PERTURBATION THEORY OF SUPERCONDUCTING MICRONETWORKS II. SECOND ORDER AND SELF-INDUCTION EFFECTS

J. I. Castro and A. López

Centro Atómico Bariloche, Comisión Nacional de Energía Atómica,

8400 S.C. de Bariloche, Río Negro, Argentina

Abstract

The perturbation method to treat the non-linear Ginzburg-Landau equations for micronetworks previously derived is extended to higher orders in the perturbation parameters, the temperature ΔT and magnetic field ΔH measured from the phase boundary. In the present paper these two parameters are treated independently and the perturbation equations to all orders are analyzed. Perturbation theory allows for consideration of self-inductive effects. The importance of these for the interpretation of experimental results in micronetworks made of Type I and Type II materials is pointed out. The Landau critical points have been discussed. Expressions for the relevant thermodynamic quantities are obtained. In particular it is shown that the phase boundary satisfies the Ehrenfest-Keesom relations for second order transitions. The theory has been explicitly worked out for simple geometries and compared with exact results which have been obtained numerically . The lines separating different modes have been evaluated to second order for these cases.

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I - Introduction

Superconductive micronetworks in a magnetic field[1-2] are known to show second order phase transitions between the normal and superconductive states, whose characteristics depend on the topology and the geometry of each specimen. Regular as well as disordered structures have been studied both theoretically and experimentally[3].

The details of the phase boundary can be predicted using the linearized version of the Ginzburg-Landau equations (GLE) derived by de Gennes and Alexander[1] for these systems.

Properties of the superconductive state can only be described either numerically or by approximate methods. To this end Fink, Rodrígues and López have deviced a perturbation scheme[4] and Wang et al.[5] have applied variational calculations. The lowest order calculation of Ref.4 used, for simplicity, a single perturbation parameter which linked together the temperature and magnetic field differences ΔT and ΔH measured from the second order phase transition boundary. It can be seen however that such a scheme cannot be easily extended to higher orders.

In this paper we present results of a perturbation approach valid to all orders, that uses as independent perturbation parameters ΔT and ΔH. The relevant physical quantities are expanded in powers of two variables x and y directly related to ΔT and ΔH. The perturbative equations are obtained by substituting this expansions into both GLE.

The perturbation equations obtained can in principle be applied to any network, finite or infinite. In the first case the equations are manageable and can be applied to simple systems. At the end of the paper we discuss and compare with exact results the case of the bare ring and a particular two loop circuit, which is a special case of the Wheatstone bridge discussed in the literature[1]. For infinite circuits the situation is still simple in the case of periodic boundary conditions. At present, work on the infinite ladder is in progress.

The approach and general scope of this paper can be compared with Abrikosov's theory of the mixed state in Type II superconductors. The differences lie on the fact that here we vary both T and H and also in that the topology of the network imposes special conditions not present in the bulk case. In particular the perturbation equations for networks contain three combinations of averages of the order parameter rather than the single Abrikosov's β_A of the bulk case. There appear averages of the squared order parameter and of its fourth power as for the bulk case, but also combinations containing the currents along branches and the self and mutual induction coefficients. It is physically interesting to see how in perturbation theory the self and mutual induction coefficients play an important role. These coefficients enter through a renormalization of the Landau-Ginzburg constant κ that could help understand some recent experimental results[8] which show different behaviour for Al or In samples.

The consistency of the perturbation expansions has been checked by showing how the equations obtained are special cases of general thermodynamic relations. In particular we shall show how the line giving the phase boundary as obtained through perturbation theory is a special case of the Ehrenfest-Keesom relations for second order transitions.

The paper is organized as follows. In Section II we give the basic perturbation equations. Section III discusses the normalization conditions. Section IV is devoted to an analysis of the self-inductive effects and their contribution to the energy. In Section V we present explicit expressions within the first order perturbation for a) the order parameter and the current density and b) the free energy and other thermodynamic quantities. Physical results like a discussion of Landau critical points and the relative stability of different modes are discussed in Section VI. Section VII is devoted to the development of the second order perturbation formalism and Section VIII contains the analysis of simple physical systems and comparisons with exact numerical calculations. Section IX summarizes the conclusions and gives some perspectives for future work. The Appendices contain explicit second order perturbation equations. These have been given in enough detail so that it can be of use for any reader interested in applying them to particular systems.

II- Perturbation equations

Throughout this paper we shall use normalized quantities as in Ref.4, calling

$$\Delta = \frac{\psi_{conv}}{\psi_\infty} \quad ; \quad \vec{H} = \frac{\vec{H}_{conv}}{\sqrt{2}\, H_c} \quad ; \quad \vec{j} = \frac{4\,\pi\,\lambda}{c\,\sqrt{2}\,H_c}\,\vec{j}_{conv}$$

$$\vec{A} = \frac{\vec{A}_{conv}}{\sqrt{2}\,H_c\lambda} \quad ; \quad \phi = \frac{2\,\pi\,\phi_{conv}}{\Phi_o} \quad ; \quad \Delta G = \frac{4\,\pi}{H_c^2\,S\,\xi_o}\,\Delta G_{conv}$$

Here the label "conv" means quantities in conventional cgs units. ψ_∞ is the zero field bulk order parameter, S is the cross section of the wires in the network assumed uniform and Φ_o is the flux quantum. All lengths will be normalized by ξ_o, the coherence length at the phase boundary.

We refer the perturbation parameters to a point at the phase boundary choosing x such that $x^2 = \Delta T/(T_c - T_o)$ where $\Delta T = T - T_o$, T_o being the phase transition temperature of the network and T_c the critical temperature of the bulk material. Defining $\tau = 1 - \frac{T}{T_c}$ we have

$$x^2 = \frac{\Delta\tau}{\tau_o}$$

The other perturbation parameter is related to the magnetic field. Calling $\vec{A}_o$ the vector potential of the applied field at the phase transition boundary, the total vector potential $\vec{A}$ can be written as

$$\vec{A} = \vec{A}_e + \vec{A}_i = \vec{A}_0 + \vec{A}_1 + \vec{A}_i \tag{1}$$

where $\vec{A}_1$ is the departure of the applied (external) field $\vec{A}_e$ from its value at the phase boundary and $\vec{A}_i$ the contribution from the induced (internal) currents. We choose as the other expansion parameter the quantity y such that

$$y^2 = \frac{1}{\xi} \oint_c A_1(s)ds = \Delta\phi$$

where ξ is the GL coherence length and c is a reference loop of area S_c in the network. It should be noted that y^2 is the incremental external magnetic flux $\Delta\phi$ through c, and is temperature independent.

Following Ref.4 we define a complex modified order parameter (mop) for a branch ab by (see Fig.1)

$$f(a,s) = e^{i\gamma(a,s)}\Delta(s) \tag{2}$$

where

$$\gamma(a,s) = \frac{1}{\xi} \int_0^S A(s')ds'$$

This mop has the advantage of simplifying the equations although it is not single valued. It depends on the definition of a reference point (a) and on the choice of the route. For each branch we take it to be defined from its starting node a, for which s = 0.

As stated above we assume an expansion of all physical quantities in powers of x and y:

$$\Delta(s) = \sum_{kl} \Delta_{kl}(s)\ x^k\ y^l \qquad (k+l = 1,3,5,\ldots,2p-1,\ldots) \tag{3a}$$

$$f(a,s) = \sum_{kl} f_{kl}(a,s)\ x^k\ y^l \qquad (k+l = 1,3,5,\ldots,2p-1,\ldots) \tag{3b}$$

$$j = \sum_{kl} j_{kl}\ x^k\ y^l \qquad (k+l = 2,4,6,\ldots,2p,\ldots) \tag{3c}$$

$$A_1(s) = \sum_{kl} A_{kl}(s)\ x^k\ y^l \qquad (k+l = 2,4,6,\ldots,2p,\ldots) \tag{3d}$$

$$\Delta G = \sum_{kl} G_{kl}\ x^k\ y^l \qquad (k+l = 2,4,6,\ldots,2p+2,\ldots) \tag{3e}$$

Here $p \geq 1$ stands for the perturbation order to be considered. Conventionally, we will call "zero" order perturbation the results of the linearized theory(1-2). For the complex conjugate of the mop f it can be seen that

$$f^*(a,s) = \sum_{kl} \sigma_l f^*_{kl}(a,s)\ x^k\ y^l \qquad (k+l = 1,3,5,\ldots,2p-1,\ldots)$$

where $\sigma_l = \sigma_0$ or σ_1 according to whether l is even or odd. In turn $\sigma_0 = \text{sign}\ (\Delta\tau)$ and $\sigma_1 = \text{sign}\ (\Delta\phi)$.

For the coherence length $\xi(T) = \xi(x)$ and penetration depth $\lambda(T) = \lambda\ (x)$ we have

$$\xi = 1 - \frac{x^2}{2} + \frac{3}{8}x^4 - \ \ldots\ = \sum_k (-)^k C_k x^{2k} \tag{4}$$

where

$$C_k = \frac{1.3.5.\dots.(2k-1)}{2.4.6.\dots.(2k)}$$

and

$$\lambda^{-2} = \frac{1 + x^2}{\kappa^2} \tag{5}$$

where κ is the temperature independent GL constant.

For the circulation of the external applied vector potential we have the relation which follows from the definition of the perturbation parameter y:

$$\gamma_e(a,s) = \frac{1}{\xi}\int_0^s A_e(s')ds' = \gamma_0(a,s) + \gamma_1(a,s)\ y^2$$

where

$$\gamma_0(a,s) = \frac{1}{\xi}\int_0^s A_0(s')ds' \quad \text{and} \quad \gamma_1(a,s) = \frac{1}{\xi\Delta\phi}\int_0^s A_1(s')ds'$$

It is clear that $\gamma_1(a,s)$ does not depend on the strength of $\Delta\phi$ but on the configuration of the incremental field $\vec{A}_1$ and on the geometry of the network. Taking this into account, the expansion of $\gamma(a,s)$, the circulation of the total vector potential, is

$$\gamma(a,s) = \frac{1}{\xi}\int_0^s A(s')ds' = \frac{1}{\xi}\int_0^s \left[A_e(s')+A_1(s')\right]ds' =$$

$$= \sum_{kl}\gamma_{kl}(a,s)x^k y^l \qquad (k+l = 2,4,6,\dots,2p,\dots)$$

where

$$\gamma_{00}(a,s) = \gamma_0(a,s)$$
$$\gamma_{02}(a,s) = \gamma_1(a,s) + \int_0^s A_{02}(s')ds'$$

Otherwise

$$\gamma_{kl}(a,s) = \sum_{2m\leq k} (-)^{m-1} \frac{C_m}{2m-1} \int_0^s A_{k-2m,l}(s')ds'$$

It should be noticed that in general we have

$$f_{kl}(a,s) = e^{i\gamma_0(a,s)} \sum_{\substack{p\leq k \\ q\leq l}} \Gamma_{pq}(a,s)\ \Delta_{k-p,l-q}(s) \tag{6}$$

where Γ_{pq} can be readily obtained from γ_{pq}, as indicated in Appendix I.

In order to arrive at the perturbation equations we must replace the above expansions in both GLE

$$\xi^2 \ddot{f} + (1 - |f|^2)f = 0 \tag{7}$$

$$\vec{\nabla}\mathrm{x}(\vec{\nabla}\mathrm{x}\ \vec{A}) = \frac{1}{\lambda^2} \vec{j} \tag{8}$$

where $\vec{j}$ in terms of f is given by

$$\vec{j} = i\,\frac{\xi}{2}\,(f^* \dot{f} - f\,\dot{f}^*)\ \check{s} \tag{9}$$

$\check{s}$ being the unit vector along the wire. Here the dot stands for the derivative with respect to s. From the first GL equation we obtain for a given branch ab

$$\ddot{f}_{kl}(a,s) + f_{kl}(a,s) = \omega_{kl}(a,b,s) \tag{10}$$

where

$$\omega_{kl}(a,b,s) = \sum_{\substack{p\leq k,\, q\leq l \\ m\leq k-p \\ n\leq l-q}} \sigma_q f_{k-p-m,l-q-n} f^*_{pq} f_{mn} - \sum_{\substack{p\leq k-2 \\ (k-p\ \text{even})}} (-)^{\frac{k-p}{2}} (\omega_{pl} - f_{pl}) \tag{11}$$

The dependence of ω_{kl} on b comes through the lower order mop's at the nodes according to Eqs.(13a) and (13b). We shall leave the analysis of the perturbation equations that follows from (8) for the next sections, where we discuss in detail the self-inductive effects which become important at higher orders.

If the network has n nodes, Eqs.(10) have to be solved using the boundary conditions

$$\sum_{(\beta)} \dot{f}_{kl}(a,0) = 0 \qquad (a = 1,\ldots,n) \tag{12}$$

where (β) stands for all branches connected to a given node a.

As in Ref.4, with this new perturbation scheme we can solve Eq.(10) by the variation of constants method

$$f_{kl}(a,s) = C_{kl}(a,b,s)\sin(L-s) + D_{kl}(a,b,s)\sin s \tag{13a}$$

where L is the length of the branch and

$$C_{kl}(a,b,s) = \frac{f_{kl}(a,0) - Z_{kl}(a,b,s)}{\sin L}$$

$$D_{kl}(a,b,s) = \frac{f_{kl}(a,L) - Z_{kl}(b,a,L-s)}{\sin L} \tag{13b}$$

$$Z_{kl}(a,b,s) = \int_0^s \omega_{kl}(a,b,s')\sin s'\,ds'$$

Here we must stress that ω_{kl} has the important property

$$\omega_{kl}(b,a,L-s) = \omega_{kl}(a,b,s)$$

When the form (13a) for the mop is replaced in the boundary conditions (12) a set of equations determining the mop's at the nodes $f_{kl}(a,0)$ in a given order (k,l) in perturbation theory is obtained. In terms of the normalized order parameter $\Delta(a)$ at each node a we have

$$\Delta_{kl}(a) \sum_{(\beta)} \cot L_\beta - \sum_{(\beta)} \frac{e^{i\gamma_o(a,L_\beta)} \Delta_{kl}(\beta)}{\sin L_\beta} = \Pi_{kl}(a) \quad (a=1,\ldots,n) \qquad (14)$$

where

$$\Pi_{kl}(a) = \sum_{(\beta)} \frac{1}{\sin L_\beta} \left[e^{i\gamma_o(a,L_\beta)} \sum_{\substack{p<k \\ q<l \\ (p+q>0)}} \Gamma_{pq}(a,L_\beta) \Delta_{k-p,\,l-q}(\beta) - Z_{kl}(\beta,a,L_\beta) \right] \qquad (15)$$

These equations are similar to Eqs.(13) of Ref.4. The difference lies on the fact that now we have two independent expansion parameters x and y. For this reason all perturbation coefficients have here two indices (k,l) giving the order in x or y. It can be shown that Eqs.(14) are compatible in the Rouché-Frobenius sense.

III- Normalization of the order parameter

The lowest order equations (10) or the discrete form (14) are homogeneous equations and therefore do not fix the amplitude of the order parameter. The situation is similar to that encountered by Abrikosov in the analysis of the mixed state of Type II superconductors.

As discussed in Ref.4 the orthogonality condition in the form used by Abrikosov for the differential equation cannot be applied to networks because of the non hermiticity of the corresponding operator in the domain [a,b].

There are two equivalent ways to overcome this difficulty. One consists of writting down the scalar product between the solution of the homogeneous differential equation and the inhomogeneous term in each order and finding its value[4]. The other, alternative procedure, consists of using the linear system of equations (14) for the order parameter at the nodes. In that case it can be seen that the corresponding matrix is hermitian in such a way that an orthogonality condition similar to Abrikosov's holds in the discrete case. The resulting conditions that determine the amplitude of the order parameter are the same whichever way is followed.

For each order of perturbation theory Eqs.(14) can be written as

$$\mathbf{A}\,\vec{\Delta}_{kl} = \vec{\Pi}_{kl} \qquad , \qquad \mathbf{A} = \left[\mathcal{A}(a,b)\right]$$

where $\mathbf{A}$ is an nxn hermitian matrix with components $\mathcal{A}(a,b)$. To lowest order we have $\vec{\Pi}_{10} = \vec{\Pi}_{01} \equiv 0$ and the corresponding equations are homogeneous, identical to the de Gennes-Alexander linearized equations[1]. Calling $\vec{\Delta}_o$ the solutions of the homogeneous equations, we have for the complex scalar product of the vectors $\vec{\Delta}_o$ and $\vec{\Pi}_{kl}$

$$(\vec{\Delta}_0, \vec{\Pi}_{kl}) = (\vec{\Delta}_0, \mathbb{A}\,\vec{\Delta}_{kl}) = (\mathbb{A}^+\vec{\Delta}_0, \vec{\Delta}_{kl}) = (\mathbb{A}\,\vec{\Delta}_0, \vec{\Delta}_{kl}) = 0,$$

a relation similar to Abrikosov's for the mixed state. The ensuing relations for the modified order parameter are

$$\sum_{\beta}\left\{ \sum_{\substack{p\le k\\ q\le l\\ (p+q>0)}} \Gamma_{pq}(a,L_\beta)\left[\dot{f}_{k-p,l-q}(\beta,0)\;\dot{f}^*_o(\beta,0) - f^*_o(\beta,0)\dot{f}_{k-p,l-q}(\beta,0)\right] - \int_0^{L_\beta} f^*_0(a,s)\,\omega_{kl}(a,\beta,s)\,ds\right\} = 0 \qquad (16)$$

where the summation is over all branches in the network. Here $f(b,s)$ stands for f measured from the node b as in Eq.5 of Ref.4 and

$$f_o(a,s) = \frac{f_o(a,0)\sin(L-s) + f_o(a,L)\sin s}{\sin L} \qquad (17)$$

These relations allow us to determine the amplitude of the order parameter at the different nodes and, from Eq.(13), the amplitude at any place in the network.

To see how Eq.(16) works, we give now the explicit form for the lowest order. The first amplitudes to be determined correspond to the normalization of the "zero" order solution (17). Choosing a reference node a = 1 at which $f_0(1,0)= 1$ all other amplitudes associated with the linearized GLE are determined from the homogeneous Eqs.(14) and from Eq.(17). The physically relevant order parameter to lowest order is

$$f_1(a,s) = f_{10}(a,s)x + f_{01}(a,s)y$$

where $f_{10}(a,s) = \alpha_{10}f_0(a,s)$ and $f_{01}(a,s) = \alpha_{01}f_0(a,s)$, α_{10} and α_{01} being constant amplitude factors. The normalization conditions allow us to determine α_{10} and α_{01}. For this order the values of k and l to be used in Eq.(16) are such that k + l = 3. From Eq. (11) we have

$$\omega_{30}= \left[\sigma_0|f_{10}|^2-1\right]f_{10} \quad , \quad \omega_{21}= \left[2\sigma_0|f_{10}|^2-1\right]f_{01} + \sigma_1 f_{10}^2 f_{01}^* ,$$

$$\omega_{12}= 2\sigma_1|f_{01}|^2 f_{10} + \sigma_0 f_{01}^2 f_{10}^* \quad , \quad \omega_{03}= \sigma_1|f_{01}|^2 f_{01}$$

Now, in terms of these variables and functions we get from Eq.(16) four relations:

$$21\ \alpha_{10}\sum_\beta \Gamma_{20}(a,L_\beta)\ j_{0\beta} + \sigma_0|\alpha_{10}|^2\alpha_{10}\sum_\beta \int_0^{L_\beta}|f_0(a,s)|^4 ds -$$

$$- \alpha_{10}\sum_\beta \int_0^{L_\beta}|f_0(a,s)|^2 ds = 0 \qquad (18a)$$

$$2i\ \alpha_{01}\sum_{\beta} \Gamma_{20}(a,L_\beta)\ j_{0\beta} + 2i\ \alpha_{10}\sum_{\beta} \Gamma_{11}(a,L_\beta)\ j_{0\beta} +$$

$$+ \left[2\ \sigma_0|\alpha_{10}|^2\alpha_{01} + \sigma_1\alpha_{10}^2\alpha_{01}^*\right] \sum_{\beta} \int_0^{L_\beta} |f_0(a,s)|^4 ds -$$

$$- \alpha_{01}\sum_{\beta} \int_0^{L_\beta} |f_0(a,s)|^2 ds = 0 \qquad (18b)$$

$$2i\ \alpha_{10}\sum_{\beta} \Gamma_{02}(a,L_\beta)\ j_{0\beta} + 2i\ \alpha_{01}\sum_{\beta} \Gamma_{11}(a,L_\beta)\ j_{0\beta} +$$

$$+ \left[2\ \sigma_1|\alpha_{01}|^2\alpha_{10} + \sigma_0\alpha_{01}^2\alpha_{10}^*\right] \sum_{\beta} \int_0^{L_\beta} |f_0(a,s)|^4 ds = 0 \qquad (18c)$$

$$2i\ \alpha_{01}\sum_{\beta} \Gamma_{02}(a,L_\beta)\ j_{0\beta} + \sigma_1|\alpha_{01}|^2\alpha_{01} \sum_{\beta} \int_0^{L_\beta} |f_0(a,s)|^4 ds = 0 \qquad (18d)$$

where

$$j_{0\beta} = \frac{i}{2}\left[f_0^*(a,0)\dot{f}_0(a,0) - f_0(a,0)\dot{f}_0^*(a,0)\right] =$$

$$= \frac{i}{2}\left[f_0(b,0)\dot{f}_0^*(b,0) - f_0^*(b,0)\dot{f}_0(b,0)\right] \qquad (19)$$

is the "zero" order current density in each branch. Since the α's are complex we need the four Eqs.(18) to determine α_{10} and α_{01}.

For higher orders we have from Eqs.(14) and (6) relations of the form

$$f_{kl}(1,0) = \alpha_{kl}$$

$$f_{kl}(a,0) = \left[\alpha_{kl} - X_{kl}(a)\right] f_0(a,0) \qquad (a \neq 1) \qquad (20)$$

and

$$f_{kl}(a,L) = \Big[\alpha_{kl} - X_{kl}(b) + Y_{kl}(a,b,L)\Big] f_0(a,L) \qquad (21)$$

Here

$$X_{kl}(a) = \frac{\begin{vmatrix} \mathcal{A}(2,2)\ldots\mathcal{A}(2,a-1) & \Pi_{kl}(2) & \mathcal{A}(2,a+1)\ldots\mathcal{A}(2,n) \\ \mathcal{A}(3,2)\ldots\mathcal{A}(3,a-1) & \Pi_{kl}(3) & \mathcal{A}(3,a+1)\ldots\mathcal{A}(3,n) \\ \vdots \quad\quad \vdots & \vdots & \vdots \quad\quad \vdots \\ \mathcal{A}(n,2)\ldots\mathcal{A}(n,a-1) & \Pi_{kl}(n) & \mathcal{A}(n,a+1)\ldots\mathcal{A}(n,n) \end{vmatrix}}{\begin{vmatrix} \mathcal{A}(2,2)\ldots\mathcal{A}(2,a-1) & \mathcal{A}(2,1) & \mathcal{A}(2,a+1)\ldots\mathcal{A}(2,n) \\ \mathcal{A}(3,2)\ldots\mathcal{A}(3,a-1) & \mathcal{A}(3,1) & \mathcal{A}(3,a+1)\ldots\mathcal{A}(3,n) \\ \vdots \quad\quad \vdots & \vdots & \vdots \quad\quad \vdots \\ \mathcal{A}(n,2)\ldots\mathcal{A}(n,a-1) & \mathcal{A}(n,1) & \mathcal{A}(n,a+1)\ldots\mathcal{A}(n,n) \end{vmatrix}} \qquad (22)$$

and $Y_{kl}(a,b,L)$ stands for the sum

$$Y_{kl}(a,b,L) = \sum_{\substack{p\le k \\ q\le l \\ (p+q>0)}} \Gamma_{pq}(a,L) \Big[\alpha_{k-p,l-q} - X_{k-p,l-q}(b)\Big]$$

The constant amplitude factors α_{kl} can now be determined using the normalization condition Eq.(16) in the same fashion as previously done in lowest order. Knowledge of $f_{kl}(a,0)$ and $f_{kl}(a,L)$ allows us to determine $f_{kl}(a,s)$ along any branch ab using Eq.(13).

A word of caution should be said about Eqs. (20) and (21). Because of our definition of the mop the value of f at a given node is not uniquely determined. Eq. (20) gives $f_{kl}(a,0)$ for a node (a), which is chosen as starting point for the phase counting procedure as implied by the definition Eq.(5). Eq.(21) gives the mop at the other end of a branch starting at a. As mentioned above, this gives an ambiguity in the definition of the f's. It can be shown however that this ambiguity is limited, in all orders in perturbation theory, to a phase difference.

IV - Self-inductive effects

At the phase transition boundary the superconductive currents are zero and the only magnetic field acting on the system is the external one. As one moves away from the phase boundary the currents become finite, creating induced fields. Within the perturbation scheme it is important to take proper care of these induced fields and to study their contribution to the free energy.

The magnetic fields created by the currents are determined by the second GL equation (Eqs. (8) and (9)). As pointed out following Eq. (1), the total vector potential is the sum of the applied field $\vec{A}_e$ plus the induced field $\vec{A}_i$. This last part is formally given, as solution of Eq. (8), by (see Fig. 2)

$$\vec{A}_i(\vec{\rho}) = \frac{1}{4\pi\lambda^2} \int_V \frac{\vec{j}(\vec{\rho}\,')}{|\vec{\rho} - \vec{\rho}\,'|} \, d^3\vec{\rho}\,'$$

In general we are interested in the component of $\vec{A}(\vec{\rho})$ along a given wire in the network. For $\vec{A}_i$ this is given by a sum over all branches of the network:

$$A_i(s) = \vec{A}_i(\vec{\rho}).\check{s} = \frac{S}{4\pi\lambda^2} \sum_{\beta'} j_{\beta'} \int_0^{L_{\beta'}} \frac{\check{s}\,.\,\check{s}'}{|\vec{\rho} - \vec{\rho}\,'|} \, ds' \tag{23}$$

Here $\vec{\rho}$ is a vector giving the position at which the field is to be calculated, s is a curvilinear coordinate and $\check{s}$ a unit vector along the wire; as above S is the cross section of the wires. Using Eqs.(23), (3) and (5) we can find the following relation between the perturbation coefficients for field and current density

$$A_{kl}(s) = \frac{S}{4\pi\kappa^2}\sum_{\beta'}\left[(j_{\beta'})_{kl} + (j_{\beta'})_{k-2,l}\right]\int_0^{L_{\beta'}} \frac{\check{s}\cdot\check{s}'}{|\vec{\rho}-\vec{\rho}'|}\,ds'$$

The coefficients j_{kl} are independent of s because of current conservation along each branch. In a given branch β, j_{kl} is related to the mop through

$$j_{kl} = \frac{i}{2}\sum_{\substack{p\le k\\ q\le l}}\ \sum_{2m\le k-p} (-)^m \sigma_q C_m \Big[f^*_{pq}(a,0)\dot{f}_{k-p-2m,l-q}(a,0) - f_{k-p-2m,l-q}(a,0)\,\dot{f}^*_{pq}(a,0)\Big] \qquad (24)$$

The free energy of the system, measured from the normal state is given by

$$\Delta G = S^{-1}\int_{\text{all space}}\left[-|\Delta|^2 + \frac{|\Delta|^4}{2} + \xi^2\left|\left(i\nabla - \frac{\vec{A}}{\xi}\right)\Delta\right|^2 + |\vec{h} - \vec{H}|^2\right]dV \qquad (25)$$

where $\vec{H}$ is the applied magnetic field and $\vec{h}$ is the total magnetic field, $\vec{h} = \lambda\,\vec{\nabla}\times\vec{A}$. Written in terms of the mop, ΔG becomes

$$\Delta G = \sum_{\beta}\int_0^{L_\beta}\left[-|f|^2 + \frac{|f|^4}{2} + \xi^2|\dot{f}|^2\right]ds + S^{-1}\int_{\text{all space}}|\vec{h} - \vec{H}|^2\,dV \qquad (26)$$

The inductive effects are contained in the last term and also in the $\xi^2|\dot{f}|^2$ term in (26). In particular the last term can be written as

$$\Delta G_1 = S^{-1}\int_{\text{all space}} |\vec{h} - \vec{H}|^2 dV = S^{-1}\int_V \vec{j} \cdot \vec{A}_1 \, dV$$

where V is the volume of the network. This equation in cgs units reads $\Delta G_1 = (2c)^{-1} \int_V \vec{j}.\vec{A}_1 \, dV$. Furthermore, this can be transformed into

$$\Delta G_1 = \sum_\beta j_\beta \int_0^{L_\beta} A_1(s)ds = \frac{S}{4\pi\lambda^2}\sum_{\beta\beta'} j_\beta j_{\beta'} K_{\beta\beta'} \tag{27}$$

where

$$K_{\beta\beta'} = \int_0^{L_\beta}\int_0^{L_{\beta'}} \frac{\check{s} \cdot \check{s}'}{|\vec{\rho} - \vec{\rho}'|} ds'ds$$

We shall see that the purely geometrical factors $K_{\beta\beta'}$ are related to the self and mutual inductances of the network. In fact, the sum over branches in (27) can be replaced by a sum over loops μ:

$$\begin{aligned}\Delta G_1 &= \frac{S}{4\pi\lambda^2}\sum_{\mu\mu'} j_\mu j_{\mu'} \oint_\mu \oint_{\mu'} \frac{\check{s} \cdot \check{s}'}{|\vec{\rho} - \vec{\rho}'|} ds'ds = \\ &= \frac{S}{\lambda^2}\left[\frac{1}{2}\sum_\mu \Lambda_\mu j_\mu^2 + \sum_{\substack{\mu \\ \mu'>\mu}} M_{\mu\mu'} j_\mu j_{\mu'}\right]\end{aligned} \tag{28}$$

For a given network with n nodes and N_β branches, there are $N_\mu = N_\beta - n + 1$ loops that can be independently chosen. Each branch must be included at least once in the set of N_μ loops. The expression (28) is nothing but the Neumann formula for the magnetic energy of a system of linear currents. Λ_μ and $M_{\mu\mu'}$ are the self and mutual induction coefficients

$$M_{\mu\mu'} = \frac{1}{2\pi} \oint_\mu ds \oint_{\mu'} ds' \frac{\check{s} \cdot \check{s}'}{|\vec{\rho} - \vec{\rho}'|} \quad , \quad \Lambda_\mu = M_{\mu\mu}$$

Remembering that from Eq. (3) we have for the total free energy the expansion $\Delta G = \sum_{kl} G_{kl} x^k y^l$ we can write

$$G_{kl} = \sum_\beta \int_0^{L_\beta} \left[-|f|^2_{kl} + \frac{|f|^4_{kl}}{2} + \left[\xi^2 |\dot{f}|^2\right]_{kl} \right] ds +$$

$$+ \frac{S}{4 \pi \kappa^2} \sum_{\beta\beta'} \sum_{\substack{p \le k \\ q \le l}} \left[(j_\beta)_{k-p,l-q} + (j_\beta)_{k-p-2,l-q} \right] (j_{\beta'})_{pq} K_{\beta\beta'} \tag{29a}$$

or equivalently

$$G_{kl} = \sum_\beta \int_0^{L_\beta} \left[-|f|^2_{kl} + \frac{|f|^4_{kl}}{2} + \left(\xi^2 |\dot{f}|^2\right)_{kl} \right] ds +$$

$$+ \frac{S}{2 \kappa^2} \sum_{\mu\mu'} \sum_{\substack{p \le k \\ q \le l}} \left[(j_\mu)_{k-p,l-q} + (j_\mu)_{k-p-2,l-q} \right] (j_{\mu'})_{pq} M_{\mu\mu'} \tag{29b}$$

It can be seen here that the self induction effects can be neglected whenever $S/\kappa^2 \longrightarrow 0$, i.e. when one assumes that the wires of the network are really one dimensional elements or the material is extreme Type II. When L/ξ_0 is small the $M_{\mu\mu'}$'s become small, and the self induction effects can also be neglected.

V - First order perturbation results

a) Order parameter and current density

In this section we shall discuss in detail the first order perturbation results within the present scheme, in order to compare them with the results of Ref.4. In the next sections we shall consider the second order results. From the normalization conditions (18), we obtain

$$|\alpha_{10}|^2 = \sigma_0 \frac{B_1}{D} \quad , \quad |\alpha_{01}|^2 = \sigma_1 \frac{B_2}{D} \quad , \quad D = B_3^2 - \frac{S}{\kappa^2} B_4^2 \tag{30}$$

The following relation also obtains

$$\sigma_0 \alpha_{01} \alpha_{10}^* + \sigma_1 \alpha_{01}^* \alpha_{10} = 0 \tag{31}$$

which is essential to assure that $|f|^2$ is real. In these equations

$$B_1 = \sum_\beta \int_0^{L_\beta} |f_0|^2 \, ds \tag{32a}$$

$$B_2 = 2 \sum_\beta \gamma_1(a, L_\beta) j_{0\beta} \tag{32b}$$

$$B_3^2 = \sum_{\beta} \int_0^{L_\beta} |f_0|^4 \, ds \tag{32c}$$

$$B_4^2 = \frac{1}{2\pi} \sum_{\beta\beta'} j_{o\beta}\, j_{o\beta'}\, K_{\beta\beta'} = \sum_{\mu} \Lambda_\mu\, j^2_{o\mu} + 2 \sum_{\substack{\mu \\ \mu' > \mu}} M_{\mu\mu'}\, j_{o\mu}\, j_{o\mu'} \tag{32d}$$

Here $j_{o\beta}$ is the current density in branch β associated with the zero order solution as in Eq.(19). The coefficients B above are very similar to the β's obtained in Ref.4. The main difference lies on the fact that B_4^2 as written above shows explicitly the self and mutual induction effects.

From Eqs. (30) and (31) we can see that one can determine for both α_{10} and α_{01} their square amplitude and their relative phase. Now, for the mop to first order we have explicitly

$$|f(a,s)|^2 \simeq \frac{\left[B_1 \left(\Delta\tau/\tau_0\right) + B_2\, \Delta\phi\right]}{D} |f_0(a,s)|^2 = \alpha^2 |f_0(a,s)|^2 \tag{33}$$

For the current density using Eq.(24) we obtain similarly

$$j \simeq \alpha^2 j_0 \tag{34}$$

It can be seen that the perturbation coefficients B can be interpreted in terms of different contributions to the lowest order energy. So, taking $\xi \simeq 1$ at the phase boundary, the kinetic energy of the supercurrents is related to B_1:

$$\frac{\Delta\tau}{\tau_0}\frac{1}{S}\int_V \Delta_0^* \frac{\hat{p}^2}{2}\Delta_0 dV = \frac{\Delta\tau}{\tau_0}\frac{1}{2S}\int_V \Delta_0^*\left[i\vec{\nabla} - \vec{A}_0\right]^2\Delta_0 dV =$$

$$= \frac{1}{2}\frac{\Delta\tau}{\tau_0}\sum_\beta \int_0^{L_\beta} |f_0|^2 ds = \frac{1}{2}\frac{\Delta\tau}{\tau_0} B_1 \quad ,$$

the interaction energy of the currents with the external field increment is related to B_2:

$$\frac{1}{S}\int_V \vec{j}_0\cdot\vec{A}_1\, dV = \Delta\phi \sum_\beta \gamma_1(a,L_\beta) j_{0\beta} = \frac{1}{2}\Delta\phi\, B_2 \quad ,$$

the superconductive condensation energy is

$$\frac{1}{2S}\int_V |\Delta_0|^4 dV = \frac{1}{2}\sum_\beta \int_0^{L_\beta} |f_0|^4 ds = \frac{B_3^2}{2}$$

and the magnetic energy of the induced field is

$$\frac{S}{\kappa^2}\left(\frac{1}{2}\sum_\mu \Lambda_\mu j_{0\mu}^2 + \sum_{\substack{\mu \\ \mu'>\mu}} M_{\mu\mu'} j_{0\mu} j_{0\mu'}\right) = \frac{S}{2\kappa^2} B_4^2 \; .$$

Finally, D/2 can be identified with the equilibrium free energy of the system: $D/2 = -\Delta G_0$.

b) The Ehrenfest-Keesom equations for the phase transition boundary

Equation (33) gives the order parameter at a point separated from the phase boundary a certain distance $\Delta\phi$ and $\Delta\tau$. If these values were chosen such that the end point is again on the phase boundary, the order parameter must vanish. This gives a relation between $\Delta\phi$ and $\Delta\tau$, which can be shown to give rise to a differential equation for the phase boundary. This reads

$$\left(\frac{d\phi}{d\tau}\right)_0 = - \frac{1}{\tau_0} \frac{B_1}{B_2} \tag{35}$$

When written in terms of $L = \dfrac{L_{conv}}{\xi_0(T_0)}$ we have

$$\frac{dL}{d\phi_0} = - \frac{L}{2} \frac{B_2}{B_1} \tag{36}$$

This equation can be identified with one of the Ehrenfest-Keesom relations[6] which are to second order phase transitions what the Clausius-Clapeyron equation is to first order transitions. Written with the variables we are using the Ehrenfest-Keesom equations are

$$\frac{1}{S_c} \left(\frac{d\phi}{d\tau}\right)_0 = \frac{\Delta\alpha}{\Delta\eta_\tau} \quad , \quad \frac{1}{S_c} \left(\frac{d\phi}{d\tau}\right)_0 = \frac{1}{(1-\tau_0)} \frac{\Delta C_\phi}{\Delta\alpha} \tag{37}$$

where α is the coefficient for the thermal variation of the magnetization, defined by

$$\alpha = - \frac{1}{V} \left(\frac{\partial \mathcal{M}}{\partial \tau}\right)_\phi \tag{38}$$

η_τ is the isothermal susceptance defined by

$$\eta_\tau = \frac{S_c}{V}\left(\frac{\partial \mathcal{M}}{\partial \phi}\right)_\tau \tag{39}$$

and C_ϕ is the specific heat at constant field defined by

$$C_\phi = \frac{(\tau - 1)}{V}\left(\frac{\partial \mathcal{S}}{\partial \tau}\right)_\phi \tag{40}$$

In turn

$$\mathcal{M} = - S_c\left(\frac{\partial G}{\partial \phi}\right)_\tau \quad \text{and} \quad \mathcal{S} = \left(\frac{\partial G}{\partial \tau}\right)_\phi \tag{41}$$

are the total magnetic dipole moment and the entropy respectively. The symbol Δ in (37) means the difference in the values of the corresponding quantity between the superconducting and the normal states. From (37) it follows that

$$\Delta\alpha = \frac{1}{S_c}\left(\frac{d\phi}{d\tau}\right)_0 \Delta\eta_\tau \quad , \quad \Delta C_\phi = (1 - \tau_0)\left[\frac{1}{S_c}\left(\frac{d\phi}{d\tau}\right)_0\right]^2 \Delta\eta_\tau$$

Since we are using two independent expansion parameters to describe the properties of the superconductive phase a given distance away from the phase boundary, the question arises whether this description is unique. In fact, as shown in Fig.3, a given point P in the superconductive phase can be reached from more than one point at the phase boundary. The order parameter at P can be written as

$$|f|_A^2 \simeq \frac{\left[B_{1A}\left(\Delta\tau/\tau_0\right)_A + B_{2A}\Delta\phi_A\right]}{D_A}|f_0|_A^2$$

or as

$$|f|_B^2 \simeq \frac{\left[B_{1B}\left(\Delta\tau/\tau_0\right)_B + B_{2B}\Delta\phi_B\right]}{D_B}|f_0|_B^2$$

according to which starting point is taken. Neglecting second order terms in $\Delta\tau$ and $\Delta\phi$, it can be seen that the Ehrenfest-Keesom equation (35) implies that these two expressions are equal.

c) Free energy and thermodynamic relations

The free energy to lowest order is given by Eqs.(3) and (29):

$$\Delta G \simeq G_{20}x^2 + G_{11}xy + G_{02}y^2 + G_{40}x^4 + G_{31}x^3y + G_{22}x^2y^2 + G_{13}xy^3 + G_{04}y^4$$

where the explicit forms for G_{kl} are

$$G_{20} = 0,\ G_{11} = 0,\ G_{02} = 0,\ G_{31} = 0,\ G_{13} = 0,$$

$$G_{40} = -\frac{1}{2}\ \frac{B_1^2}{D},\ G_{22} = -\frac{B_1B_2}{D},\ G_{04} = -\frac{1}{2}\ \frac{B_2^2}{D}$$

With these coefficients, the free energy in first order perturbation becomes

$$\Delta G \simeq -\frac{D}{2}\left[\left(\frac{B_1}{D}\right)\frac{\Delta\tau}{\tau_0} + \left(\frac{B_2}{D}\right)\Delta\phi\right]^2 \qquad (42)$$

This result, when the self-induction effects are neglected, reduces to Eq.(21) of Ref.4.

We can calculate the values of the different thermodynamic variables in the superconductive state relative to the normal state using Eqs. (38)-(41) and (42):

$$\Delta\mathcal{M} = - S_c\left[\frac{\partial\ \Delta G}{\partial\phi}\right] \simeq S_c\left[\left[\frac{B_1}{D}\right]\frac{\Delta\tau}{\tau_0} + \left[\frac{B_2}{D}\right]\Delta\phi\right]B_2 \tag{43a}$$

$$\Delta\alpha = \frac{S_c}{V}\left[\frac{\partial^2\Delta G}{\partial\tau\partial\phi}\right] \simeq - \frac{S_c}{V\tau_0}\ \frac{B_1\ B_2}{D} \tag{43b}$$

$$\Delta\eta_\tau = - \frac{S_c^2}{V}\left[\frac{\partial^2\Delta G}{\partial\phi^2}\right] \simeq \frac{S_c^2}{V}\ \frac{B_2^2}{D} \tag{43c}$$

$$\Delta\mathcal{S} = \left[\frac{\partial\ \Delta G}{\partial\tau}\right] \simeq - \left[\left[\frac{B_1}{D}\right]\frac{\Delta\tau}{\tau_0} + \left[\frac{B_2}{D}\right]\Delta\phi\right]\frac{B_1}{\tau_0} \tag{43d}$$

$$\Delta C_\phi = \frac{(\tau-1)}{V}\left[\frac{\partial^2\Delta G}{\partial\tau^2}\right] \simeq \frac{(1-\tau)}{V\tau_0^2}\ \frac{B_1^2}{D} \tag{43e}$$

Since ΔG above refers to the whole system, $\mathcal{M}$ in (43a) is the total magnetic moment of the network. For networks there is no meaning for a local magnetic moment so the relevant thermodynamic quantity is the susceptance and not the susceptibility.

For the sake of comparison with Abrikosov's theory for Type II bulk materials, we can give the expressions of the jump in the specific heat, thermal variation of magnetization coefficient and magnetic susceptance across the phase boundary in cgs units:

$$(\Delta C_H)_{bulk} = \frac{T_0}{16\,\pi^3\beta_A(2\kappa^2-1)}\,\frac{\Phi_0^2}{\xi^4(0)T_c^2}$$

$$(\Delta C_H)_{network} = \frac{T_0}{16\,\pi^3\beta'_A(2\tilde{\kappa}^2-1)}\,\frac{\Phi_0^2}{\xi^4(0)T_c^2},$$

$$(\Delta\alpha)_{bulk} = \frac{T_0}{8\,\pi^2\beta_A(2\kappa^2-1)}\,\frac{\Phi_0}{\xi^2(0)T_c}$$

$$(\Delta\alpha)_{network} = \frac{T_0}{8\,\pi^2\beta''_A\,(2\tilde{\kappa}^2-1)}\,\frac{\Phi_0}{\xi^2(0)T_c},$$

$$(\Delta\eta_\tau)_{bulk} = \frac{1}{4\,\pi\,\beta_A(2\kappa^2-1)}$$

$$(\Delta\eta_\tau)_{network} = \frac{1}{4\,\pi\,\beta'''_A(2\tilde{\kappa}^2-1)}$$

where β_A is Abrikosov's factor, $\xi(0)$ is Gor'kov's coherence length and

$$\beta'_A = \frac{V}{2}\,\frac{B_4^2}{B_1^2},\quad \beta''_A = -\,2V\,\frac{B_4^2}{B_1 S_c B_2},\quad \beta'''_A = \frac{V}{2}\,\frac{B_4^2}{S_c^2 B_2^2},\quad \tilde{\kappa} = (2S)^{-1/2}\,\frac{B_3}{B_4}\,\kappa \qquad (47)$$

Here the expressions for $(\Delta C_H)_{bulk}$ and $(\Delta\alpha)_{bulk}$ have been obtained from $(\Delta\eta_\tau)_{bulk}$ using the Ehrenfest-Keesom relations. We can see that for networks instead of a single factor like Abrikosov's β_A there are three different factors related to ratios of different contributions to the energy which are specific to the network structure. Besides, the topology and the geometry of the network require a renormalization of the GL constant κ. The three factors β'_A, β''_A and β'''_A are not independent; they are related by the Ehrenfest-Keesom equation (35):

$$\beta''_A = 4\,\frac{\tau_0}{S_c}\left(\frac{d\phi}{d\tau}\right)_0 \beta'_A \quad , \quad \beta'''_A = \frac{\tau_0^2}{S_c^2}\left(\frac{d\phi}{d\tau}\right)_0^2 \beta'_A$$

Here we must stress that $(S_c B_2)$ does not depend on the loop c selected to measure the magnetic flux. Finally, the specific heat at constant magnetization can be obtained from the usual thermodynamic relation

$$C_{\mathcal{M}} - C_\phi = 4\,\pi\,\frac{(1-\tau)\alpha^2}{\eta_\tau} \quad ;$$

in fact, using Eqs.(43) we have, for the specific heat $\Delta C_{\mathcal{M}}$ in the supercondutive state relative to the normal state

$$\Delta C_{\mathcal{M}} = (1 + 4\pi)\Delta C_\phi$$

As a check of the perturbation expansions we can see relating Eqs.(43) and (35) that the thermodynamic quantities, independently calculated, satisfy the Ehrenfest-Keesom relations:

$$\frac{\Delta C_\phi}{\Delta \eta_\tau} = \frac{(1 - \tau_0)}{\tau_0^2 \, S_c^2} \, \frac{B_1^2}{B_2^2} = (1 - \tau_0) \left[\frac{1}{S_c} \left(\frac{d\phi}{d\tau} \right)_0 \right]^2$$

$$\frac{\Delta \alpha}{\Delta \eta_\tau} = - \frac{1}{S_c \tau_0} \, \frac{B_1}{B_2} = \frac{1}{S_c} \left(\frac{d\phi}{d\tau} \right)_0$$

VI - Physical results

a) Landau critical points (LCP).

We can see from (33) and (42) that whenever $D(\tau_0, \phi_0) = 0$ there is an interesting change in the behaviour of the solutions. In fact, it can be seen from (33) that $|f|^2$ is always positive when the increments $\Delta\tau$ and $\Delta\phi$ bring us from a point at the phase boundary to another point inside the superconducting region. A special situation arises when $D(\tau_0, \phi_0)$ changes sign. In that case, in order to have a positive definite $|f|^2$, $\Delta\tau$ and $\Delta\phi$ must bring us out of the superconducting region and into the "normal" region. We can see from (42) that when this happens the free energy as a function of $\Delta\tau$ (or $\Delta\phi$) undergoes a change in curvature. As seen in Fig.4, this indicates that the system goes through a Landau critical point. The transition changes to first order and the perturbation approach gives only the supercooling limit. It can be seen that the change in sign in D allows for the classification of the transitions as follows:

$$D > 0 \quad , \quad \frac{S \, B_4^2}{\kappa^2 \, B_3^2} < 1 \quad , \quad \text{2nd. order}$$

$$D = 0 \quad , \quad \frac{S\,B_4^2}{\kappa^2\,B_3^2} = 1 \quad , \quad \text{LCP} \tag{48}$$

$$D < 0 \quad , \quad \frac{S\,B_4^2}{\kappa^2\,B_3^2} > 1 \quad , \quad \text{1st. order}$$

We see from (48) that S/κ^2 and B_4^2, which contains the self-induction effects are important to determine the kind of transition to be expected. Physically what happens is that the positive energy associated with the magnetic field of the induced currents overcomes the negative contribution from the condensation energy. Superconductivity is only possible with a finite and large order parameter so the system makes a jump to meet that condition.

We have checked these results against exact calculations for a hollow cylinder by Fink and Grünfeld[7]. With the notation given in Fig.5, we have for the self-induction coefficient $\Lambda = L^2/\,2\pi L'$ and the relevant perturbation coefficients are

$$B_3^2 = \int_0^L |f_0|^4 ds = L \quad , \quad B_4^2 = \Lambda\, j_0^2 = \Lambda$$

The condition (48) for a LCP reads in this case $Ld\,/\,2\pi\kappa^2 = 1$ which in cgs units means

$$\frac{1}{2\,\pi}\,\frac{L\,d}{\lambda^2(0)}\left(1 - \frac{T_{LCP}}{T_c}\right) = 1$$

Using the values given in Fig.7 of Ref.7 this gives

$T_{LCP} \simeq 0.961 T_c$. This result is slightly different from that in Ref.7 where the authors get $T_{LCP} \simeq 0.987\ T_c$ due to the fact that the exact calculation takes into account field expulsion effects (Meissner currents) which are neglected in the de Gennes-Alexander model which we use.

It can be seen from (48) that $1/\kappa^2$ amplifies the self-inductive effects given by B_4^2. Recent experimental work on the critical currents and I-V characteristics in the resistive transition of micronetworks suggests a strong dependence on the type of material used to prepare the samples[8]. In particular in Ref.8 In and Al are used, which have κ values differing by a factor of 10. Although we have not addressed the question of critical currents in this paper, our Eq. (48) suggests that the induction effects help understand the difference in the experimental results of Ref.8 for In and Al.

b) Relative stability of different modes

In general, for any network, different condensation modes are characterized by the value of the integer n in the fluxoid quantization condition

$$2 \pi n = \frac{1}{\xi} \oint_c \frac{j}{|f|^2} ds + \varphi \tag{49}$$

where c is some conveniently chosen loop and φ is the total magnetic flux linked by c. In Ref.4 the relative stability of different condensation modes was discussed. In particular the case studied in Ref.4 in some detail was the ladder for which, when periodic boundary conditions after N strands are chosen, n takes on the values $0 \le n \le N$. Each value of n gives the highest possible transition temperature over a certain region along the phase boundary. Here we give more details about this point and discuss the effect of the inductive contributions.

Within the superconductive region a given mode will be stable when its associated free energy is lower than that of any other mode. The boundaries separating different modes are determined by the condition

$$\Delta G^{(n)}(\phi,\tau) = \Delta G^{(n+1)}(\phi,\tau) \tag{50}$$

Using the variable $\ell = L_{conv}/\xi(T)$ it can be shown from (42) that, to first order in perturbation theory, the line along which (50) is satisfied obeys the equation

$$\phi = \phi_o + \left[\frac{B_1^{(n)}\sqrt{D^{(n+1)}} - B_1^{(n+1)}\sqrt{D^{(n)}}}{B_2^{(n)}\sqrt{D^{(n+1)}} - B_2^{(n+1)}\sqrt{D^{(n)}}}\right]\left(1 - \frac{\ell^2}{L^2}\right) \tag{51}$$

The stability lines drawn in Figs.2 and 3 of Ref.4 were obtained numerically from Eq.(51) neglecting self-inductive effects.

It can be shown further that the three slopes that meet at each point on the phase boundary where the stability line starts are related through

$$\frac{d\phi}{d\ell} = \left[\frac{B_1^{(n)}\sqrt{D^{(n+1)}} - B_1^{(n+1)}\sqrt{D^{(n)}}}{B_1^{(n)}\left[\frac{dL}{d\phi_0}\right]^{(n)}\sqrt{D^{(n+1)}} - B_1^{(n+1)}\left[\frac{dL}{d\phi_0}\right]^{(n+1)}\sqrt{D^{(n)}}}\right]$$

where $d\phi/d\ell$ is the slope associated with the mode stability line given by (51) and $dL/d\phi_0$ refers to the phase boundary for each mode as in Eq.(36).

VII - Second order perturbation formalism

a) Second order expansions

In the previous paragraphs we have discussed the lowest order perturbation expansion in detail, having obtained explicit results for f_{kl}, Δ_{kl}, j_{kl}, A_{kl} and G_{kl}, coefficients in the expansion for the order parameter, current, vector potential and free energy.

From Ref.4, where equivalent results were obtained, we know that a comparison of this results with exact calculations shows that very good agreement is obtained in first order for the slopes of current and order parameter as function of temperature or field. This is also true for the phase separation lines for different modes in the (H,T) plane.

In order to explore the superconducting region further apart from the phase boundary it is necessary to go up to second order terms. From these we expect to predict the curvature of the different variables quoted above as functions of field and temperature. Once the formalism is developed and checked against exact results in some simple cases the theory is equipped with a trustful procedure to study more complicated systems.

The second order contribution to the mop follows from Eqs. (11) and (13) with $k + \ell = 3$ ($k\ell$ = 30, 21, 12, 03); if we denote this contribution by f_3 we have from Eq. (3)

$$f_3 = f_{30}(a,s)\left[\frac{\Delta\tau}{\tau_0}\right]^{\frac{3}{2}} + f_{21}(a,s)\frac{\Delta\tau}{\tau_0}\sqrt{\Delta\phi} + f_{12}(a,s)\sqrt{\frac{\Delta\tau}{\tau_0}}\,\Delta\phi +$$

$$+ f_{03}(a,s)(\Delta\phi)^{3/2}$$

Detailed expressions for the coefficients $f_{kl}(a,s)$ are given in Appendix II. For the squared order parameter, the expansion up to second order is thus

$$|\Delta|^2 \propto \text{Eq. (33)} + |\Delta|^2_{40}\left[\frac{\Delta\tau}{\tau_0}\right]^2 + |\Delta|^2_{22}\ \frac{\Delta\tau}{\tau_0}\,\Delta\phi + |\Delta|^2_{04}(\Delta\phi)^2 \qquad (52)$$

and for the currents the result is

$$j \propto \text{Eq. (34)} + j_{40}\left[\frac{\Delta\tau}{\tau_0}\right]^2 + j_{22}\,\frac{\Delta\tau}{\tau_0}\,\Delta\phi + j_{04}(\Delta\phi)^2\ ; \qquad (53)$$

the coefficients $|\Delta|^2_{kl}$ in (52) and j_{kl} in (53) are given in Appendix II. It should be noted that $|\Delta|^2_{31} = |\Delta|^2_{13} = 0$ and $j_{31} = j_{13} = 0$, a result which ensures that the squared order parameter and the current density are always real.

The free energy reads

$$\Delta G \simeq \text{Eq.}(42) + G_{60}\left[\frac{\Delta\tau}{\tau_0}\right]^3 + G_{42}\left[\frac{\Delta\tau}{\tau_0}\right]^2\Delta\phi + G_{24}\,\frac{\Delta\tau}{\tau_0}\,(\Delta\phi)^2 + G_{06}(\Delta\phi)^3 \qquad (54)$$

In this case $G_{51} = G_{33} = G_{15} = 0$, and ΔG is always real. The explicit expressions for the coefficients are given at the end of Appendix III.

In the following sections we shall apply these perturbation calculations to simple systems and compare the ensuing results with numerical calculations.

b) Normalization of the order parameter

The normalization coefficients α_{kl} ($k + \ell = 3$) which give the part of the mop which is proportional to the "zero" order solution are determined using the normalization conditions (16) with $k + \ell = 5$. After some algebra, the following relations are obtained

$$\begin{aligned}
&\sigma_0(\alpha_{10}\,\alpha^*_{30} + \alpha^*_{10}\,\alpha_{30}) = A \\
&\sigma_0(\alpha_{01}\,\alpha^*_{30} + \alpha^*_{10}\,\alpha_{21}) + \sigma_1(\alpha_{10}\,\alpha^*_{21} + \alpha^*_{01}\,\alpha_{30}) = 0 \\
&\sigma_0(\alpha_{10}\,\alpha^*_{12} + \alpha^*_{10}\,\alpha_{12}) + \sigma_1(\alpha_{01}\,\alpha^*_{21} + \alpha^*_{01}\,\alpha_{21}) = B \qquad (55) \\
&\sigma_1(\alpha_{10}\,\alpha^*_{03} + \alpha^*_{01}\,\alpha_{12}) + \sigma_0(\alpha_{01}\,\alpha^*_{12} + \alpha^*_{10}\,\alpha_{03}) = 0 \\
&\sigma_1(\alpha_{01}\,\alpha^*_{03} + \alpha^*_{01}\,\alpha_{03}) = C
\end{aligned}$$

The explicit expressions of the right hand side terms A, B and C will be left for Appendix III, where it is seen that they can be fully calculated with the "zero" order perturbation results. In principle, the linear system of equations (55) can be solved for α_{30}, α_{21}, α_{12} and α_{03}, but this is not necessary because all physically observable quantities ($|\Delta|^2$, j, ΔG, etc.) depend on the linear combinations of the α_{kl} coefficients given by (55) and it is possible to make use of these equations without any more algebra.

VIII - <u>Application to simple systems</u>

a) Superconducting bare ring

Fink and Grünfeld[9] have explicitly solved the GL equations for a simple wire ring of length L. The wire is assumed to be thin enough so the order parameter is constant across the whole section; $|f|$ is also uniform around the ring and it is related to the temperature and magnetic flux through

$$|f|^2 = 1 - (\varphi - 2\pi n)^2 \frac{\xi^2}{L^2} \qquad \text{(n integer)} \tag{56}$$

where

$$\varphi = \phi + \phi_i = \phi_0 + \Delta\phi + \phi_i$$

is the total magnetic flux linked by the ring, sum of the external flux ϕ and the induced flux ϕ_i; in turn, ϕ is the sum of the flux ϕ_0 at the phase boundary and the incremental applied flux $\Delta\phi$ from the phase boundary to a given point in the superconductive region. n is the phase winding number for the order parameter as in Eq.(49) (see Fig.6). The dependence on the reduced temperature τ is given implicity through the coherence length:

$$\xi = \xi(\tau) = \sqrt{\frac{\tau_0}{\tau}} = \frac{1}{\sqrt{1+\frac{\Delta\tau}{\tau_0}}} \tag{57}$$

where τ_0 is the phase boundary temperature and $\Delta\tau$ is the temperature increment from the phase boundary to the point under consideration in the superconductive region.

It can be shown that the following relation between current density in the wire and order parameter holds[9]:

$$j = \mp |f|^2 \sqrt{1 - |f|^2} \; ; \tag{58}$$

the -(+) sign must be taken when $\phi_0 < n\,\Pi$ ($\phi_0 > n\,\Pi$). The induced magnetic flux is related to the current density through

$$\phi_1 = \frac{(L - D)}{2} \frac{j}{\xi^3} \tag{59}$$

where

$$D = L - \frac{S}{\kappa^2} \Lambda$$

with Λ the self-induction coefficient for the ring:

$$\Lambda = \frac{L}{\pi} \left[\ln \frac{4\,L}{\sqrt{\pi}\,S} - \frac{7}{4} \right]$$

This expression is different from Fock's formula[10] because here the current is assumed to be uniformly distributed in the cross section S of the wire. Eqs.(56), (58) and (59) allow us to find all relevant quantities as function of τ and ϕ exactly, including self-induction effects.

In order to apply perturbation theory we recall results from the linearized de Gennes-Alexander theory applied to the ring[1]. The phase transition lines in the (ℓ, ϕ) plane are given by

$$L = |\phi_0 - 2\, n\, \pi| \qquad ; \tag{60}$$

in Fig.6 we show the lines for n = 0 and n = 1. The modulus and phase of the "zero" order normalized order parameter around the ring are

$$|\Delta_0(s)| \equiv 1 \qquad , \qquad \theta_0(s) = -\, 2\, \pi\, n\, \frac{s}{L} \tag{61}$$

For the "zero" order perturbation mop we have

$$f_0(a,s) = e^{\pm\, i\, s}$$

where the +(-) sign must be taken for $\phi_0 < n\,\pi$ ($\phi_0 > n\,\pi$). In the notation of Appendix II and III we have $a = 1$, $b = e^{\pm iL}$ and $\dot{a} = \pm\, i$, and the normalized "zero" order current density is

$$j_0 = \frac{i}{2}\,(a^*\dot{a} - a\,\dot{a}^*) = \mp\, 1$$

The first order perturbation coefficients can be easily calculated from Eqs.(32); they are

$$B_1 = B_3^2 = L \quad , \quad B_2 = 2\, j_0 = \mp\, 2 \quad , \quad B_4^2 = \Lambda\, j_0^2 = \Lambda$$

Using Eqs. (33) and (34) in first order perturbation we have

$$|f|^2 \simeq \frac{L}{D}\,\frac{\Delta\tau}{\tau_0} \mp \frac{2}{D}\Delta\phi \quad , \qquad j \simeq \mp\,\frac{L}{D}\,\frac{\Delta\tau}{\tau_0} + \frac{2}{D}\Delta\phi \tag{62}$$

If the self-induction effects are moderate $D > 0$, $|f|^2$ is always positive in the superconductive region and j changes sign according to the phase border line considered. The Ehrenfest-Keesom equation (36) takes now the simple form

$$\frac{dL}{d\phi_0} = \pm 1 \quad \text{or} \quad L = \pm\,\phi_0 + \text{constant}$$

which is just the characteristic equation for the phase transition lines (60).

For the free energy difference in first order perturbation we have from (42)

$$\Delta G \simeq -\,\frac{L^2}{2D}\left[\frac{\Delta\tau}{\tau_0} \mp \frac{2}{L}\,\Delta\phi\right]^2 \tag{63}$$

As discussed in Section VI, the condition for a LCP in this model is $D = 0$, which in cgs units corresponds to

$$T_{LCP} = T_c\left[1 - \frac{\pi\,\lambda^2(0)}{S\left[\ell n\,\dfrac{4L}{\sqrt{\pi S}} - \dfrac{7}{4}\right]}\right]$$

The lines separating regions of stability for different modes (different values of n), as obtained from Eq.(51), are given by

$$\phi = \phi_0$$

meaning that the lines are parallel to the T axis.

Now, we will consider second order perturbation results, including self-induction effects. For a network with only one node, as is the case for the ring, the system of equations (14) is homogeneous to all perturbation orders because the condition for existence of solution in the lowest order requires the determinant of **A** to be zero: $|\mathbf{A}| = \mathcal{A}(1,1) = 0$; thus, for the existence of finite solutions in higher order kl, we must have $\pi_{kl}(1) \equiv 0$. This result can be confirmed in second order perturbation by explicit calculation of Eq.(15) using the expressions for $Z_{kl}(a,b,s)$ given in Appendix II. The second order perturbation coefficients for the bare ring can be evaluated using the formulas in Appendix III; we have

$$\begin{aligned}
R_1 &= \frac{L}{2}\,(L \cot L - 1) \quad , \quad & R_2 &= \mp\,(L \cot L - 1) \\
R_3 &= 2 \cot L \quad , \quad & R_4 &= -\,\frac{L}{2}\,(L \cot L - 1) \\
R_5 &= \Lambda\,(L \cot L - 1) \quad , \quad & R_6 &= \pm\,(L \cot L - 1) \\
R_7 &= \mp\, 2\,\Lambda \cot L \quad , \quad & R_8 &= \frac{L}{2}\,(L \cot L - 1) \\
R_9 &= -\,\Lambda\,(L \cot L - 1) \quad , \quad & R_{10} &= 2\,\Lambda^2 \cot L
\end{aligned} \tag{64}$$

The normalization relations (55) are

$$\sigma_0\left[\alpha_{10}\alpha^*_{30} + \alpha^*_{10}\alpha_{30}\right] = -\,\frac{(3L^3 - 6L^2D + 7\,LD^2)}{4D^3}$$

$$\sigma_0\left[\alpha_{01}\alpha^*_{30} + \alpha^*_{10}\alpha_{21}\right] + \sigma_1\left[\alpha_{10}\alpha^*_{21} + \alpha^*_{01}\alpha_{30}\right] = 0$$

$$\sigma_0\left[\alpha_{10}\alpha^*_{12} + \alpha^*_{10}\alpha_{12}\right] + \sigma_1\left[\alpha_{01}\alpha^*_{21} + \alpha^*_{01}\alpha_{21}\right] = \pm \frac{(3L^2 - 4LD + 3D^2)}{D^3}$$

$$\sigma_1\left[\alpha_{10}\alpha^*_{03} + \alpha^*_{01}\alpha_{12}\right] + \sigma_0\left[\alpha_{01}\alpha^*_{12} + \alpha^*_{10}\alpha_{03}\right] = 0 \tag{65}$$

$$\sigma_1\left[\alpha_{01}\alpha^*_{03} + \alpha^*_{01}\alpha_{03}\right] = - \frac{(3L - 2D)}{D^3}$$

The modulus of the order parameter is uniform around the ring; using (52), the formulas of Appendix II and the relations (65) we have for $|f|^2$ <u>up to second order perturbation</u> the result

$$|f|^2 \simeq \frac{L}{D}\frac{\Delta\tau}{\tau_o} \mp \frac{2}{D}\Delta\phi - \frac{(3L^3 - 6L^2D + 7\,LD^2)}{4D^3}\left[\frac{\Delta\tau}{\tau_0}\right]^2 \pm$$

$$\pm \frac{(3L^2 - 4LD + 3D^2)}{D^3}\frac{\Delta\tau}{\tau_o}\Delta\phi - \frac{(3L - 2D)}{D^3}(\Delta\phi)^2 \tag{66}$$

The current density in the wire can be found from the formulas (A4) of Appendix II; there, the second terms on the right hand side are

$$\frac{B_1}{D}\gamma_{20}(a,L)\left[\frac{ab^* + ba^*}{2\sin L}\right] + \mathrm{Im}\left[\frac{a\,\sigma_0\alpha_{10}Z^*_{30}(b,a,L)}{\sin L}\right] = \mp\frac{(L^2 - LD)}{2\,D^2}$$

$$\left[\frac{B_1}{D}\gamma_{02}(a,L) + \frac{B_2}{D}\gamma_{20}(a,L)\right]\left[\frac{ab^* + ba^*}{2\sin L}\right] +$$

$$+ \mathrm{Im}\left\{\frac{a\left[\sigma_0\alpha_{10}Z^*_{12}(b,a,L) + \sigma_1\alpha_{01}Z^*_{21}(b,a,L)\right]}{\sin L}\right\} = \frac{2\,L - D}{D^2}$$

$$\frac{B_2}{D}\,\gamma_{02}(a,L)\left[\frac{ab^* + ba^*}{2\sin L}\right] + \mathrm{Im}\left[\frac{a\,\sigma_1\alpha_{10}Z^*_{03}(b,a,L)}{\sin L}\right] = \mp\,\frac{2}{D^2}$$

and the third terms are zero because $X_{kl}(1) = 0$; using (53) we have for j <u>up to second order perturbation</u>

$$j \simeq \mp\,\frac{L}{D}\frac{\Delta\tau}{\tau_o} + \frac{2}{D}\,\Delta\phi \pm \frac{(3L^3 - 4L^2D + 7\,LD^2)}{4D^3}\left(\frac{\Delta\tau}{\tau_o}\right)^2 - \tag{67}$$
$$- \frac{(3L^2 - 2LD + 3D^2)}{D^3}\,\frac{\Delta\tau}{\tau_o}\,\Delta\phi \pm \frac{3L}{D^3}\,(\Delta\phi)^2$$

In first order perturbation, the coherence length ξ and the phase of the order parameter θ are, according to (57) and (61)

$$\xi \simeq 1 - \frac{1}{2}\frac{\Delta\tau}{\tau_o} \quad , \quad \theta(s) \simeq \theta_o(s) = -\,\frac{2\,\pi\,n}{L}\,s\ ; \tag{68}$$

for the circulation of the total magnetic vector potential we have

$$\gamma(a,s) \simeq \gamma_o(a,s)+\gamma_{20}(a,s)\frac{\Delta\tau}{\tau_o}+\gamma_{02}(a,s)\Delta\phi = \left[\frac{\phi_o}{L}\mp\frac{(L-D)}{2D}\,\frac{\Delta\tau}{\tau_o}+\frac{\Delta\phi}{D}\right]s \tag{69}$$

Remembering that the supercurrent velocity field is given by

$$q = \xi\left[\frac{d\theta}{ds} + \frac{d\gamma}{ds}\right]$$

and using Eqs. (68), (69) and (60) we have in first order perturbation

$$q \simeq \pm 1 \mp \frac{L}{2D}\frac{\Delta\tau}{\tau_0} + \frac{\Delta\phi}{D} \tag{70}$$

Using Eqs. (66), (67) and (70) we can verify that, neglecting powers higher than $(\Delta\tau,\ \Delta\phi)^2$, the relation $j = -q|f|^2$ holds[11]. Having calculated j, q and $|f|^2$ independently, this relation proves the consistency of the perturbation expansion. Here we must stress that at the phase boundary j is zero but q remains finite, as shown by Eqs. (67) and (70).

For the free energy, using the formulas (A6) of Appendix III and the coefficients R given in (64), we have <u>up to second order perturbation</u>

$$\Delta G \simeq -\frac{L}{2}\left[\frac{L}{D}\left(\frac{\Delta\tau}{\tau_0}\right)^2 \mp \frac{4}{D}\frac{\Delta\tau}{\tau_0}\Delta\phi + \frac{4}{LD}(\Delta\phi)^2 - \right.$$
$$- \frac{(L^3 - 2L^2D + 5LD^2)}{2D^3}\left(\frac{\Delta\tau}{\tau_0}\right)^3 \pm \frac{(3L^3 - 4L^2D + 9LD^2)}{LD^3}\left(\frac{\Delta\tau}{\tau_0}\right)^2\Delta\phi -$$
$$\left. - \frac{2(3L^3 - 2L^2D + 4LD^2)}{L^2D^3}\frac{\Delta\tau}{\tau_0}(\Delta\phi)^2 \pm \frac{4}{D^3}(\Delta\phi)^3\right] \tag{71}$$

Using the expressions (66), (67) and (71) it is possible to verify that, neglecting powers higher than $(\Delta\tau,\ \Delta\phi)^3$, the relation[11]

$$\Delta G = -\frac{1}{2S}\int_V |\Delta|^4 dV + \frac{S}{\lambda^2}\frac{\Lambda\, j^2}{2} = -\frac{L}{2}\left[|f|^4 - \frac{(L - D)}{L\xi^2}\, j^2\right] \tag{72}$$

holds, which links the equilibrium free energy to the order parameter and current density. As above, this relation verifies the consistency of the perturbation expansion.

In Fig.7 we have compared our results given by the perturbative equations (62), (63), (66), (67) and (71) with the results obtained from the theory of Fink and Grünfeld[9] using Eqs. (56), (58), (59) and the exact relation (72).

b) Superconducting ying-yang

The superconducting ying-yang is a particular case of the Wheatstone bridge discussed in the literature[1]. In the linearized theory it is possible to analyze its solution from the point of view of group theory. The balanced Wheatstone bridge is a planar two loop network with point symmetry group D_{2h} (see Fig.8(a)) in absence of external magnetic field; when the Wheatstone bridge is immersed in a magnetic field we must take into account that $\vec{H}$ is a pseudovector and the system remains invariant only under those point operations that leave $\vec{H}$ invariant[12] and the new point symmetry group is C_{2h}. In fact, the only acceptable point groups for regular superconducting networks are of the form C_s, C_n, C_{nh} and S_n in the Schönflies notation[12]. Recent theoretical work on the second order phase transition of regular infinite networks has invoked mirror symmetries in p4gm and c2mm space groups to explain cusps in the phase boundary[13], but the pseudovector character of $\vec{H}$ prevents such an explanation.

In the presence of a magnetic field the ying-yang and the balanced Wheatstone bridge have the same symmetry C_{2h} (see Fig.8(b)). The ying-yang adds the advantage of equal length branches to the equal area elementary loops already present. The phase boundary is determined by two eigenvalues of the linearized equations which cross one another as shown in Fig.9; each eigenvalue corresponds to different modes of the "zero" order normalized order parameter $\Delta_0(s)$. The possible modes of $\Delta_0(s)$ transform under symmetry operations as the irreducible representations A_g (symmetric) or B_u (antisymmetric) of the space point group C_{2h}. The characteristic equation for the phase boundary lines is

$$\cos L = \pm \left[\frac{1 + 2\cos\phi_0}{3}\right] \tag{73}$$

where L is the length of the branches (normalized by ξ_0) and ϕ_0 is the external magnetic flux linked by one elementary loop (the ying or the yang). The +(-) sign holds when the "winding number" n for the order parameter around the external ring is even (odd) as we show in the inserts of Fig.9. From now on, the upper sign corresponds to n even and the lower one to n odd.

The "zero" order perturbation mop is

$$f_0(1,s) = \begin{cases} \dfrac{\cos\left(\frac{L}{2} - s\right)}{\cos\frac{L}{2}} & \text{(n even)} \\ \dfrac{\sin\left(\frac{L}{2} - s\right)}{\sin\frac{L}{2}} & \text{(n odd)} \end{cases}$$

for the central branch,

$$f_0(1,s) = \frac{\sin(L-s) \pm e^{i\phi_0}\sin s}{\sin L}$$

for the right branch and

$$f_0(1,s) = \frac{\sin(L-s) \pm e^{-i\phi_0}\sin s}{\sin L}$$

for the left branch. The corresponding normalized order parameter is

$$\Delta_0(s) = e^{-i\gamma_0(1,s)} f_0(1,s)$$

where

$$\gamma_0(1,s) = \begin{cases} \dfrac{\phi_0}{4\pi}\left[\sin\left(\dfrac{2\pi s}{L}\right) + \dfrac{2\pi s}{L}\right] & (0 \le s \le \frac{L}{2}) \\ \dfrac{\phi_0}{2} - \dfrac{\phi_0}{4\pi}\left[\sin\left(\dfrac{2\pi s}{L}\right) + \dfrac{2\pi s}{L}\right] & (\frac{L}{2} \le s \le L) \end{cases}$$

for the central branch and

$$\gamma_0(1,s) = \frac{\phi_0 s}{L} \quad , \quad \gamma_0(1,s) = -\frac{\phi_0 s}{L}$$

for the right and left branches respectively. It is interesting to point out that $\left[\Delta_0(s)\right]_{left} = (-1)^n\left[\Delta_0(L-s)\right]_{right}$ as we expect from the C_{2h} symmetry. The "zero" order current densities at node 1 are, for the central branch $j_0 \equiv 0$, and for the right and left branches

$$j_0 = \mp \frac{\sin \phi_0}{\sin L} \quad , \quad j_0 = \pm \frac{\sin \phi_0}{\sin L}$$

The first order perturbation coefficients (32) can be calculated analytically; the results are

$$B_1 = 3L \quad , \quad B_2 = \mp 4 \frac{\sin \phi_0}{\sin L} ,$$

$$B_3^2 = \frac{9}{2} L - \frac{3}{4} \frac{(1 \mp \cos L)^2}{\sin^4 L}\left[(L-\sin L \cos L) \pm 4(L \pm \sin L) \cos L\right] , \qquad (74)$$

$$B_4^2 = \frac{\sin^2\phi_0}{\sin^2 L} \Lambda = \frac{2}{\pi} \frac{\sin^2\phi_0}{\sin^2 L} L \left(\ln \frac{8L}{\sqrt{\pi S}} - \frac{7}{4}\right) ;$$

here Λ is the self-induction coefficient for the external loop of the ying-yang, where the current circulates. The Ehrenfest-Keesom equation (36) reads in this case

$$\frac{dL}{d\phi_0} = \pm \frac{2}{3} \frac{\sin \phi_0}{\sin L}$$

and, when integrated, gives the characteristic equation (73). The LCP can be found using the condition (48) with the coefficients (74).

Let us consider now the second order perturbation results. The symmetry properties of the ying-yang imply that the equations (14) are homogeneous to all perturbation orders, i.e. $\Pi_{kl}(1)=\Pi_{kl}(2)=0$; obviously, from (22) we have $X_{kl}(1) = X_{kl}(2) = 0$. To calculate the second order perturbation coefficients we have deviced a numerical routine that calculates the integrals of Appendix III.We shall show the type of results obtained when we apply the perturbation scheme to a ying-yang made of an extreme Type II material, a case whose exact solution can be obtained from numerical calculations[13] since self-induction effects are negligible.

The coefficients $|\Delta|^2_{kl}$ in Eq. (52) for the nodes of the ying-yang, the coefficients j_{kl} in Eq.(53) for the external branches and the coefficients G_{kl} in Eq. (54) for the whole ying-yang, have been calculated for two values of ϕ_0 ($\phi_0 = \frac{2\pi}{3}$ and $\phi_0 = \pi$) and for two modes of the normalized order parameter (n = 0 and n = 1). The results can be summarized in the following table:

Table I

Starting point (ϕ_0, L)	$\left(\frac{2\pi}{3}, \frac{\pi}{2}\right)$		$(\pi, 1.911)$	$(\pi, 1.231)$
Mode n	0	1	0	1
Character at starting point	Degenerate		Unstable	Stable
$\|\Delta\|^2_{40}$	- 0.817	- 0.746	- 0.507	- 0.766
$\|\Delta\|^2_{22}$	0.606	- 0.515	0	0
$\|\Delta\|^2_{04}$	- 0.013	- 0.256	0.188	- 0.457
j_{40}	0.529	- 0.686	0	0
j_{22}	- 0.118	- 1.159	0.540	- 0.844
j_{04}	- 0.193	- 0.724	0	0
G_{60}	3.683	3.756	2.887	2.933
G_{42}	- 4.888	5.513	0	0
G_{24}	1.254	3.337	- 1.088	1.688
G_{06}	0.257	0.966	0	0

These results and those corresponding to other values of ϕ_0 have been checked by comparison with exact calculations made by Fink[14], as can be seen in Figs. 10 and 11. An interesting feature of the behaviour of j as a function of $\Delta\phi$ at constant temperature ($\Delta\tau = 0$) for n = 0 is that, at $\phi_0 \simeq 1.9$, $j_{04} = 0$. Thus $j = j(\Delta\phi)$ has no quadratic term $(\Delta\phi)^2$; this means that at this point in the phase boundary j grows into the superconductive region without initial curvature. For ϕ_0 values higher (lower) than 1.9 the initial curvature produces a higher (lower) current density.

The point $\phi_0 = \pi$ at the phase boundary corresponds to one fluxon linked by the external ring and the lowest free energy is associated with the mode n = 1. The results shown in Table I are consistent with the expected physical behaviour of the network: the supercurrent is zero on the line $\phi = \phi_0 = \pi$ where the external magnetic field remains constant; the only way to have a current is to change the external field and the temperature at the same time. It should be noted that $|f|^2$ depends quadratically on the external flux change $\Delta\phi$; at constant temperature ($\Delta\tau = 0$) we have $|f|^2 < 0$ whatever the sign of $\Delta\phi$: the network goes into the normal state.

Finally, using Eq.(54) for the free energy and the coefficients given in Table I, we have studied the stability line separating both modes in the superconductive region. This line begins at the point $\phi_0 = \frac{2\pi}{3}$ on the phase boundary and goes initially parallel to the temperature axis; then it bends into values of ϕ_0 greater than $\frac{2\pi}{3}$, as we have indicated in Fig.9.

IX - Conclusions and perspectives

In this paper we have shown how a consistent perturbation scheme can be deviced to treat the superconductive phase of micronetworks. The procedure can be extended to all orders in ΔT and ΔH and applied to any type of network.

The complications of the algebra suggest however that beyond second order some other procedure should be applied if additional physical properties of the networks are sought.

In this sense we feel that a variational approach of the kind developed by Wang et al.[5], if self induction effects are properly included, could be more adequate to describe the vortex structure. In this particular aspect it would also be desirable to have experimental results obtained through decoration techniques to compare with.

The subject of this paper is relevant to the theory of high T_c superconductors considered as granular systems, in which there is an intrinsic network structure, with superconductive islands interconnected via weak links. This would bring the theory back to the original ideas of de Gennes and Alexander, who proposed their theory as a model for heterogeneous superconductors.

A possible theory[15] for granular high T_c superconductors interprets the phase diagram of these materials using averaged GL theories with renormalized coherence length and penetration depth. This approach is succesful in describing the static properties of the magnetic flux within the material. Recent experimental results[16] however show that such an approach cannot account, neither qualitatively nor quantitatively, for the dissipative effects caused by flux motion induced by a transport current. It has been suggested[17] that both the depinning and the motion of the vortices are related to the topological characteristics of the intrinsic network, which are not included in an averaged theory.

For this reason a possible extension of the work of this paper is towards the inclusion of transport currents and a study of the induced vortex movement. Work in that direction is in progress.

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Appendix I: The Γ matrix

To second order the matrix $\Gamma_{pq}(a,s)$ which appears in Section II, Eq. (6), is given by

$$
\begin{array}{c|cccccc}
\Gamma_{pq} & q \longrightarrow & & & & & \\
 & 0 & 1 & 2 & 3 & 4 & \\
\hline
p\ \downarrow\ 0 & 1 & 0 & i\gamma_{02} & 0 & i\gamma_{04} - \dfrac{\gamma_{02}^2}{2} & \ldots \\
1 & 0 & i\gamma_{11} & 0 & i\gamma_{13} - \gamma_{11}\gamma_{02} & \ldots & \\
2 & i\gamma_{20} & 0 & i\gamma_{22} - \gamma_{20}\gamma_{02} - \dfrac{\gamma_{11}^2}{2} & \ldots & & \\
3 & 0 & i\gamma_{31} - \gamma_{20}\gamma_{11} & \ldots & & & \\
4 & i\gamma_{40} - \dfrac{\gamma_{20}^2}{2} & \ldots & & & & \\
 & \ldots & & & & &
\end{array}
$$

It is possible to reduce this matrix further by taking into account that the normalization conditions (16) imply $\gamma_{kl} \equiv 0$ for k (or l) odd.

Appendix II: Second order perturbation formalism

In this Appendix we will use the notation $f_0(a,0) \equiv a$, $f_0(a,L) \equiv b$ and $\dot{f}_0(a,0) = \dot{a}$ in order to make the formulation short and clear.

The general explicit form of $f_{kl}(a,s)$ for a branch is

$$f_{kl}(a,s) = \alpha_{kl} f_0(a,s) - \Big[Z_{kl}(a,b,s) + a\, X_{kl}(a)\Big]\frac{\sin\,(L-s)}{\sin L} - \Big\{Z_{kl}(b,a,L-s) + b\,\Big[X_{kl}(b) - Y_{kl}(a,b,L)\Big]\Big\}\,\frac{\sin s}{\sin L} \tag{A1}$$

In second order perturbation (kl = 30, 21, 12, 03) we have

$$Y_{30}(a,b,L) = i\;\alpha_{10}\gamma_{20}(a,L) \quad , \quad Y_{21}(a,b,L) = i\;\alpha_{01}\gamma_{20}(a,L)$$

$$Y_{12}(a,b,L) = i\;\alpha_{10}\gamma_{02}(a,L) \quad , \quad Y_{03}(a,b,L) = i\;\alpha_{01}\gamma_{02}(a,L)$$

and

$$Z_{kl}(a,b,s) = \frac{\sin L}{(ab^* - ba^*)}\left[U_{kl}\left(a\int_0^s |f_0|^4 ds' - a^*\int_0^s |f_0|^2 f_0^2 ds'\right) - V_{kl}\left(a\int_0^s |f_0|^2 ds' - a^*\int_0^s f_0^2 ds'\right)\right] \tag{A2}$$

where

$$U_{30} = \frac{B_1}{D}\;\alpha_{10}\;,\quad U_{21} = \frac{B_1}{D}\;\alpha_{01}\;,\quad U_{12} = \frac{B_2}{D}\;\alpha_{10}\;,\quad U_{03} = \frac{B_2}{D}\;\alpha_{01}$$

$$V_{30} = \alpha_{10} \quad , \quad V_{21} = \alpha_{01} \quad , \quad V_{12} = V_{03} = 0$$

The coefficients $|\Delta|^2_{kl}$ in expression (52) for $|\Delta|^2 = |f|^2$ are obtained using the expansion Eq.(3) and the second order normalization relations (55):

$$|\Delta|^2_{40} = \sigma_0\left[f_{10}\, f^*_{30} + f^*_{10}\, f_{30}\right]$$

$$|\Delta|^2_{22} = \sigma_0\left[f_{10}\, f^*_{12} + f^*_{10}\, f_{12}\right] + \sigma_1\left[f_{01}\, f^*_{21} + f^*_{01}\, f_{21}\right]$$

$$|\Delta|^2_{04} = \sigma_1\left[f_{01}\, f^*_{03} + f^*_{01}\, f_{03}\right] \tag{A3}$$

$$|\Delta|^2_{31} = |\Delta|^2_{13} = 0$$

For the current density coefficients j_{kl} in (53), using Eq.(24) and the previously obtained expressions (A1) for $f_{kl}(a,s)$ we obtain

$$j_{40} = \left[\sigma_0\left(\alpha_{10}\alpha^*_{30} + \alpha^*_{10}\alpha_{30}\right) - \frac{B_1}{2D}\right] j_0 - \left\{\frac{B_1}{D}\,\gamma_{20}(a,L)\left(\frac{ab^* + ba^*}{2\sin L}\right) + \right.$$

$$+ \operatorname{Im}\left[\frac{a\sigma_0\alpha_{10}Z^*_{30}(b,a,L)}{\sin L}\right]\Bigg\} +$$

$$+ \operatorname{Im}\left\{\sigma_0\alpha_{10}\left[a^* X^*_{30}(a)(\dot{a} + a\cot L) - \frac{ab^* X^*_{30}(b)}{\sin L}\right]\right\}$$

$$j_{22} = \left\{\left[\sigma_0\left(\alpha_{10}\alpha^*_{12} + \alpha^*_{10}\alpha_{12}\right) + \sigma_1\left(\alpha_{01}\alpha^*_{21} + \alpha^*_{01}\alpha_{21}\right)\right] - \frac{B_2}{2D}\right\} j_0 -$$

$$-\left\{\left[\frac{B_1}{D}\gamma_{02}(a,L)+\frac{B_2}{D}\gamma_{20}(a,L)\right]\left(\frac{ab^*+ba^*}{2\sin L}\right)+\right. \tag{A4}$$

$$+\ \mathrm{Im}\left\{\frac{a\left[\sigma_0\alpha_{10}Z^*_{12}(b,a,L)+\sigma_1\alpha_{01}Z^*_{21}(b,a,L)\right]}{\sin L}\right\}\right\}+$$

$$+\ \mathrm{Im}\left\{\sigma_0\alpha_{10}\left[a^*X^*_{12}(a)(\dot{a}+a\cot L)-\frac{ab^*X^*_{12}(b)}{\sin L}\right]+\right.$$

$$\left.+\ \sigma_1\alpha_{01}\left[a^*X^*_{21}(a)(\dot{a}+a\cot L)-\frac{ab^*X^*_{21}(b)}{\sin L}\right]\right\}$$

$$j_{04}=\left[\sigma_1\left(\alpha_{01}\alpha^*_{03}+\alpha^*_{01}\alpha_{03}\right]j_0-\left\{\frac{B_2}{D}\gamma_{02}(a,L)\left(\frac{ab^*+ba^*}{2\sin L}\right)+\right.$$

$$\left.+\ \mathrm{Im}\left[\frac{a\sigma_1\alpha_{01}Z^*_{03}(b,a,L)}{\sin L}\right]\right\}+$$

$$+\ \mathrm{Im}\left\{\sigma_1\alpha_{01}\left[a^*X^*_{03}(a)(\dot{a}+a\cot L)-\frac{ab^*X^*_{03}(b)}{\sin L}\right]\right\}$$

$$j_{31}=j_{13}=0$$

where $Z_{kl}(b,a,L)$ is given by (A2) with the adequate values of the variables. All the quantities in this Appendix can be calculated from the "zero" and the first order perturbation results, with the exception of the linear combinations for α_{30}, α_{21}, α_{12} and α_{03} which we can know from Appendix III.

Appendix III: Second order perturbation coefficients

In this Appendix we will use the same notation as in Appendix II. The second order normalization relations (55) can be explicitly written in the form

$$\sigma_0(\alpha_{10}\alpha^*_{30} + \alpha^*_{10}\alpha_{30}) = -\frac{C_1}{D} - \frac{B_1C_4}{D^2} - \frac{B_1^2C_6}{D^3}$$

$$\sigma_0(\alpha_{01}\alpha^*_{30} + \alpha^*_{10}\alpha_{21}) + \sigma_1(\alpha_{10}\alpha^*_{21} + \alpha^*_{01}\alpha_{30}) = 0$$

$$\sigma_0(\alpha_{10}\alpha^*_{12} + \alpha^*_{10}\alpha_{12}) + \sigma_1(\alpha_{01}\alpha^*_{21} + \alpha^*_{01}\alpha_{21}) = \tag{A5}$$

$$= -\frac{C_2}{D} - \frac{(B_1C_5+B_2C_4)}{D^2} - \frac{2B_1B_2C_6}{D^3}$$

$$\sigma_1(\alpha_{10}\alpha^*_{03} + \alpha^*_{01}\alpha_{12}) + \sigma_0(\alpha_{01}\alpha^*_{12} + \alpha^*_{10}\alpha_{03}) = 0$$

$$\sigma_1(\alpha_{01}\alpha^*_{03} + \alpha^*_{01}\alpha_{03}) = -\frac{C_3}{D} - \frac{B_2C_5}{D^2} - \frac{B_2^2C_6}{D^3}$$

where

$$C_1 = B_1 + \frac{R_1}{2} \quad , \quad C_2 = B_2 + R_2 \quad , \quad C_3 = \frac{R_3}{2} \quad ,$$

$$C_4 = 2R_4 + \frac{S}{\kappa^2} R_5 \quad , \quad C_5 = 2R_6 + \frac{S}{\kappa^2} R_7 \quad ,$$

$$C_6 = \frac{3}{2} R_8 + \frac{3}{2} \frac{S}{\kappa^2} R_9 + \frac{3}{8} \left(\frac{S}{\kappa^2}\right)^2 R_{10}$$

In turn

$$R_1 = (N_1 + N_1)^* + (N_2 + N_2^*) - N_3 - N_4$$

$$R_2 = i\left[N_5 - (N_6 - N_6^*)\right]$$

$$R_3 = N_7$$

$$R_4 = N_8 + (N_8 + N_9^*) - (N_{10} + N_{10}^*) - (N_{11} + N_{11}^*) + (N_{12} + N_{12}^*)$$

$$R_5 = i\left[N_{13} - (N_{14} - N_{14}^*)\right]$$

$$R_6 = -\, i\left[N_{15} - (N_{16} - N_{16}^*)\right]$$

$$R_7 = N_{17}$$

$$R_8 = (N_{18} + N_{18}^*) + (N_{19} + N_{19}^*) - N_{20} - N_{21}$$

$$R_9 = -\, i\left[N_{22} - (N_{23} - N_{23}^*)\right]$$

$$R_{10} = N_{24}$$

and

$$N_1 = \sum_\beta \left(\frac{\sin L_\beta}{ab^* - ba^*}\right) \int_0^{L_\beta} \int_0^{s} f_0^{*2}(a,s) f_0^2(a,s')\,ds'\,ds$$

$$N_2 = 2 \sum_\beta \frac{a^* b^* \sin L_\beta}{(ab^* - ba^*)^2} \left(\int_0^{L_\beta} f_0^2\,ds\right) \left(\int_0^{L_\beta} |f_0|^2\,ds\right)$$

$$N_3 = \sum_\beta \frac{(ab^* + ba^*) \sin L_\beta}{(ab^* - ba^*)^2} \left(\int_0^{L_\beta} |f_0|^2\,ds\right)^2$$

$$N_4 = \sum_\beta \frac{(ab^* + ba^*)\sin L_\beta}{(ab^* - ba^*)^2} \left| \int_0^{L_\beta} f_0^2 \, ds \right|^2$$

$$N_5 = \sum_\beta \left[\frac{ab^* + ba^*}{ab^* - ba^*}\right] \gamma_1(a, L_\beta) \int_0^{L_\beta} |f_0|^2 ds$$

$$N_6 = \sum_\beta \left[\frac{a^* b^*}{ab^* - ba^*}\right] \gamma_1(a, L_\beta) \int_0^{L_\beta} f_0^2 \, ds$$

$$N_7 = \sum_\beta \left[\frac{ab^* + ba^*}{\sin L_\beta}\right] \gamma_1^2(a, L_\beta)$$

$$N_8 = \sum_\beta \frac{(ab^* + ba^*)\sin L_\beta}{(ab^* - ba^*)^2} \left[\int_0^{L_\beta} |f_0|^2 ds\right] \left[\int_0^{L_\beta} |f_0|^4 ds\right]$$

$$N_9 = \sum_\beta \frac{a^* b^* \sin L_\beta}{(ab^* - ba^*)^2} \left[\int_0^{L_\beta} |f_0|^2 ds\right] \left[\int_0^{L_\beta} |f_0|^2 f_0^2 \, ds\right]$$

$$N_{10} = \sum_\beta \frac{a^* b^* \sin L_\beta}{(ab^* - ba^*)^2} \left[\int_0^{L_\beta} f_0^2 ds\right] \left[\int_0^{L_\beta} |f_0|^4 ds\right]$$

$$N_{11} = \sum_\beta \left(\frac{\sin L_\beta}{ab^* - ba^*}\right) \int_0^{L_\beta} \int_0^{s} |f_0(a,s)|^2 f_0^{*2}(a,s) f_0^2(a,s') \, ds' \, ds$$

$$N_{12}=\sum_{\beta} \frac{ab^{*}\sin L_{\beta}}{(ab^{*}-ba^{*})^{2}} \left[\int_{0}^{L_{\beta}} f_{0}^{2}ds\right]\left[\int_{0}^{L_{\beta}} |f_{0}|^{2}f_{0}^{*2}ds\right]$$

$$N_{13} = \frac{1}{2\pi}\sum_{\beta\beta'} \left(\frac{ab^{*}+ba^{*}}{ab^{*}-ba^{*}}\right)\left[\int_{0}^{L_{\beta}} |f_{0}|^{2}ds\right] J_{0\beta'}K_{\beta\beta'}$$

$$N_{14} = \frac{1}{2\pi}\sum_{\beta\beta'} \left(\frac{a^{*}b^{*}}{ab^{*}-ba^{*}}\right)\left[\int_{0}^{L_{\beta}} f_{0}^{2}\, ds\right] J_{0\beta'}K_{\beta\beta'}$$

$$N_{15} = \sum_{\beta} \left(\frac{ab^{*}+ba^{*}}{ab^{*}-ba^{*}}\right) \gamma_{1}(a,L_{\beta})\int_{0}^{L_{\beta}} |f_{0}|^{4}ds$$

$$N_{16} = \sum_{\beta} \left(\frac{a^{*}b^{*}}{ab^{*}-ba^{*}}\right) \gamma_{1}(a,L_{\beta})\int_{0}^{L_{\beta}} |f_{0}|^{2}f_{0}^{2}\, ds$$

$$N_{17} = \frac{1}{2\pi}\sum_{\beta\beta'} \left(\frac{ab^{*}+ba^{*}}{\sin L_{\beta}}\right)\gamma_{1}(a,L_{\beta})J_{0\beta'}K_{\beta\beta'}$$

$$N_{18}=\sum_{\beta} \left(\frac{\sin L_{\beta}}{ab^{*}-ba^{*}}\right)\int_{0}^{L_{\beta}}\int_{0}^{s} |f_{0}(a,s)|^{2}f_{0}^{*2}(a,s)|f_{0}(a,s')|^{2}\ f_{0}^{2}(a,s')\ ds'ds$$

$$N_{19}= 2\sum_{\beta} \frac{a^{*}b^{*}\sin L_{\beta}}{(ab^{*}-ba^{*})^{2}}\left[\int_{0}^{L_{\beta}} |f_{0}|^{2}f_{0}^{2}ds\right]\left[\int_{0}^{L_{\beta}} |f_{0}|^{4}\ ds\right]$$

$$N_{20}=\sum_{\beta}\frac{(ab^{*}+ba^{*})\sin L_{\beta}}{(ab^{*}-ba^{*})^{2}}\left[\int_{0}^{L_{\beta}}|f_{o}|^{4}\,ds\right]^{2}$$

$$N_{21}=\sum_{\beta}\frac{(ab^{*}+ba^{*})\sin L_{\beta}}{(ab^{*}-ba^{*})^{2}}\left|\int_{0}^{L_{\beta}}|f_{o}|^{2}f_{o}^{2}\,ds\right|^{2}$$

$$N_{22}=\frac{1}{2\pi}\sum_{\beta\beta'}\left(\frac{ab^{*}+ba^{*}}{ab^{*}-ba^{*}}\right)\left[\int_{0}^{L_{\beta}}|f_{o}|^{4}ds\right]J_{o\beta'}K_{\beta\beta'}$$

$$N_{23}=\frac{1}{2\pi}\sum_{\beta\beta'}\left(\frac{a^{*}b^{*}}{ab^{*}-ba^{*}}\right)\left[\int_{0}^{L_{\beta}}|f_{o}|^{2}f_{o}^{2}\,ds\right]J_{o\beta'}K_{\beta\beta'}$$

$$N_{24}=\frac{1}{4\pi^{2}}\sum_{\beta\beta'\beta''}\left(\frac{ab^{*}+ba^{*}}{\sin L_{\beta}}\right)J_{o\beta'}\;J_{o\beta''}\;K_{\beta\beta'}\;K_{\beta\beta''}$$

The coefficients G_{kl} in the expression (54) can now be written as

$$G_{60}=\frac{B_{1}C_{1}}{D}+\frac{B_{1}^{2}C_{4}}{2D^{2}}+\frac{B_{1}^{3}C_{6}}{3D^{3}}$$

$$G_{42}=\frac{(B_{1}C_{2}+B_{2}C_{1})}{D}+\frac{(B_{1}^{2}C_{5}+2B_{1}B_{2}C_{4})}{2D^{2}}+\frac{B_{1}^{2}B_{2}C_{6}}{D^{3}}\qquad\text{(A6)}$$

$$G_{24}=\frac{(B_{2}C_{2}+B_{1}C_{3})}{D}+\frac{(B_{2}^{2}C_{4}+2B_{1}B_{2}C_{5})}{2D^{2}}+\frac{B_{2}^{2}B_{1}C_{6}}{D^{3}}$$

$$G_{06} = \frac{B_2C_3}{D} + \frac{B_2^2C_5}{2D^2} + \frac{B_2^3C_6}{3D^3}$$

$$G_{51} = G_{33} = G_{15} = 0$$

Figure captions

Fig. 1: Branch ab of length L and arc s used in the definition of the modified order parameter f(a,s) from the normalized order parameter $\Delta(s)$.

Fig. 2: Branches ab and a'b' of the network and geometry used in the calculation of the self and mutual induction effects.

Fig. 3: Portion of the phase diagram indicating normal (N) and superconductive (S) regions. A point like P in region S can be reached from a point A at the phase transition line through increments $\Delta\tau_A$ in the reduced temperature and $\Delta\phi_A$ in flux or from point B through increments $\Delta\tau_B$ and $\Delta\phi_B$.

Fig. 4: Instabilities of the order parameter and free energy near a Landau Critical Point. The LCP is given by the condition D = 0 (see text). $\Delta\tau(+)$ means a positive temperature change from the phase boundary.

Fig. 5: Geometry used to evaluate the LCP of a hollow cylinder, for which exact results are compared with perturbation results.

Fig. 6: Phase diagram in the variables $\ell = (L_{conv}/\xi(0))\sqrt{1 - T/Tc}$, $\phi/ 2\pi = \phi_{conv}/\Phi_0$ for a bare ring. The inserts show the behaviour of the normalized order parameter around the ring for the modes n = 0 and n = 1. The phase is represented by the angle between the arrow and the vertical axis.

Fig. 7: Comparison of exact and perturbation theory results for a bare ring. (a), (b) and (c) show the variations of normalized order parameter $|\Delta|^2$, current density j and free energy $\Delta G/L$ at constant temperature as a function of flux. The insert in (a) shows the path followed in the phase diagram. (d), (e) and (f) correspond to the same quantities at constant flux and as a function of the variable $\frac{\ell}{L} = \sqrt{\frac{T_c - T}{T_c - T_0}}$; the insert in (d) shows the path in the phase diagram. In these plots $n = 0$, $\kappa = 1/\sqrt{2}$, and $D = 0.9\pi$, $\phi_0 = \pi$ for (a), (b) and (c), and $D = 0.4875\ \pi$, $\phi_0 = \pi/2$ for (d), (e) and (f). In all cases the solid lines are the exact results of Ref.(9), the dashed lines are first order and the dotted lines are second order.

Fig. 8: Shown are (a) the balanced Wheatstone bridge and (b) the ying-yang.

Fig. 9: Phase diagram for the ying-yang. The inserts show the behaviour of the normalized order parameter along the branches for the modes $n = 0$, $n = 1$ and $n = 2$; the external ring and the central branch are shown separately for the sake of clarity. For $n = 1$ the central branch has a phase flip center with zero order parameter. Notice that for n even the twisting of the phase at the central point increases with n.

Fig. 10: Comparison of exact and perturbation theory results for the ying-yang at fixed temperature. (a) Plot of the

current density versus applied flux for the mode $n = 0$. (b) Same plot as above for $n = 1$. The solid lines are the exact results of Fink; the dashed lines and the dotted lines are first and second order perturbation results respectively. The inserts show the paths in the phase diagram. In these plots the starting point is at $\phi_0 = \frac{2\Pi}{3}$.

Fig. 11: Comparison of exact and perturbation theory results for the ying-yang at fixed magnetic field. (a) Plot of the current density versus $\frac{\ell}{L} = \sqrt{\frac{T_c - T}{T_c - T_0}}$ for the mode $n = 0$. (b) Same plot as above for $n = 1$. The solid lines are the exact results of Fink; the dashed lines and the dotted lines are first and second order perturbation results respectively. The inserts show the paths in the phase diagram. In these plots $\phi_0 = 1.79$ for (a) and $\phi_0 = 2.47$ for (b).

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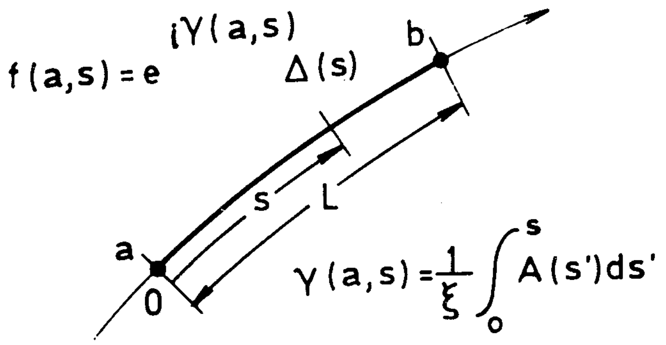

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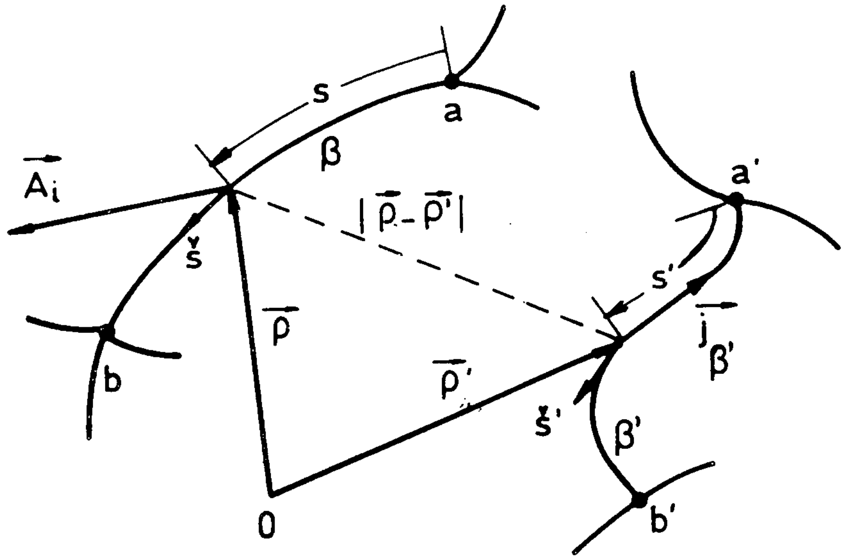

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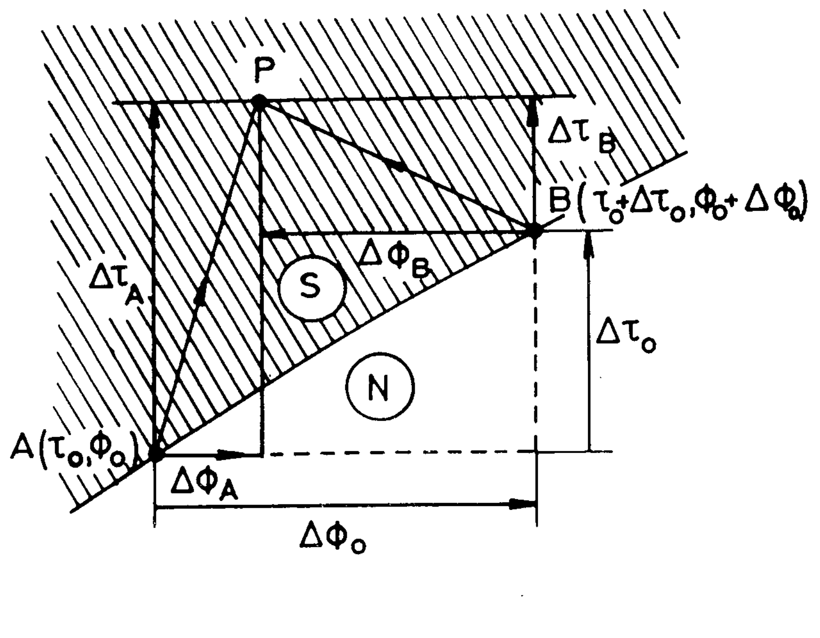

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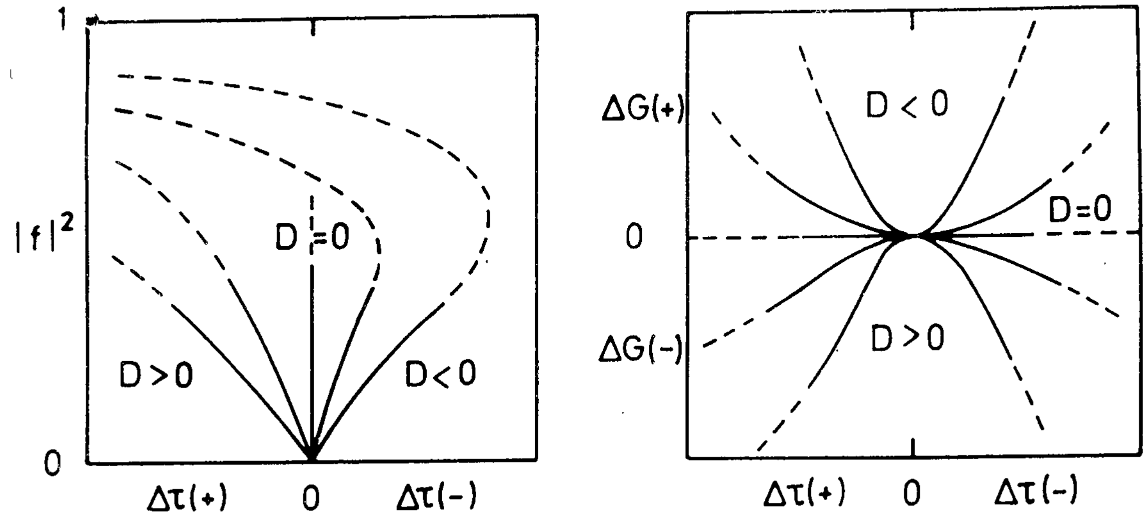

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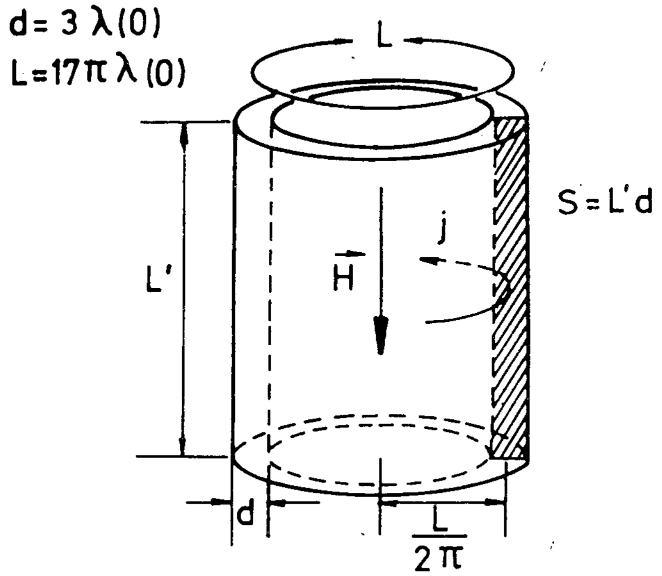

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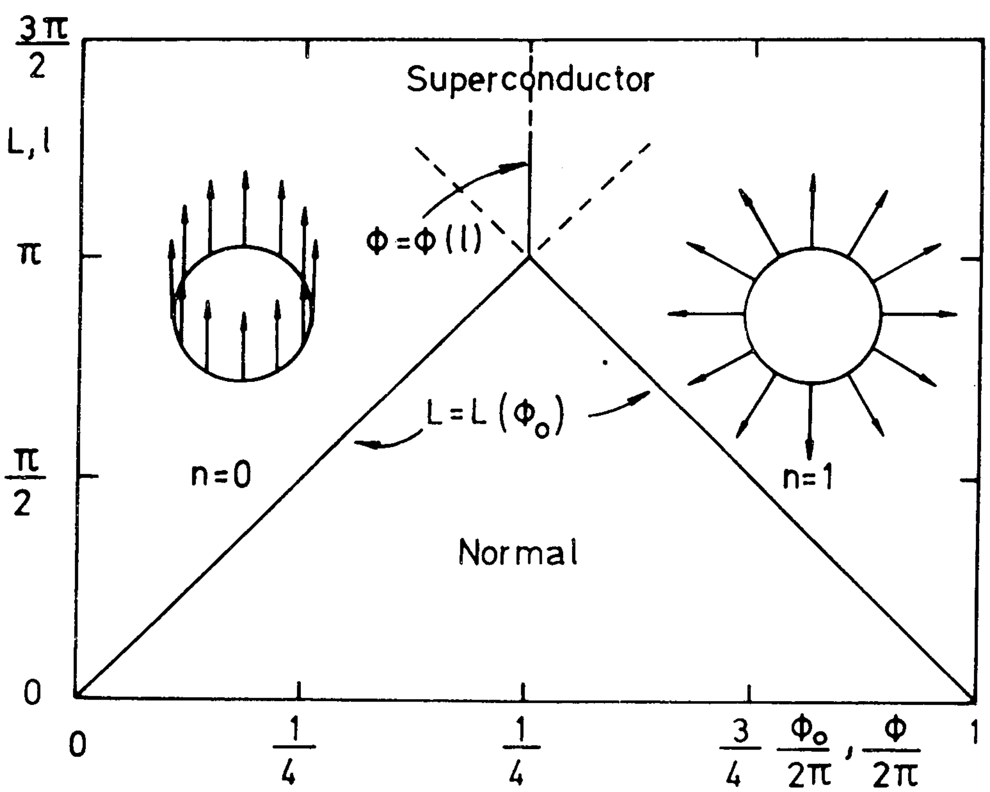

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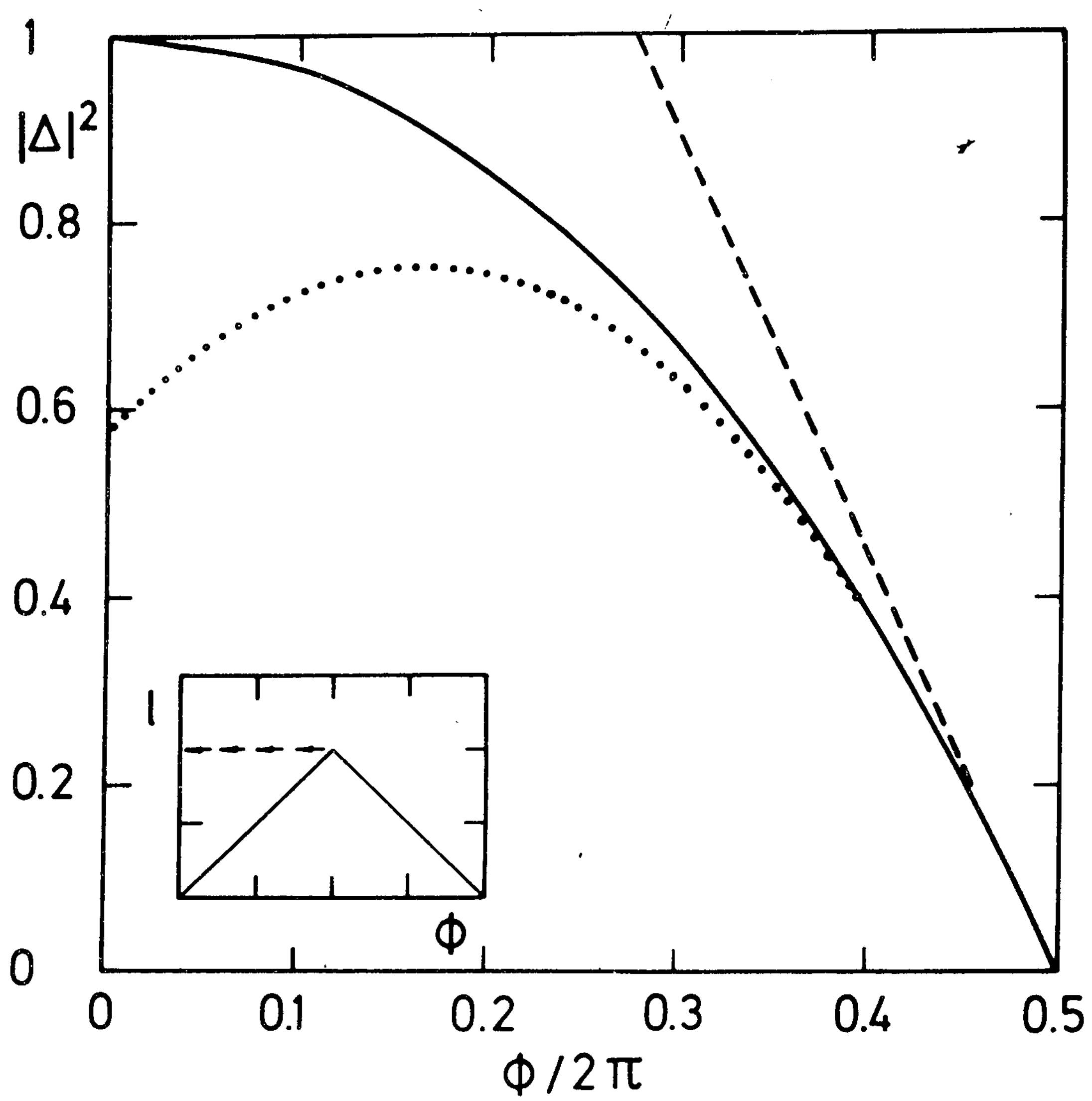

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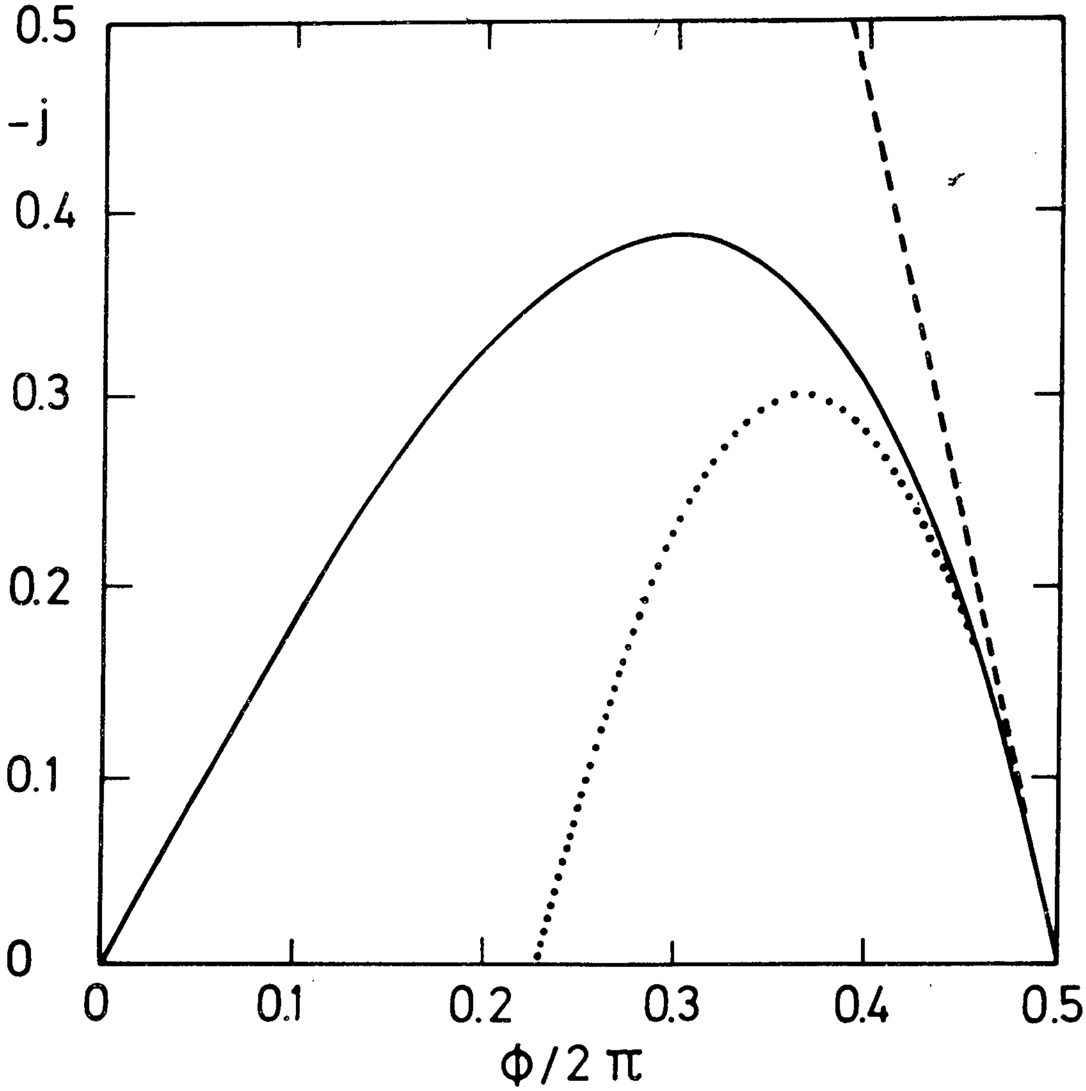

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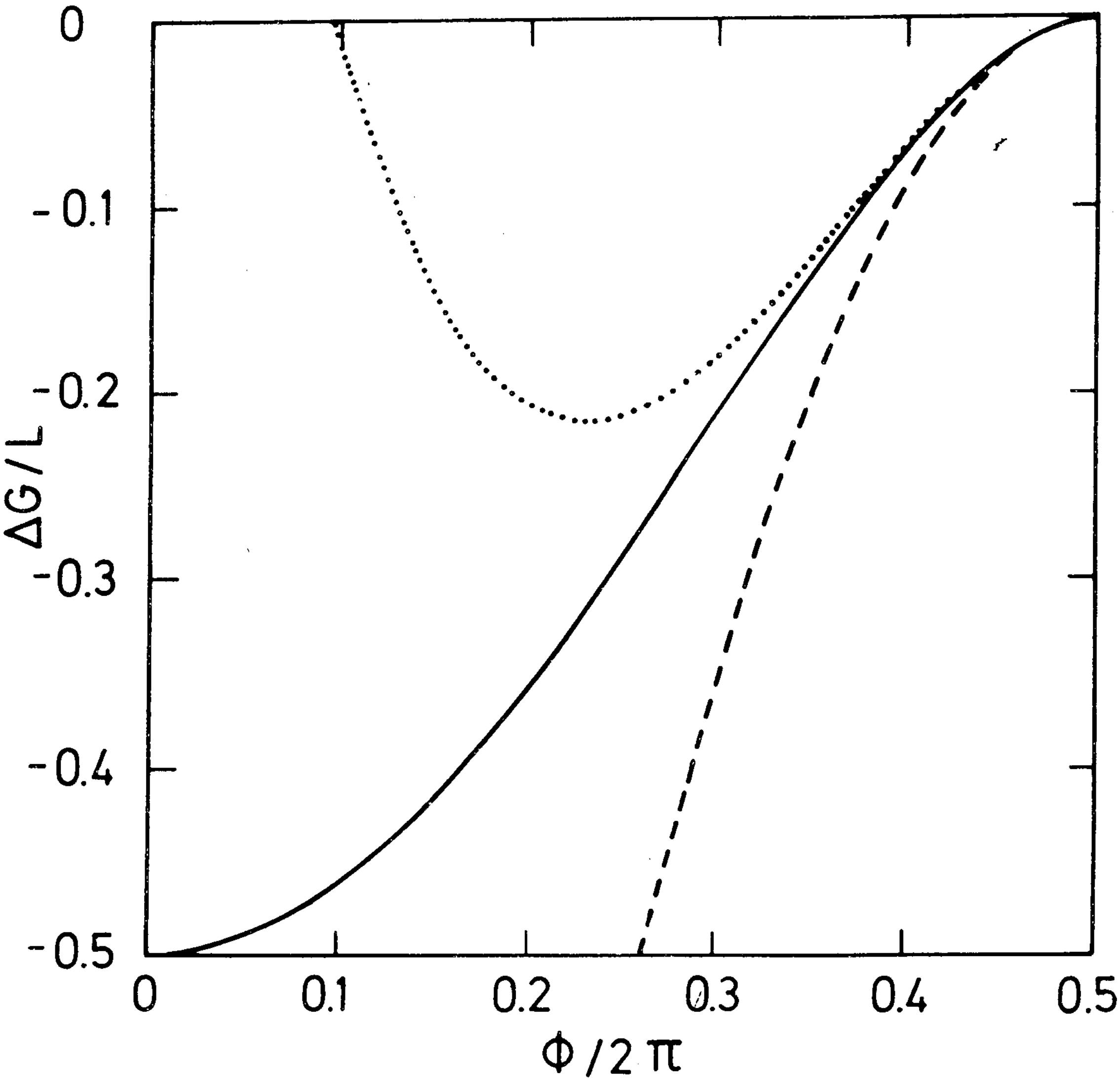

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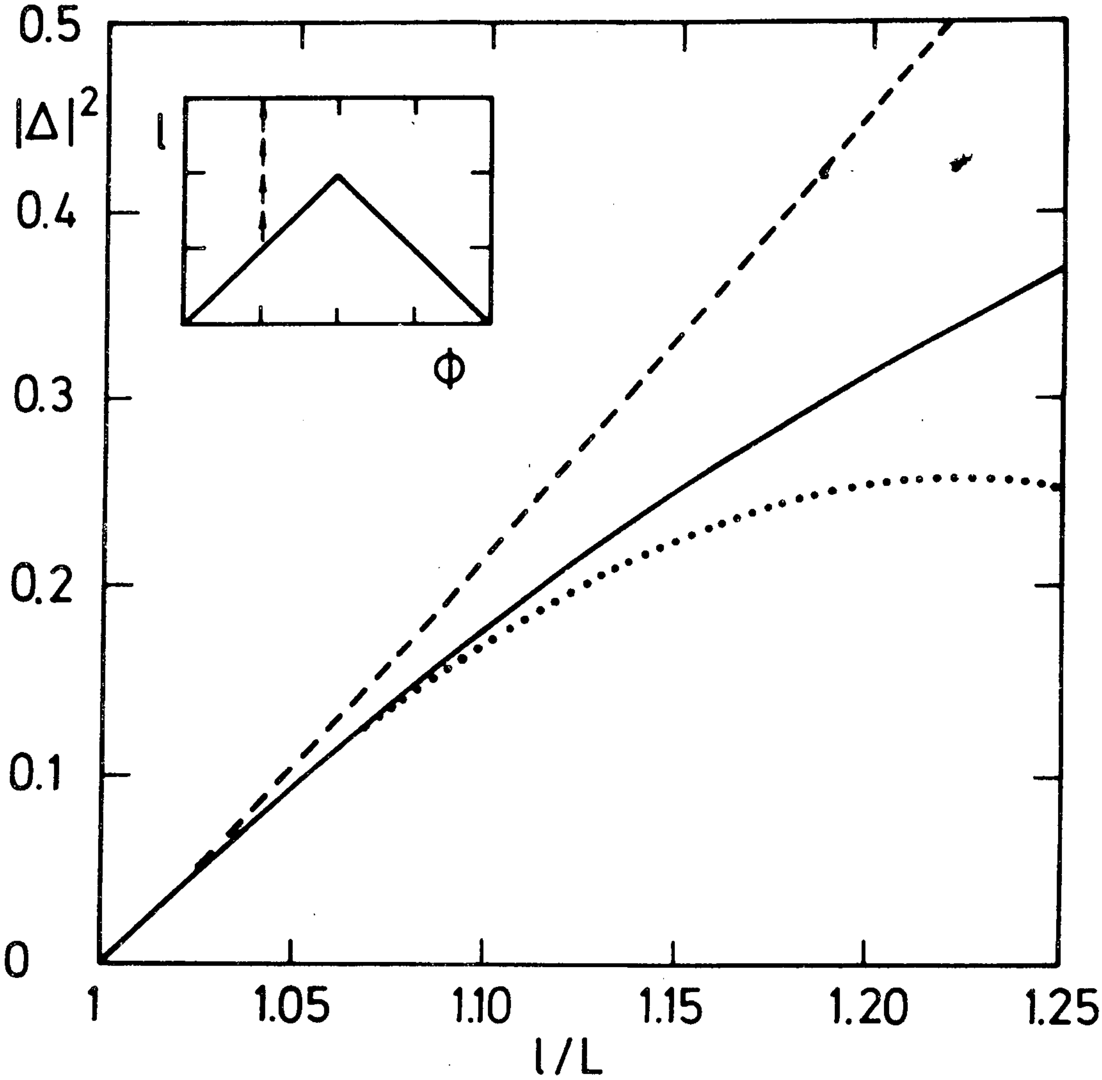

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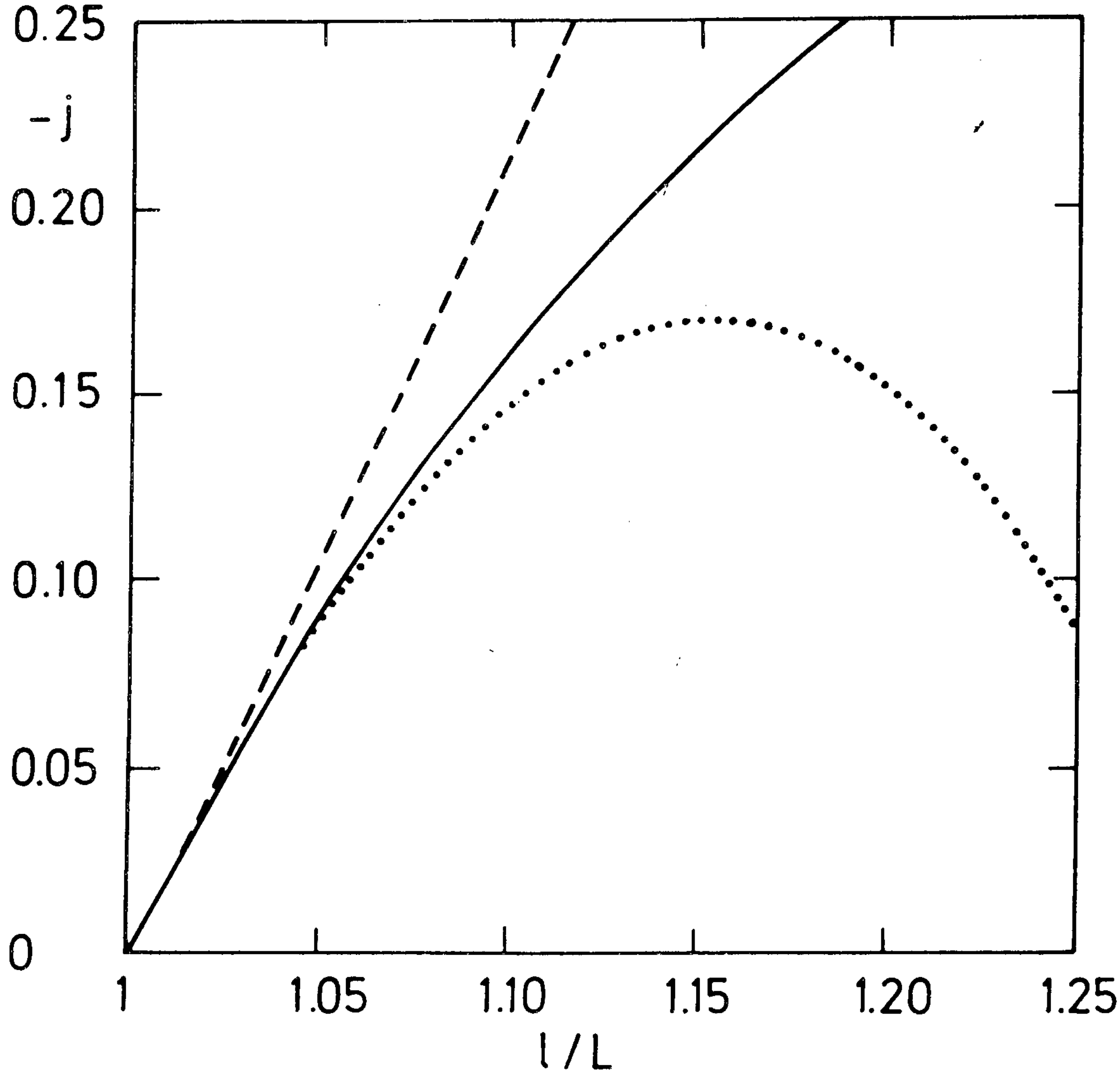

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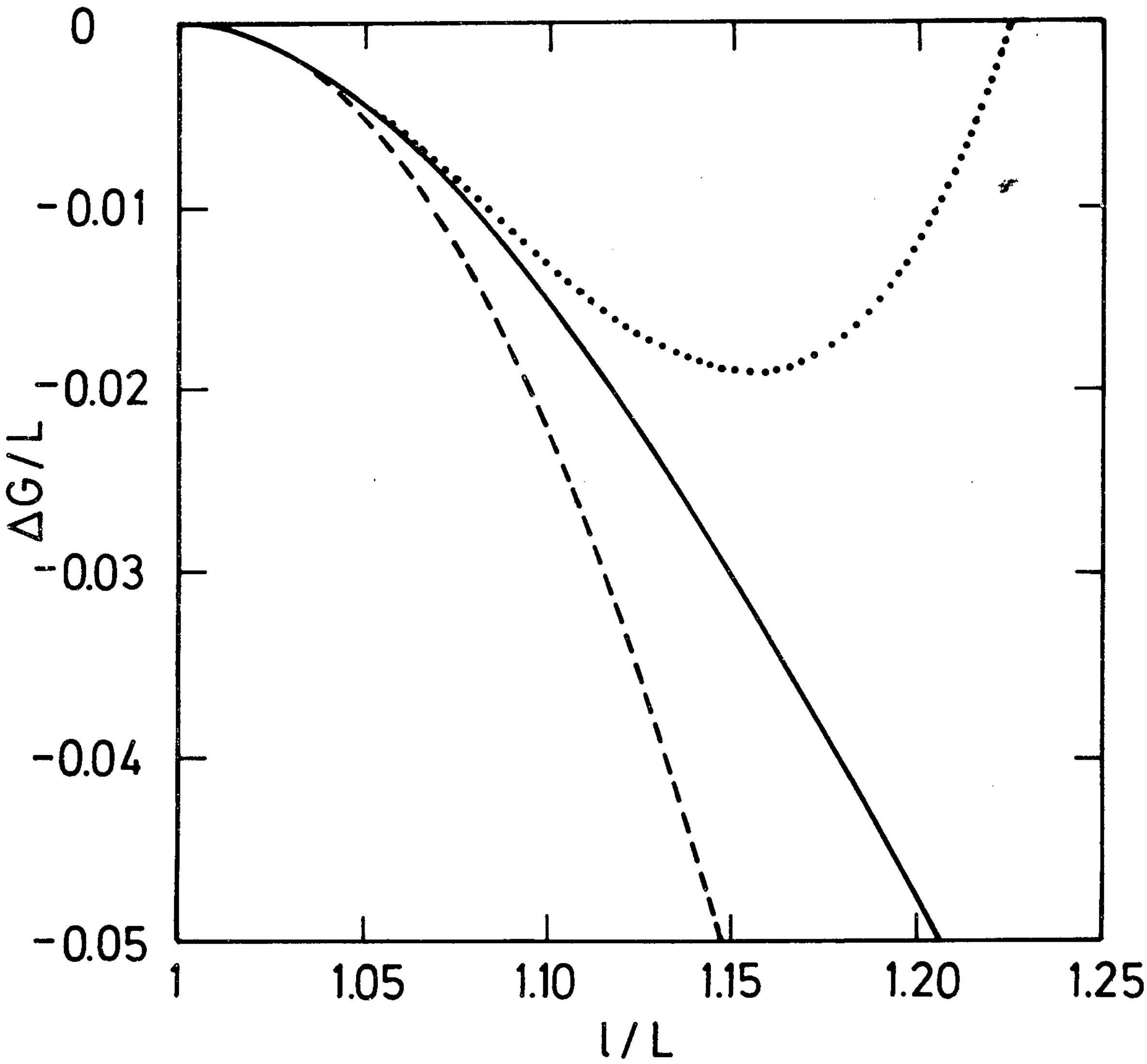

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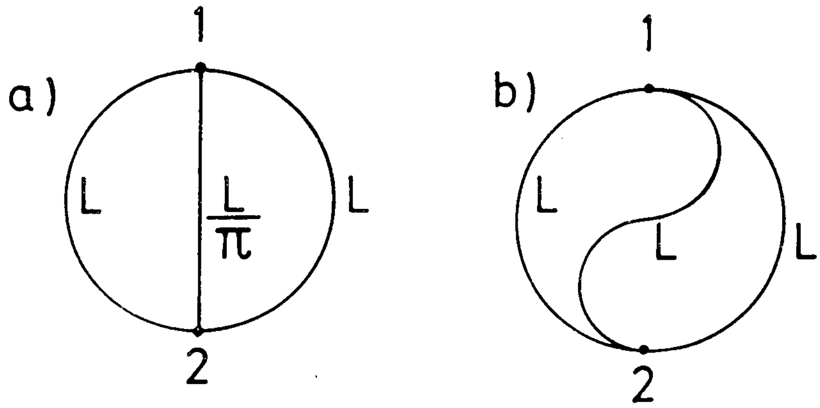

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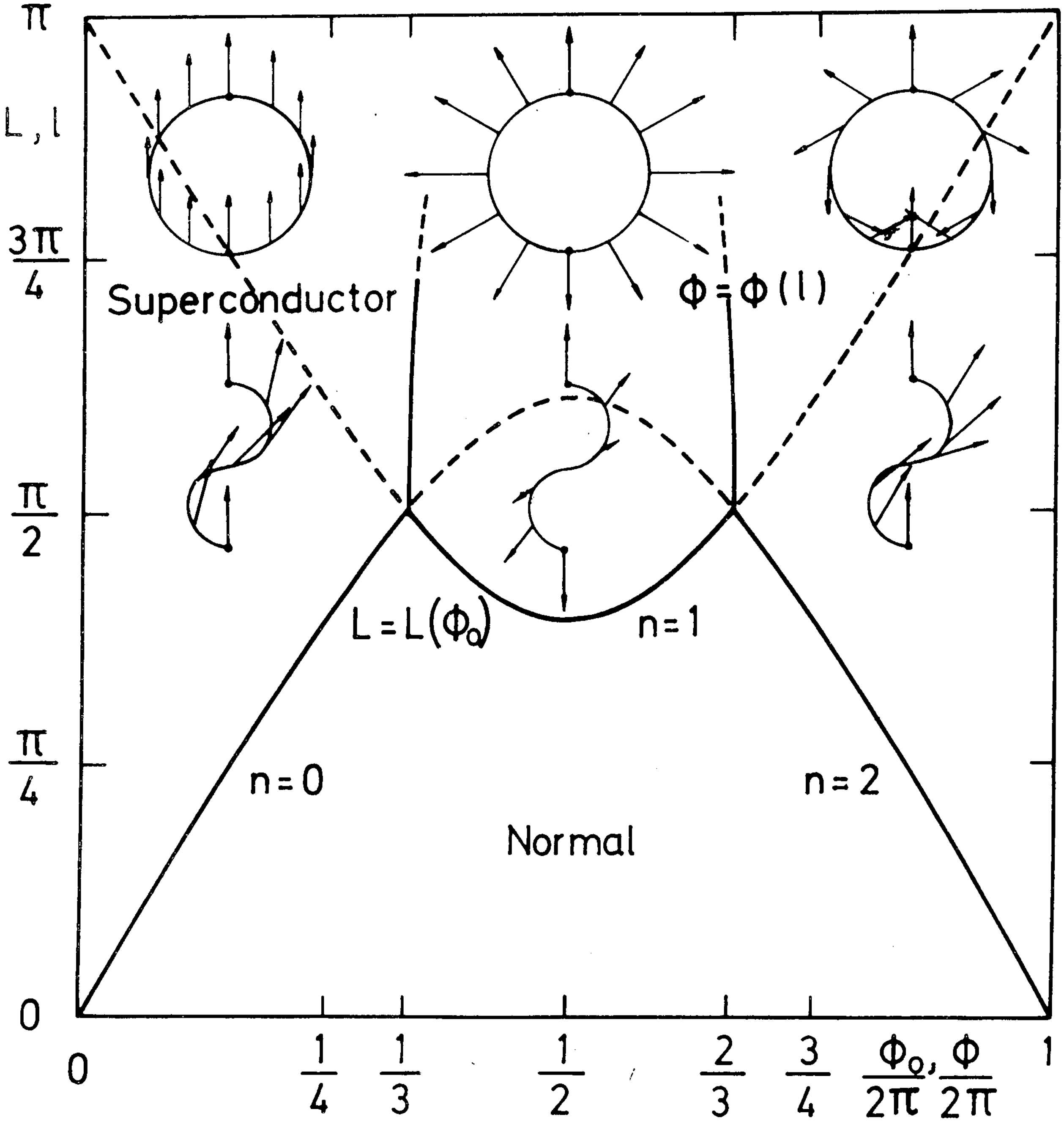


π
L , l
3π/4
π/2
π/4
0
Superconductor
Φ = Φ(l)
L = L(Φ₀)
n=1
n=0
n=2
Normal
0
1/4
1/3
1/2
2/3
3/4
Φ₀/2π, Φ/2π
1

Figure 9

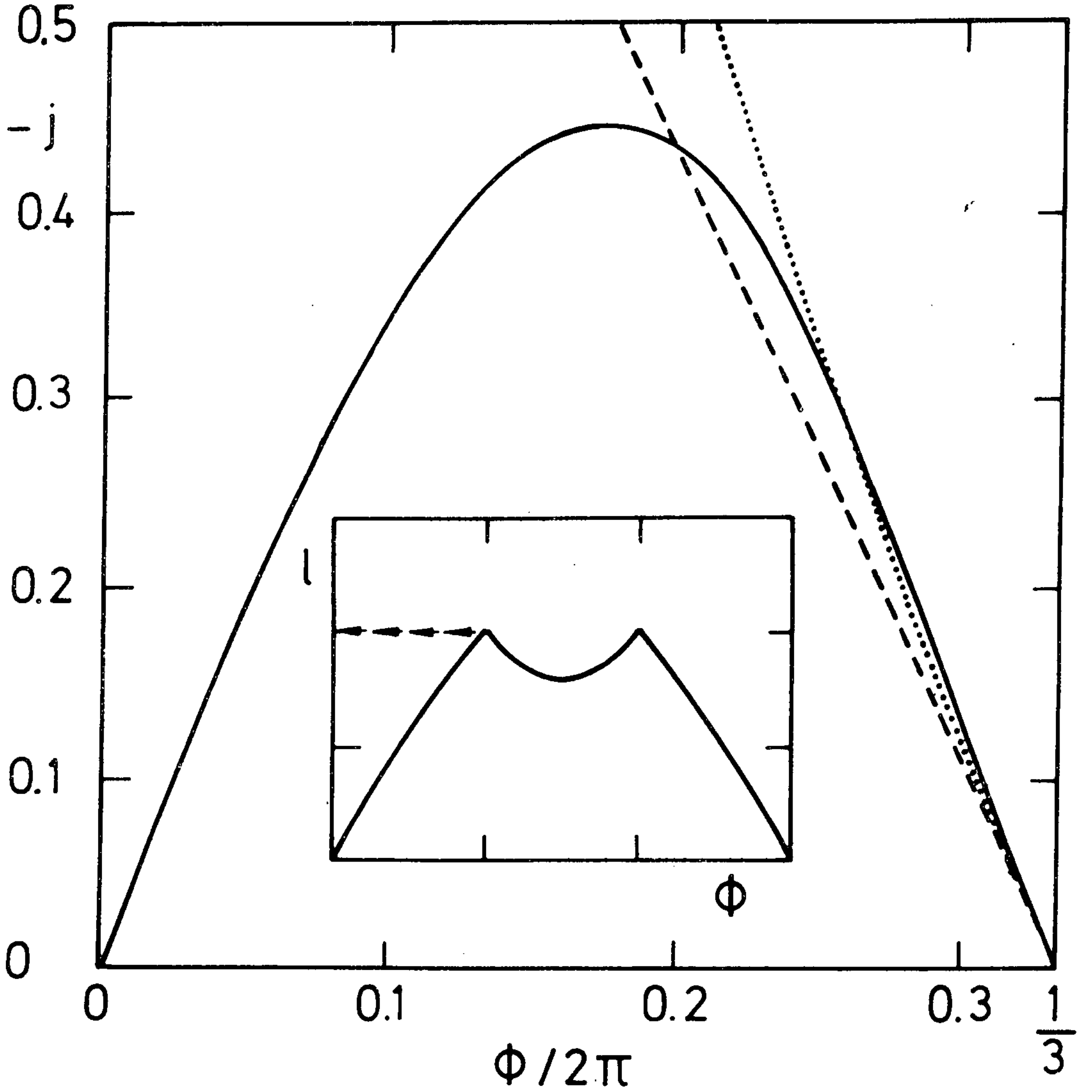
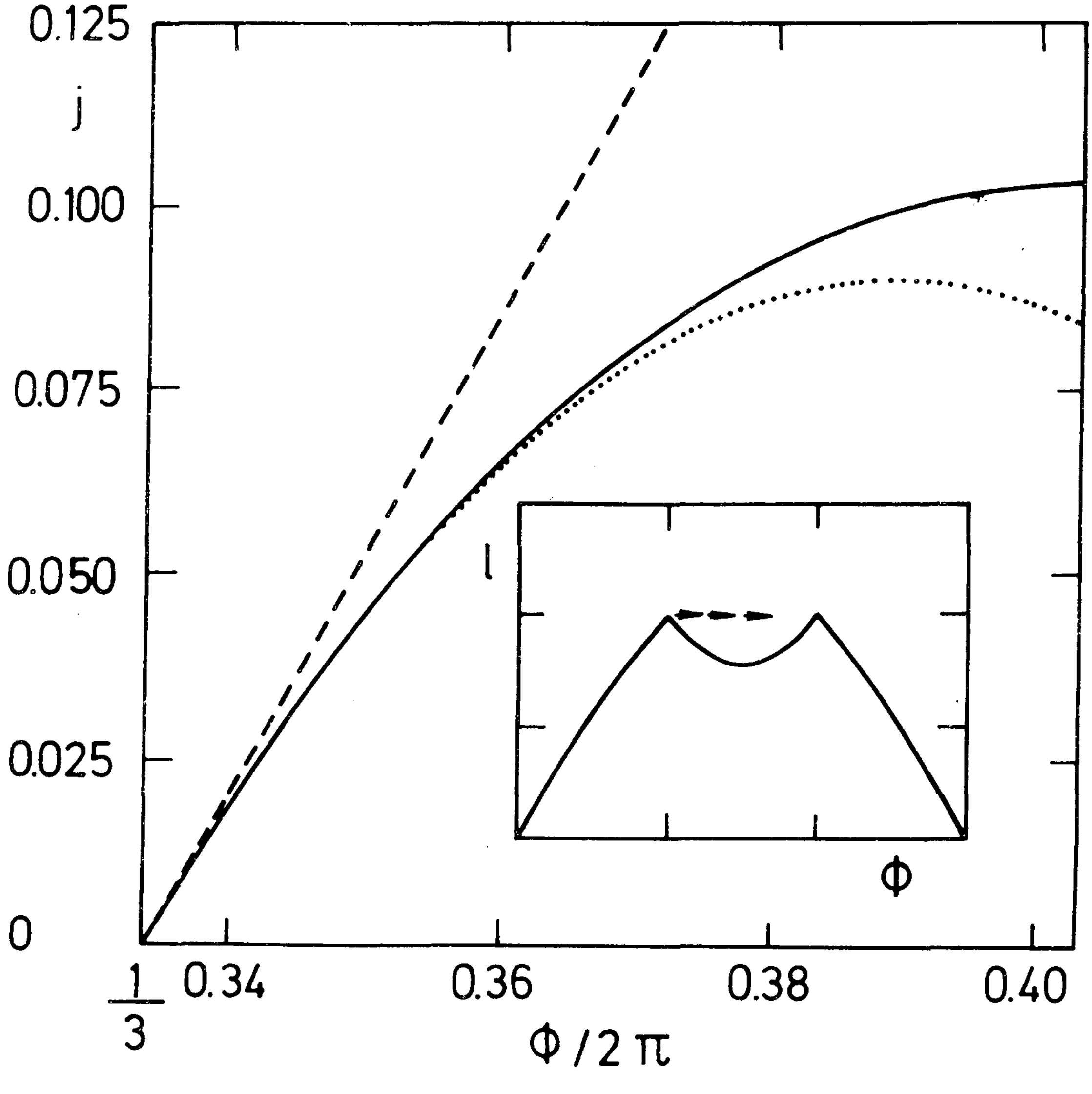


Figure 10 a



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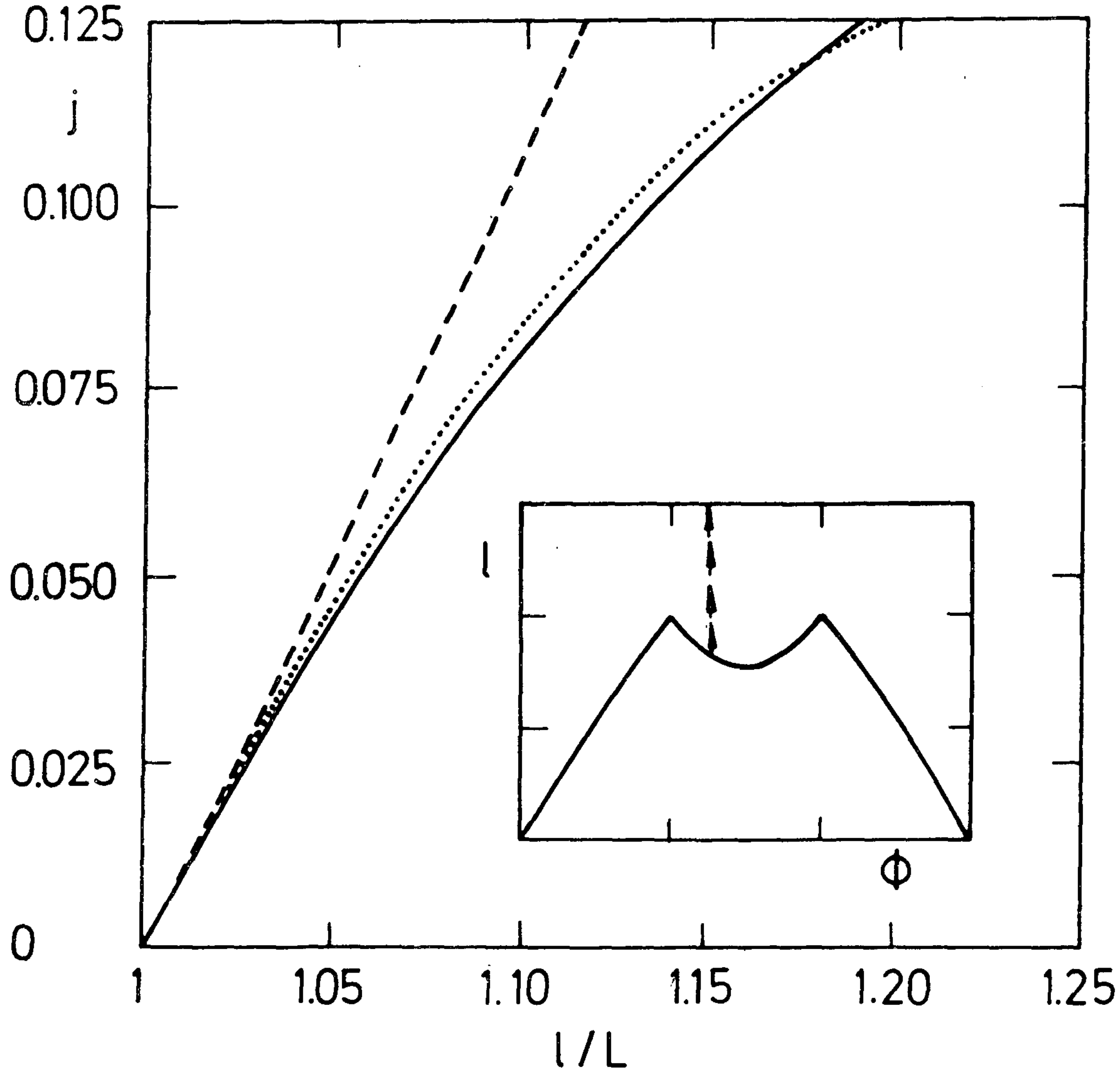

Figure 11 a

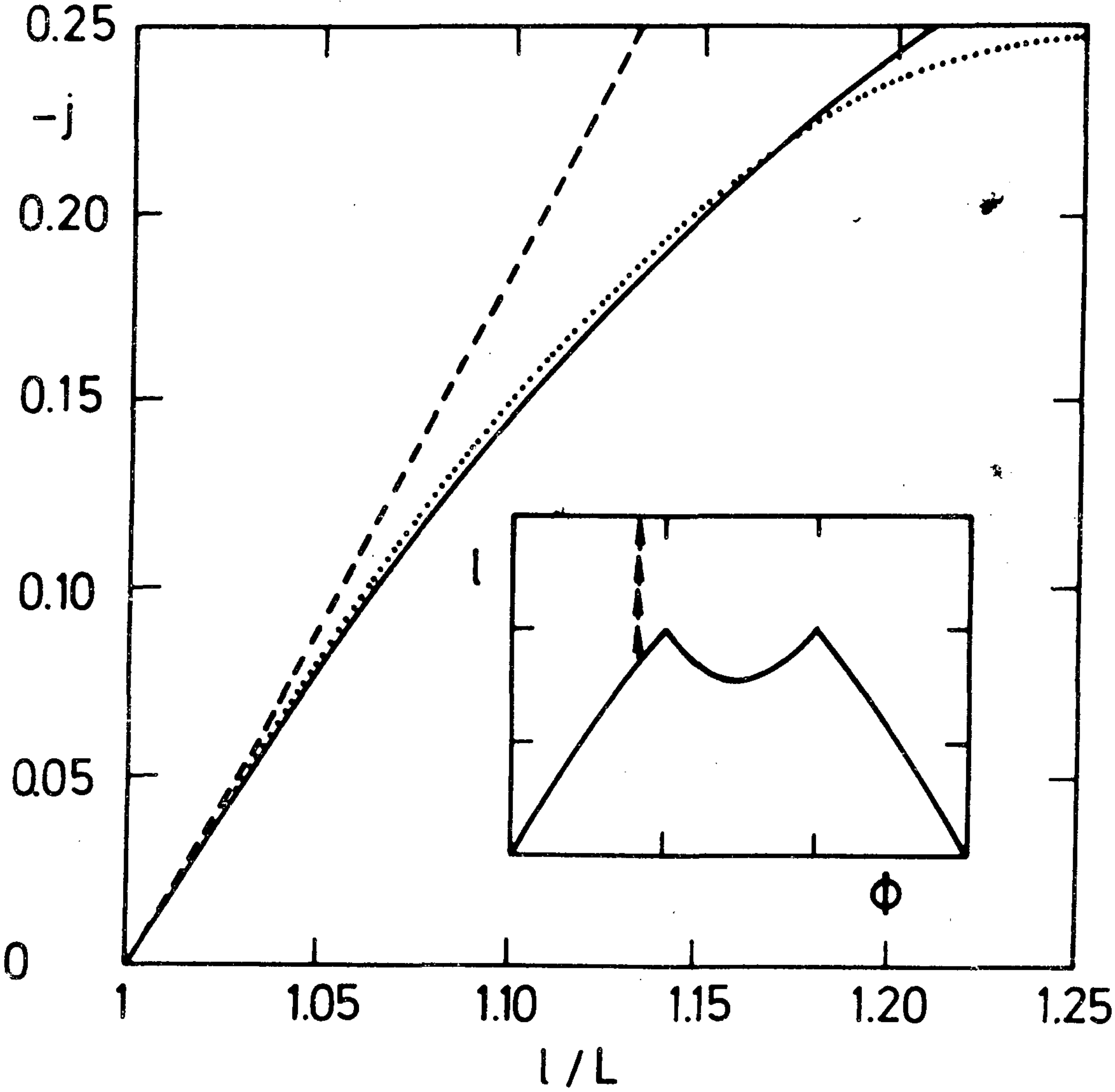

Figure 11b