ by (9) for an adequate $L$.

It is not difficult to verify that the $\mathscr{H}$ so defined establishes the following correspondence between monoparametric groups ([12]):

$$\begin{cases} U_{\alpha\beta}(t) = \mathbf{1}\cos\tfrac{1}{2}t - \sigma_\alpha\sigma_\beta\sin\tfrac{1}{2}t \overset{\mathscr{H}}{\longrightarrow} R_{\alpha\beta}(t), & \alpha \neq \beta, \\ U_{0\alpha}(t) = \mathbf{1}\cosh\tfrac{1}{2}t - \sigma_\alpha\sinh\tfrac{1}{2}t \overset{\mathscr{H}}{\longrightarrow} R_{\alpha 0}(t), & \alpha, \beta = 1, 2, 3, \end{cases} \tag{17}$$

(*) We abreviate $\hat{\sigma}_k$ for the matrix of components $\hat{\sigma}_k{}^{A\dot{B}}$ and $\hat{\sigma}^k$ for $\hat{\sigma}^k{}_{A\dot{B}}$.

where the $R_{ij}(t)$ are the usual rotations of angle $t$ on the plane $ij$ from the axis $i$ to $j$; $i, j = 0, 1, 2, 3$.

We shall adopt the next basis for the Lie algebra $\mathscr{G}_{SL_{2,C}}$:

$$G_{\alpha\beta} = -\frac{i}{2}\sigma_\gamma\,, \qquad G_{0\alpha} = -\frac{1}{2}\sigma_\alpha\,, \qquad \alpha, \beta, \gamma = 1, 2, 3 \text{ in ciclic order}\,, \tag{18}$$

and for $\mathscr{G}_{0_{1,0}}$ the basis $\xi_{ij} = \mathrm{d}\mathscr{H}\cdot G_{ij}$, induced by $\mathscr{H}$.

## 3. – Spinor manifolds.

In this section we define the notion of spinor field in a curved space-time following essentially the work of GEROCH (16) and CHOQUET-BRUHAT (17), but within a different approach than that of CASTAGNINO (18). The geometrical aspects of the covariant derivative of spinors have been emphasized by JHANGIANI (19).

Let $x$ be a point of the space-time $M$. A spinor on $x$ will be given by the equivalence class of triplets $(q, \Lambda, \Phi)$ (17) (*), such that

$$(q, \Lambda, \Phi) \sim (q', \Lambda', \Phi') \Leftrightarrow \begin{cases} \Phi' = (\Lambda')^{-1}\Lambda\Phi\,, \\ q' = q(\tilde{H}\Lambda^{-1}\Lambda')\ (^{**})\,, \end{cases} \tag{19}$$

$q, q'$: orthonormal oriented frames at $x$; $\Lambda, \Lambda' \in SL_{2,C}$; $\Phi, \Phi' \in \mathscr{S}$.

This definition corresponds essentially to that given in (16).

A pair $(q, \Lambda)$ will be referred to as a spinor normal frame at $x$ for reasons given later. Any two such frames $(q, \Lambda)$, $(q', \Lambda')$ will belong to the same family if the second relation of the r.h.s. member of (19) holds for them. They are then linked by a $SL_{2,C}$ transformation $\hat{\Lambda} = \Lambda^{-1}\Lambda'$:

$$(q', \Lambda') = (q, \Lambda)\hat{\Lambda} \stackrel{\text{def}}{=} (q\tilde{\mathscr{H}}\hat{\Lambda}, \Lambda\hat{\Lambda})\,.$$

When the class of equivalence (19) is run over, the same is done for some family of frames. In the approach of (16) a spinor frame is defined as a pair $(v, \alpha)$, where $v$ is a Lorentz frame and $\alpha$ is a path in the fibre of these frames at $x \in M$, from $v$ to some reference frame $w$. Two pairs $(v, \alpha)$ and $(u, \beta)$ are considered identical if $u = v$ and if the path $\alpha$ can be continuously distorted into the path $\beta$ while keeping the end points fixed. Each $w$ then determines a family of such frames.

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(16) R. GEROCH: *J. Math. Phys. (N. Y.)*, **9**, 1739 (1968).
(17) Y. CHOQUET-BRUHAT: *Géométrie différentielle et systèmes extérieurs* (Paris, 1968).
(18) M. CASTAGNINO: *Ann. Inst. Henri Poincaré Sec. A*, **16**, 293 (1972).
(19) V. JHANGIANI: *Found. Phys.*, **7**, 111 (1977).
(*) This reference treats the four-component spinor case.
(**) Right action of the Lorentz group on the frame $q$.

The problem arises whether it is possible to globally define in a continuous way for each point of the space-time $M$ a unique family of spinor frames, *e.g.* a spinor structure for $M$. It is not difficult to see that a sufficient condition for that is the existence on $M$ of a global system of Lorentz frames ([17]). The important question is that for a noncompact $M$ this is also a necessary condition ([16]) and that the most relevant solutions of Einstein's equations lead to space-time manifolds which are compatible with this apparently restrictive requirement ([20,21]). We assume for $M$ such two conditions.

The spinors defined in (19) have a structure of vector space, namely, if $\Psi = (q, \Lambda, \Phi)^\sim$ and $\Psi' = (q, \Lambda, \Phi')^\sim$,

$$\alpha\Psi + \beta\Psi' \overset{\text{def}}{=} (q, \Lambda, \alpha\phi + \beta\phi')^\sim , \qquad \alpha, \beta \in \mathbf{C}. \tag{20}$$

This vector space $F_x$ will be referred to as the spinor tangent space at $x \in M$.

The « metric » $\varepsilon$ of $\mathscr{S}$ induces in $F_x$ a skew-symmetric bilinear form also designated by $\varepsilon$:

$$\begin{cases} \varepsilon_{F_x}(\Psi, \Psi') \overset{\text{def}}{=} \varepsilon_{\mathscr{S}}(\Phi, \Phi') , \\ \Psi = (q, \Lambda, \Phi)^\sim \in F_x , \qquad \Psi' = (q, \Lambda, \Phi') \in F_x . \end{cases} \tag{21}$$

Clearly this definition is not dependent on the chosen member of the equivalence classes for $\Psi$ and $\Psi'$, whenever the same pair $(q, \Lambda)$ is used as the definition states.

Let $\lambda_A$, $A = 1, 2$, be a basis of $\mathscr{S}$ such that $\varepsilon(\lambda_A, \lambda_B) = \varepsilon_{AB}$. It is not difficult to show that each basis with the just-defined property establishes a one-to-one correspondence between the pairs $(q, \Lambda)$ and the « normal frames » $\{\xi_A\}$ of $F_x$, *i.e.* those for which $\varepsilon(\xi_A, \xi_B) = \varepsilon_{AB}$ holds. For instance, in one sense

$$(q, \Lambda) \to (q, \Lambda, \lambda_A)^\sim \overset{\text{def}}{=} \xi_A , \qquad A = 1, 2. \tag{22}$$

One sees that the condition for the existence of a spinor structure for $M$ implies at the same time the existence of global systems of normal spinor frames usually called systems of dyads or zweibeins.

Let $SP_2$ be the principal fibre bundle space ([10,17,22]) of normal spinor frames for two-component spinors. The base of this bundle is the space-time $M$ and the typical fibre the group $SL_{2,\mathbf{C}}$. We can introduce the associated fibre bundle of spinor fields by using the representation of this group on $\mathscr{S}$. This construction is essentially the one just made when describing a spinor at $x \in M$.

([20]) R. GEROCH: *J. Math. Phys.* (*N. Y.*), **11**, 343 (1970).
([21]) K. K. LEE: *Gen. Rel. Grav.*, **4**, 421 (1973).
([22]) S. KOBAYASHI and K. NOMIZU: *Foundations of Differential Geometry* (New York, N. Y., 1963).

A $\mathscr{G}_{SL_{2,C}}$-valued form of connection (*op. cit.*) $\omega$ given on $SP_2$ defines a « metric » connection for the « metric » $\varepsilon$ of the spinor fields. Let $\varrho(x)$ be a cross-section on $SP_2$. On $M$ we define the $\mathscr{G}_{SL_{2,C}}$-valued 1-form $\tilde{A}_{(\varrho)}$ as the reciprocal image of $\omega$ by $\varrho$:

$$\tilde{A}_{(\varrho)} \overset{\text{def}}{=} \varrho^* \omega . \tag{23}$$

It is immediately verified that the natural mapping $f$ between $SP_2$ and $L_0$, the principal bundle of Lorentz frames of $M$, defined as

$$(q, \Lambda) \xrightarrow{f} q \tag{24}$$

is an homomorphism between these bundles for the corresponding homomorphism $\tilde{\mathscr{H}}$ of their typical fibres $SL_{2,C}$ and $O_{1,3}$.

By taking the adopted basis of $\mathscr{G}_{SL_{2,C}}$ and the Lorentz frames $q(x) = f\cdot\varrho(x)$ of $M$, let $\tilde{A}^{ii'}_k$ ($i, i', k = 0, 1, 2, 3$; the antisymmetric label $ii'$ stands for the simple index $b$ running from 1 to 6, the dimension of $\mathscr{G}$) be the component of $\tilde{A}_{(\varrho)}$. In the directions of the frames $q$ we get the usual component expressions of the covariant derivatives for the spinor field $\Psi$ as

$$\begin{cases} \nabla_k \Psi^A \equiv \Psi^A_{;k} = \Psi^A_{,k} + B_{kB}^{\;A} \Psi^B , \\ \{B_{kB}^{\;A}\} = \frac{1}{2} \tilde{A}^{ii'}_k G_{ii'} , \qquad \Psi^A_{,k} = \partial_k \Psi^A . \end{cases} \tag{25}$$

The $G_{ii'}$ are defined in (18). For a « covariant » spinor $\Psi_A$, a minus sign must be placed before the $B_k$ (*).

Let us take the vector space $\mathscr{M} \otimes S^* \otimes \bar{\mathscr{S}}^*$ and the representation of the group $O_{0\,1,3} \times SL_{2,C}$ on it. In a standard way a vector bundle of mixed vector-spinor fields may be introduced. Consider on it a global system of frames $q(x) \otimes \varrho^*(x) \otimes \bar{\varrho}^*(x)$, where $q = f\varrho$ and $\varrho^*$ means the dual base of $\varrho$. We define now the mixed field $\tilde{\sigma}$ as a field whose constant components on $q \otimes \varrho^* \otimes \bar{\varrho}^*$ are given by the $\tilde{\sigma}^k{}_{A\dot{B}}$ of (16) and (9). That definition is correct in the sense that it is not dependent on the chosen type of bases $q \otimes \varrho^* \otimes \bar{\varrho}^*$ (whevener $q = f\varrho$ holds). This is a consequence of the property

$$\tilde{\sigma}^k{}_{A\dot{B}} = (\tilde{\mathscr{H}}\Lambda)^k_s \tilde{\sigma}^s{}_{C\dot{D}} (\Lambda^{-1})^C_A (\bar{\Lambda}^{-1})^{\dot{D}}_{\dot{B}} , \qquad \forall \Lambda \in SL_{2,C} , \tag{26}$$

deduced from (8).

If a connection is given on $SP_2$, the homomorphism $f$ induces a connection in $L_0$ [10,17,22]. In components this may be written as

$$\mathrm{d}\tilde{\mathscr{H}} \cdot (\tfrac{1}{2} \tilde{A}^{ii'}_k G_{ii'}) = \tfrac{1}{2} \tilde{A}^{ii'}_k L \xi_{ii'} L^{-1} = \tfrac{1}{2} \tilde{A}^{ii'}_k \xi_{ii'} , \tag{27}$$

(*) For spinors of $\bar{\mathscr{S}}$ we have $\bar{B}_{kB}^{\;A} = \frac{1}{2} \tilde{A}^{ii'}_k \bar{G}_{ii'}$ instead of $B_{kB}^{\;A}$.

where we have used (10) and the definition $\xi_{ij} = \mathrm{d}\mathscr{H}\cdot G_{ij}$, at the end of sect. **2**. For the matrix representation of $O_{1,3}$ one obtains

$$\xi_{ii'n}^{\;\;\;\;m} = L_q^m \xi_{ii'p}^{\;\;\;\;q}(L^{-1})_n^p = L_i^m \gamma_{i'p}(L^{-1})_n^p - L_{i'}^m \gamma_{ip}(L^{-1})_n^p . \tag{28}$$

The components of the induced connection on the frame $q = f\varrho$ are then (*)

$$A_{kn}^m = L_i^m \tilde{A}_k^{ii'} \gamma_{i'p}(L^{-1})_n^p . \tag{29}$$

The second expression (25) may be conveniently written as

$$\begin{cases} B_k = \mathscr{B}_k^\alpha \sigma_\alpha , & \alpha = 1, 2, 3, \\ \mathscr{B}_k^\alpha = -\frac{1}{2}(\tilde{A}_k^{0\alpha} + i\tilde{A}_k^{\beta\gamma}) , & (\alpha, \beta, \gamma) \text{ ciclic order.} \end{cases} \tag{30}$$

The $\sigma_\alpha$ are the three Pauli matrices, from (18).

The next result is useful in the formulation of spinor gravity. It is normally introduced in a different form ([1-8,9,15]).

*Theorem.* The absolute covariant differential of the fields $\tilde{\sigma}$ identically vanishes, if and only if the connection on $L_0$ is the induced connection from $SP_2$ by $f$.

We naturally understood that the parallel transport of the « mixed » fields derives from the parallel transport of the individual spinor and vector fields.

The proof of the theorem can be given by direct geometrical reasoning by taking into account that 1) the condition of vanishing absolute differential implies the path independence of the parallel transport of the field (this will be referred to as an autoparallel field); 2) the autoparallelism of a field is equivalent to the constancy of its components in any parallel-transported frame; 3) this is exactly what holds for the induced connection because of the property (26): any parallel-transported mixed frame may be represented as $(f\lambda(t))\,L \otimes \lambda^*(t) \otimes \bar{\lambda}^*(t)$, where $\lambda(t)$ is a parallel-transported spinor frame in some path $C(t)$ parametrized with $t$ and $L$ is a *fixed* Lorentz transformation. The components of the field $\tilde{\sigma}$ in this frame are those of its definition multiplied by the constant Lorentz matrix $L^{-1}$. They are then constant. Conversely, if the components of $\tilde{\sigma}$ remain constant on every parallel-transported frame and the frame cannot be expressed in the previous form for at least one path $C(t)$, we should have $q(t) \otimes \lambda^*(t) \otimes \bar{\lambda}^*(t) = (f\lambda(t))\,La(t) \otimes \lambda^*(t) \otimes \bar{\lambda}^*(t)$, $a(t) \in O_{1,3}$, $a(0) = e$ and $a(t) \neq e$. But this is incompatible with the assumed constancy of components following from the definition of $\tilde{\sigma}$ and (26). ∎

(*) There are just obtained for the basis $\xi_{ii'} = \mathrm{d}\mathscr{H}\cdot G_{ii'}$ of $\mathscr{G}_{O_{1,3}}$ corresponding to the $R_{ii'}(t)$ monoparametric groups (17).

In the appendix, the theorem is also proved by direct inspection of the condition $\tilde{\sigma}^k_{;s} = 0$.

For the induced connection, which obviously expresses nothing more than the natural link between $SP_2$ and $L_0$, by assuming for simplicity $\tilde{\sigma} = \hat{\sigma}$, $L = 1$ in (9), it is not difficult to prove the relation [8,19]

$$B_{kB}^{\;A} = \tfrac{1}{2}\,\hat{\sigma}_r^{\;A\dot{C}}\,\hat{\sigma}^s_{\;B\dot{C}}\,\Gamma^r_{ks}\,, \tag{31}$$

valid for the frames $\varrho(x)$ and $f\varrho(x)$.

The importance of the condition $\tilde{\sigma}^k_{;s} = 0$ in spinor gravity was extensively commented elsewhere [6,8,9,15] etc. and then we shall not insist on it.

The curvature form for $\omega$ [10], $\Omega(X, Y) \overset{\text{def}}{=} \mathrm{D}\omega \overset{\text{def}}{=} \mathrm{d}\omega(hX, hY)$, $X, Y \in T_p(SP_2)$, $h$ being the horizontal projection on $H_p$, has on the direct product basis the only nonvanishing components

$$2\Omega^{kk'}_{ij} = F^{kk'}_{ij} = \partial_i \tilde{A}^{kk'}_j - \partial_j \tilde{A}^{kk'}_i + \tilde{A}^{kq}_j \gamma_{qr} \tilde{A}^{rk'}_j - \tilde{A}^{k'q}_i \gamma_{qr} \tilde{A}^{rk}_j - c^s_{ij} \tilde{A}^{kk'}_s\,, \tag{32}$$

where the $c^s_{ij}$ are the commutation coefficients of the orthonormal basis on $T_x(M)$.

Before concluding this section we wish to point out that we would have arrived at the same results but with some grater degree of generality by working with the homorphism $f$ defined between $SP_2$ and $L$, the bundle of linear frames on $M$, instead of $L_0$. On it, $f$ would induce metric connections. Because there are the more interesting ones from the physical point of view, it suffices to use $L_0$ as we have just done. This is perhaps a more natural approach.

## 4. – Gravitation.

In a similar way as in [10] we shall derive the gauge field dynamics, gravitational field in this case, from a variational principle defined on the fundamental entity of work: the principal fibre bundle $SP_2$ of normal spinor frames.

A linear connection on $SP_2$ as a differential manifold will be needed for such purposes. It is assumed [10] that $SP_2$ is endowed with a pseudo-Riemannian metric structure by making orthogonal the horizontal $H$ and vertical $V$ tangent spaces for the connection form $\omega$ of $SP_2$ and such that on the horizontal spaces $H$ it behaves isometrically with respect to $g$ on the base $M$ for the projection onto $M$, and similarly on the vertical spaces $V$ with respect to the Killing metric $\mathring{g}$ of $\mathscr{G}_{SL_{2,C}}$ for the canonical isomorphism between $V$ and the Lie algebra.

In these conditions one could define on $SP_2$ the Christoffel-Cartan metric and torsionless linear connection for the just-described metric. As in [10] we shall not use this metric, because the derived variational principle excludes important gravitational theories as that of Einstein and Cartan. Instead the « adapted » connection introduced in that paper will be chosen. This is defined under the following four geometrical requisites:

i) conservation of the horizontal or vertical character of a vector by parallel transport,

ii) compatibility with the connection of the base $M$,

iii) compatibility with the group connection,

iv) horizontal parallelism of the fundamental fields.

The second and third requisites refer to the fact that the horizontal (vertical) projection of a parallel transport is just a parallel transport on the base $M$ (group $G$). The fourth means the vanishing of the covariant derivative of the fundamental fields on any horizontal direction.

The foregoing conditions univocally determine on $SP_2$ a linear connection starting from the connection $\Gamma$ of the base $M$ and $\overset{G}{\Gamma}$, the connection of the structural group. On the horizontal-vertical bases (*) of the tangent spaces to $SP_2$ the only nonvanishing components of it are just the $\Gamma^i_{jk}$ and $\overset{G}{\Gamma}{}^a_{bc}$. This is the origin of the name « adapted » connection. Two important consequences of requisites i) to iv) are that the parallel transport on $SP_2$ is invariant by the right action of the group $SL_{2,C}$ on $SP_2$ and that the adapted connection is metric if and only if $\Gamma$ and $\overset{G}{\Gamma}$ are metric.

We proceed now to define on $SP_2$ the total dimension form corresponding to the part of the Lagrangian related to 1) the gauge-gravitational fields and 2) the matter fields.

For the first one, the procedure outlined on [10] is followed. Two forms of total dimension present themselves as natural candidates: one is $R\eta$, $R$ being the scalar curvature of the adapted connection and $\eta$ the volume element of $SP_2$ corresponding to its metric. The other is $\Omega\Lambda{}^*\Omega$ (**), where ${}^*\Omega$ is the dual of the curvature form by the element of volume. Both are *invariant by the action of $SL_{2,C}$ on $SP_2$*. The total dimension Lagrangian form for the gravitational field will then be a real linear combination of them:

$$\mathscr{L}_G = \alpha R\eta + \beta\Omega\Lambda{}^*\Omega\,, \qquad \alpha, \beta \in \mathbf{R}\,. \tag{33}$$

In the same way, we wish now to define on $SP_2$ a similar form of total dimension for matter fields with the property of being also invariant by the

(*) The vertical part consists of the fundamental vectors at the point of $SP_2$ under consideration and the horizontal part is the horizontal lift of the base at the corresponding tangent space of $M$.

(**) This exterior product is based on the Killing metric of $SL_{2,C}$.

action of the structural group. To this end let us consider what follows: 1) if $P$ is a principal fibre bundle with structural group $G$, projection $\Pi$ and base $M$, and $E$ is an associated bundle to $P$ of typical fibre $V$ (vector space) for the representation $G \ni a \to T(a)$ (linear operator on $V$), to every local field $\tilde{v}(x)$ on the open set $U \subset M$ (local cross-section of $E$) there corresponds a differentiable function $\varphi_{\tilde{v}}: \Pi^{-1}(U) \to V$ with the property $\varphi_{\tilde{v}}(\tilde{R}_a p) = T(a^{-1})\varphi_{\tilde{v}}(p)$, $\forall p \in P$, $\forall a \in G$, and reciprocally. ($\tilde{R}$ is the right action of $G$ on $P$.) $\varphi_{\tilde{v}}(p)$ is the unique element of $V$ such that the pair $(p, \varphi_{\tilde{v}}(p))$ belongs to the equivalence class defining $\tilde{v}(\Pi p)$. 2) Let $Y$ be a real-valued differentiable function on $V$ for which $Y(T(a)v) = Y(v)$, $\forall a \in G$, $\forall v \in V$, holds. The composition $Z = Y \circ \varphi$ defines on $\Pi^{-1}(U)$ a real-valued differentiable function *constant on the fibres* $\Pi^{-1}(x)$. 3) If $P$ is endowed with an adapted metric [10] exactly similar to that defined at the beginning of this paragraph and $\eta$ is the corresponding volume element on $P$, the local form $Z\eta$ *is invariant under the action* $\tilde{R}$ *of* $G$ *on* $P$.

We particularize now to the case $P = SP_2$ taking $V = \mathscr{S} \oplus \mathscr{S} \otimes \mathscr{M}^*$ and the representation $SL_{2,\mathbf{C}} \ni \Lambda \to \Lambda \oplus \Lambda \otimes (\mathscr{H}\Lambda)^{-1}$ on $V$. A real-valued function $Y$ on the $V$ just defined with the property given in 2) will be denoted as $Y_L$ and called Lagrangian-type function. Every local cross-section (field) for the associated bundle $E$ built up from $V$ will constitute a pair of a spinorial vector field $\Psi(x)$ and a $\binom{1}{1}$ tensorial mixed field $\Phi$, and conversely. A real-valued function $L_F$ can be then defined on $E$ as

$$L_F(\Psi, \Phi) \overset{\text{def}}{=} Y_L(\varrho \oplus \mu) , \tag{34}$$

where $\varrho \oplus \mu$ means *any* representative of the equivalence class of $\Psi \oplus \Phi$.

Finally for any spinor field $\Psi$ take its absolute covariant differential $\nabla\Psi$ and the field $\tilde{v} = (\Psi, \nabla\Psi)$ on $E$. The total dimension form on $SP_2$, $\mathscr{L}_F = (Y_L \circ \varphi_{(\Psi,\nabla\Psi)})\eta$, is just the invariant form under the action of $SL_{2,\mathbf{C}}$ that we were looking for. This is now added to $\mathscr{L}_G$ in (33) to obtain the interaction Lagrangian form

$$\mathscr{L} = \mathscr{L}_G + \mathscr{L}_F = \alpha R\eta + \beta\Omega\Lambda^*\Omega + (Y \circ \varphi_{(\Psi,\nabla\Psi)})\eta , \qquad \alpha, \beta \in \mathbf{R} . \tag{35}$$

For the induced connection, according to (A.3), the action for the Lagrangian finally becomes [10]

$$\begin{aligned} S(\Gamma, g, \Psi) &= S_G + S_F = \int_{SP_2} (\mathscr{L}_G + \mathscr{L}_F) = \\ &= \int_{SP_2} \left\{ \alpha R_M + \alpha R_G + \frac{\beta}{8} F^2 + (Y_L \circ \varphi_{(\Psi,\nabla\Psi)}) \right\} \eta = \\ &= V_G \int_M \left\{ \alpha(R_M - 1) + \frac{\beta}{2} \gamma_{ij}\gamma_{i'j'} R^{ii'}_{pq} R^{jj'}_{rs} \gamma^{pr}\gamma^{qs} \right\} \overset{M}{\eta} + V_G \int_M L_F(\Psi, \nabla\Psi) \overset{M}{\eta} , \qquad \alpha, \beta \in \mathbf{R} , \end{aligned} \tag{36}$$

$V_G$: « volume » of $SL_{2,\mathbb{C}}$; $\overset{M}{\eta}$: volume form of $M$; $R = R_M + R_G$ for the adapted connection ([10]); $R_M$: scalar curvature of $M$; $R_G = -1$: scalar curvature of $SL_{2,\mathbb{C}}$ for the Killing metric $\overset{G}{g}$,

$$\Omega\Lambda^*\Omega = \frac{1}{2}\overset{G}{g}_{ii',jj'}\gamma^{kp}\gamma^{sq}\Omega^{ii'}_{ks}\Omega^{jj'}_{pq}\eta = \frac{1}{32}\overset{G}{g}_{ii',jj'}\gamma^{kp}\gamma^{sq}F^{ii'}_{ks}F^{jj'}_{pq}\eta =$$

$$= \frac{1}{8}F^2 = \frac{1}{2}\gamma_{ij}\gamma_{i'j'}R^{ii'}_{pq}R^{jj'}_{rs}\gamma^{pr}\gamma^{qs}\,,$$

where use has been made of (32) and (A.3).

The variational principle (36) coincides for the gravitational part with that of the gravitational gauge theory of the Lorentz group $O_{1,3}$. This is entirely reasonable because $SL_{2,\mathbb{C}}$ is the universal covering group of $O_{1,3}$ and the gauge theories of the gravitational field depend essentially on the Lie algebra of the structural group.

The stationary condition for the functional (36) leads, once the Lagrangian field $L_F$ has been chosen, to the coupled equations determining $\Gamma$, $g$ and $\Psi$. As in the present case $\Gamma$ is metric, generalizations of the Einstein-Cartan theory with quadratic Lagrangians and cosmological terms are obtained, for nonzero values of the parameters $\alpha$ and $\beta$. If $\alpha = 1$, $\beta = 0$, $\Gamma$ torsionless, or simply $\alpha = 1$ and $\beta = 0$, Einstein's and Einstein-Cartan's theories with cosmological terms are, respectively, recovered.

It is clear that the procedure previously outlined permits to obtain in a similar way matter Lagrangians for higher-rank spinor fields, mixed fields and so on.

The « spinor curvature tensor » ([8]), defined as an obvious generalization of the usual expression ([17]) for the tangent bundle to $M$,

(37) $$F_{kl}\Psi \overset{\text{def}}{=} [\nabla_k, \nabla_l]\Psi - \nabla_{[\partial_k,\partial_l]}\Psi$$

has components

(38) $$F_{klB}^{\ \ \ A} = \partial_k B_{lB}^{\ \ A} - \partial_l B_{kB}^{\ \ A} + B_{kC}^{\ \ A}B_{lB}^{\ \ C} - B_{lC}^{\ \ A}B_{kB}^{\ \ C} - c^m_{kl}B_{mB}^{\ \ A}\,.$$

It is related to the curvature tensor of $M$ by

(39) $$F_{klB}^{\ \ \ A} = \tfrac{1}{2}R_{kls}^{\ \ \ \ r}\hat{\sigma}_r^{\ A\dot{C}}\hat{\sigma}^s_{\ B\dot{C}} \qquad (\tilde{\sigma} \equiv \hat{\sigma} \text{ for simplicity})$$

and may also be used to form quadratic Lagrangians for the gravitational field. This possibility has been extensively explored by CARMELI and coworkers ([2-8]) and also discussed by LEIBOWITZ ([23]) and by HERRERA ([24]). The gravitational equations in the Newman-Penrose formalism are obtained.

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([23]) E. LEIBOWITZ: *Phys. Rev. D*, **15**, 2139 (1977).
([24]) L. HERRERA: *Lett. Nuovo Cimento*, **21**, 11 (1978).

## 5. – Concluding remarks.

A gauge theory of gravitation based on the group $SL_{2,C}$ has been entirely constructed with elements of the spinor frame bundle $SP_2$, within the general formalism taken from ([10]). Its final expressions are those of the theory corresponding to the Lorentz group $O_{1,3}$. It has been possible to couple the gravitational field with spinor fields for a wide class of matter Lagrangians, through a variational formalism defined on the $SP_2$ bundle.

The importance of the induced connection on the orthonormal-frame bundle $L_0$ has also been emphasized. It expresses the natural bond between the spinor and Lorentz frame, or $SP_2$ and $L_0$.

Within the same lines a similar theory for Dirac four-component spinors may be constructed, based in the $SP_4$ group.

* * *

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## APPENDIX

### Alternative proof of the theorem of sect. 3.

The verification of the condition $\tilde{\sigma}^k_{;s} = 0$ is not a difficult procedure, but a rather delicate one. It is convenient to choose a basis of the type $f\varrho \otimes \varrho^* \otimes \bar{\varrho}^*$ for the calculations. Using definitions (14)-(16), property (9), the algebra of the Pauli matrices, (29) and (30), one obtains the expressions

$$\tilde{\sigma}^k_{;s} = L^k_m[(L^{-1})^m_t \Gamma^t_{sf} L^f_p \tilde{\sigma}^p - 2\,\mathrm{Re}\,\{\mathscr{B}^\alpha_s \tilde{\sigma}_\alpha \tilde{\sigma}^m\}]\,. \tag{A.1}$$

The vanishing of $\tilde{\sigma}^k_{;s}$ is equivalent to the vanishing of the brackets and this finally leads, after elementary manipulations, to the four relations

$$\text{(A.2)}\quad \begin{cases} m = 0, \\ \Gamma^{\prime 00}_s \sigma_0 + (-\Gamma^{\prime 01}_s + A^{01}_s)\sigma_1 + (\Gamma^{\prime 02}_s - A^{02}_s)\sigma_2 + (-\Gamma^{\prime 03}_s + A^{03}_s)\sigma_3 = 0\,, \\ m = 1, \\ (\Gamma^{\prime 10}_s - A^{10}_s)\sigma_0 - \Gamma^{\prime 11}_s \sigma_1 + (\Gamma^{\prime 12}_s - A^{12}_s)\sigma_2 + (-\Gamma^{\prime 13}_s + A^{13}_s)\sigma_3 = 0\,, \\ m = 2, \\ (\Gamma^{\prime 20}_s - A^{20}_s)\sigma_0 + (-\Gamma^{\prime 21}_s + A^{21}_s)\sigma_1 + \Gamma^{\prime 22}_s \sigma_2 + (-\Gamma^{\prime 23}_s + A^{23}_s)\sigma_3 = 0\,, \\ m = 3, \\ (\Gamma^{\prime 30}_s - A^{30}_s)\sigma_0 + (-\Gamma^{\prime 31}_s + A^{31}_s)\sigma_1 + (\Gamma^{\prime 32}_s - A^{32}_s)\sigma_2 - \Gamma^{\prime 33}_s \sigma_3 = 0\,, \end{cases}$$

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$\Gamma_k^{ij} = \Gamma_{kr}^i \gamma^{ri}$: components of the connection of $L_0$ in the orthonormal frame $f_Q$;

$A_k^{ij} = A_{kr}^i \gamma^{rj}$: components of the induced connection for the same frame (29).

From (A.2) and by considering the linear independence of the $\sigma_k$, the necessary and sufficient condition

$$A_{jk}^i = \Gamma_{jk}^i \tag{A.3}$$

becomes clear. ■

## ● RESUMEN

Se presenta una teoría consistente del campo de gravitación como campo de media, basada en el grupo $SL_{2,C}$ y en el formalismo de espinores de dos componentes. Bajo este enfoque unificado que emplea la noción de espacio fibrado, la materia puede ser acoplada al campo gravitacional obteniéndose finalmente un lagrangeano lineal y/o cuádrático para estos últimos campos.

## ● RIASSUNTO (*)

Si presenta una coerente teoria di gauge della gravitazione basata sul gruppo $SL_{2,C}$ e il formalismo per lo spinore a due componenti. In questo approccio unificato su fasci di fibre, la materia può essere accoppiata al campo gravitazionale portando infine a un lagrangiano lineare e/o quadratico su tali campi.

(*) *Traduzione a cura della Redazione.*

**О спинорах и гравитации.**

**Резюме** (*). — Предлагается непротиворечивая калибровочная теория гравитации, основанная на группе $SL_{2,C}$ и двухкомпонентном спинорном формализме. В этом едином подходе семейства нитей, вещество может быть связано с гравитационным полем, что приводит к линейному или квадратичному лагранжиану для таких полей.

(*) *Переведено редакцией.*

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