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Three-Dimensional Space Charge Flow*

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The general theory of stationary beams of prescribed shape is developed taking into account the electrostatic interaction between the particles. The Hamilton-Jacobi equation, expressed in a particular system of coordinates, allows the determination of the potential distribution inside the beam. Compatibility conditions for the equations are obtained, as well as a general rule for the determination of the feasibility of arbitrarily chosen trajectories. A theoretical expression results for the proportionality coefficient k in Child's law ($J=kV^{\frac{3}{2}}$) which relates current density J with the potential difference V between electrodes. The theory was applied to the particular case of a plane beam with hyperbolic boundaries. The beam was experimentally reproduced, providing values for k in close agreement with the theoretical ones.

INTRODUCTION

The effects of space charge on electron or ion beams have not been exactly calculated except for simple cases.[1–3] Assuming rectilinear motion, Pierce[4] obtained the potential distributions for beams bounded in space and was able to find by analytical means, or with the aid of an electrolytic trough, the equipotential surfaces in the region devoid of space charge. This means, in effect, that an assumption must be made about the shape of a beam to obtain the electrodes which produce it. A noniterative method[5] has also been described to plot equipotentials in cases of axial symmetry.

Spangenberg[6] has pointed out the importance of the action function to obtain a general equation for three-dimensional beams. It is, in vector form,

$$\nabla\cdot(\nabla^2 W\cdot\nabla W)=0, \tag{1}$$

where W is the action function (Hamilton's characteristic function). However, Spangenberg was unable to solve cases other than the one-dimensional (rectilinear) one. Expansion of the vector operators for the x, y coordinates provides a 22-term fourth order, third degree partial differential equations.

We shall presently show how Eq. (1) can be generally solved, and how, at least in principle, the potential distribution inside the beam can be found once the shape of the beam is assumed to be known. This means that certain trajectories (subject to conditions which will be derived later on) are imposed, as Pierce did in the particular rectilinear case. Knowledge of the inside potential then allows the determination of the external electrodes. Thus, the production of beams of any (allowed) shape can be considered as a theoretically solved problem, but may still remain as a mathematical and technical difficulty.

In what follows we shall treat only electrostatic interactions, neglecting relativistic effects[7] such as the magnetic force exerted by the beam on itself.

THE ELECTROSTATIC POTENTIAL INSIDE THE BEAM

We *impose* a certain family of curves as the possible paths of the particles in the beam. Each of these curves is given by two equations

$$g(x,y,z)=\eta \tag{2a}$$

$$h(x,y,z)=\zeta. \tag{2b}$$

The variation of the parameters η and ζ describes the whole family. Now suppose that $g(x,y,z)$ and $h(x,y,z)$ are such that we can form an orthongonal system of coordinates (ξ,η,ζ) with $\xi=f(x,y,z)$; in which, obviously, a trajectory is defined by

$$\eta=\text{const}, \tag{3a}$$

$$\zeta=\text{const}, \tag{3b}$$

and $f(x,y,z)$ is found from the requirement that $\xi=\text{const}$ is the equation for the orthogonal surfaces to the family of curves. In this coordinate system, (1) η and ζ are constants of the motion and (2) the corresponding conjugate momenta are cyclic variables in the Hamiltonian of a particle. The Hamilton-Jacobi equation for such a particle should be written

$$H(\xi,\eta,\zeta,dW/d\xi)=E, \tag{4}$$

where E is the total energy of the particle. The action function depends only on the coordinate ξ. This is another way of stating the well-known fact that the surfaces $W=\text{const}$ are normal to the trajectory of the particle.

Let $ds^2=h_1^2d\xi^2+h_2^2d\eta^2+h_3^2d\zeta^2$ be the line element in our coordinate system. The kinetic energy of a particle moving along the coordinate line ξ is

$$T=\tfrac{1}{2}p^2/mh_1^2,$$

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7 R. D. Fowler and G. E. Gibson, Phys. Rev. **46**, 1083 (1934).

where p is the conjugate momentum to the coordinate ξ. We can write explicitly, instead of Eq. (4),

$$q\varphi+(1/2mh_1^2)(dW/d\xi)^2=E; \tag{5}$$

in which φ is the electrostatic potential; q and m, the charge and mass of the particles. It can be assumed without restriction that $E=0$, and then

$$\varphi=-\Phi/h_1^2, \tag{6}$$

where $\Phi=(1/2qm)(dW/d\xi)^2$ *depends only on* ξ.

We have thus reduced the problem of finding the electrostatic potential to that of calculating a function Φ *of only one variable* ξ. In order to do this, let us apply the Laplacian operator to both members of Eq. (6). From Poisson's equation

$$\nabla^2(-\Phi/h_1^2)=-J/v\epsilon_0. \tag{7}$$

We are using the mks system throughout this paper. J is the current density and v the velocity of the particles. But

$$\mathbf{v}=(1/m)\nabla W \tag{8a}$$

and

$$v=(1/mh_1)(dW/d\xi)=(1/h_1)[(2q/m)\Phi]^{\frac{1}{2}}. \tag{8b}$$

Therefore,

$$J=\epsilon_0[(2q/m)\Phi]^{\frac{1}{2}}(1/h_1)\nabla^2(\Phi/h_1^2). \tag{9}$$

The fact that $\nabla\cdot\mathbf{J}=0$ in the steady state can be expressed as

$$(\partial/\partial\xi)[h_2h_3(\Phi^{\frac{1}{2}}/h_1)\nabla^2(\Phi/h_1^2)]=0 \tag{10}$$

because the only component different from zero in $\mathbf{J}$ is in the direction of ξ. Equation (10) is essentially the same as Eq. (1), and has been obtained by similar steps. However, our particular choice of coordinate system enables us not only to express it directly in scalar form but to proceed immediately to a first integration,

$$(\Phi^{\frac{1}{2}}/h_1)\nabla^2(\Phi/h_1^2)=K(\eta,\zeta)/h_2h_3. \tag{11}$$

The function $K(\eta,\zeta)$ results from the partial integration of Eq. (10). Equation (9) shows its physical meaning: the right-hand member of Eq. (11) is proportional to the current density J.

When the Laplacian operator in Eq. (11) is expanded, one obtains

$$\nabla^2(\Phi/h_1^2)=(1/h_1^2)\nabla^2\Phi+2\nabla(1/h_1^2)\cdot\nabla\Phi+\Phi\nabla^2(1/h_1^2),$$

where, in turn

$$\nabla^2\Phi=(1/h_1h_2h_3)(\partial/\partial\xi)[(h_2h_3/h_1)\Phi'],$$

and

$$2\nabla\Phi\cdot\nabla(1/h_1^2)=(2/h_1^2)\Phi'(\partial/\partial\xi)(1/h_1^2)=\Phi'(\partial/\partial\xi)(1/h_1^4).$$

The primes denote total differentiation with respect to ξ. These expressions result in

$$\frac{h_2h_3}{h_1}\nabla^2\left(\frac{\Phi}{h_1^2}\right)=\frac{1}{h_1^4}\frac{\partial}{\partial\xi}\left(\frac{h_2h_3}{h_1}\Phi'\right)+\frac{h_2h_3}{h_1}\Phi'\frac{\partial}{\partial\xi}\left(\frac{1}{h_1^4}\right)+\Phi\frac{h_2h_3}{h_1}\nabla^2\left(\frac{1}{h_1^2}\right)$$

$$=\frac{\partial}{\partial\xi}\left(\frac{h_2h_3}{h_1^5}\Phi'\right)+\Phi\frac{h_2h_3}{h_1}\nabla^2\left(\frac{1}{h_1^2}\right).$$

When this equation is placed in Eq. (11), the latter is transformed into

$$\Phi^{\frac{1}{2}}[(\partial/\partial\xi)(A\Phi')+B\Phi]=K(\eta,\zeta), \tag{12}$$

where the functions A and B are given by

$$A=h_2h_3/h_1^5 \tag{13a}$$

$$B=(h_2h_3/h_1)\nabla^2(1/h_1^2). \tag{13b}$$

COMPATIBILITY AND SOLUTION OF THE FUNDAMENTAL EQUATION

We have now derived a differential equation [Eq. (12)] with *two* unknown functions in which Φ depends only on ξ; and K, on the two other coordinates. The fact that the left-hand member of Eq. (12) depends upon the three coordinates and the right-hand one, upon only two, indicates *a priori* that the functions A and B cannot be arbitrarily chosen; that is, certain conditions must hold for the allowed trajectories (which ultimately determine the h_i's and thus, A and B).

We shall now show how these "compatibility conditions" can be obtained and provide the solution for Φ. For the sake of brevity in our formulas we shall derive them for a two-dimensional case with coordinates (ξ,η). The extension to three dimensions is immediate.

It is easily shown that Eq. (12) is equivalent to a system of first-order partial differential equations. In fact, defining a new function by $\Psi=A\Phi'$, the system is

$$\partial\Phi/\partial\xi=\Psi/A \tag{14a}$$

$$\partial\Psi/\partial\xi=(K/\Phi^{\frac{1}{2}})-B\Phi \tag{14b}$$

$$\partial K/\partial\xi=0 \tag{14c}$$

$$\partial\Phi/\partial\eta=0. \tag{14d}$$

From these equations, the following two can be derived:

$$\partial\Psi/\partial\eta=\Psi\partial\ln A/\partial\eta, \tag{14e}$$

$$\frac{\partial K}{\partial\eta}=\left[\left(\frac{\partial^2\ln A}{\partial\xi\partial\eta}\right)\Psi+\left(\frac{K}{\Phi^{\frac{1}{2}}}-B\Phi\right)\frac{\partial\ln A}{\partial\eta}+\Phi\frac{\partial B}{\partial\eta}\right]\Phi^{\frac{1}{2}}. \tag{14f}$$

Equations (14) allow us to write the total differentials of the functions Ψ, Φ and K,

$$d\Psi=[(K/\Phi^{\frac{1}{2}})-B\Phi]d\xi+\Psi(\partial\ln A/\partial\eta)d\eta \tag{15a}$$

$$d\Phi=(\Psi/A)d\xi \tag{15b}$$

$$dK=\Phi^{\frac{1}{2}}\left[\Psi\frac{\partial^2 \ln A}{\partial\xi\partial\eta}+\left(\frac{K}{\Phi^{\frac{1}{2}}}-B\Phi\right)\frac{\partial \ln A}{\partial\eta}+\Phi\frac{\partial B}{\partial\eta}\right]d\eta. \tag{15c}$$

Now the necessary and sufficient condition for an expression

$$f(\xi,\eta,\omega)d\xi+g(\xi,\eta,\omega)d\eta$$

to be an exact differential $d\omega$ is that

$$(\partial f/\partial\eta)+(\partial f/\partial\omega)g=(\partial g/\partial\xi)+(\partial g/\partial\omega)f.$$

If similar conditions hold identically for the differential coefficients in the right-hand members of Eqs. (15) once the derivatives of the functions are replaced by its expressions given by the system (14), this system will be compatible. Accordingly, the restrictions on the functions A and B shall be found.

The integrability conditions, when applied to Eqs. (15), provide identities except for Eq. (15c). In this case one obtains, after some algebraic manipulations,

$$4\Phi\left(\frac{K}{\Phi^{\frac{1}{2}}}-B\Phi\right)\frac{\partial^2 \ln A}{\partial\xi\partial\eta}+\frac{\Psi^2}{A}\frac{\partial^2 \ln A}{\partial\xi\partial\eta}+\Psi\Phi\left[3\frac{(\partial B/\partial\eta)-B(\partial \ln A/\partial\eta)}{A}+2\frac{\partial^3 \ln A}{\partial\xi^2\partial\eta}\right]+2\Phi^2\left(\frac{\partial^2 B}{\partial\xi\partial\eta}-\frac{\partial B}{\partial\xi}\frac{\partial \ln A}{\partial\eta}\right)=0. \tag{16}$$

If we let $\partial^2 \ln A/\partial\xi\partial\eta=0$, then A can be written as the product of two functions

$$A=X(\xi)Y(\eta).$$

Also, if the definition of Ψ is taken into account, we can now write instead of Eq. (16), that

$$\Phi'/\Phi=-\tfrac{2}{3}(\partial/\partial\xi)\ln[(\partial B/\partial\eta)-BY'/Y]. \tag{17}$$

This equation has meaning only if the right-hand member is a function of the single variable ξ. In such a case, Eq. (17) can be immediately integrated, yielding

$$\Phi=\Phi_0[L(\xi)]^{-\frac{2}{3}}. \tag{18}$$

Here Φ_0 is a constant and $L(\xi)$ appears in

$$(\partial/\partial\eta)(B/Y)=L(\xi)M(\eta), \tag{19}$$

the necessary condition, with $M(\eta)$ an arbitrary function, in order that the right-hand member of Eq. (17) depend only on ξ. Conversely, if Eq. (19) holds, the compatibility of our system is assured with the solution (18).

Suppose now $\partial^2 \ln A/\partial\xi\partial\eta\neq 0$. We can eliminate K between Eqs. (16) and (14b) and form a new system with only two unknown functions, Φ and Ψ,

$$\partial\Phi/\partial\xi=\Psi/A \tag{20a}$$

$$\partial\Phi/\partial\eta=0 \tag{20b}$$

$$\frac{\partial\Psi}{\partial\xi}=-\left\{\left[\frac{\Psi^2}{4A\Phi}+\frac{\Psi}{4A}\right]\times\frac{3[(\partial B/\partial\eta)-B(\partial \ln A/\partial\eta)]+2A(\partial^3 \ln A/\partial\xi^2\partial\eta)}{(\partial^2 \ln A/\partial\xi\partial\eta)}+\frac{\Phi}{2}\frac{(\partial^2 B/\partial\xi\partial\eta)-(\partial B/\partial\xi)(\partial\, \ln A/\partial\eta)}{(\partial^2 \ln A/\partial\xi\partial\eta)}\right\} \tag{20c}$$

$$\partial\Psi/\partial\eta=\Psi(\partial \ln A/\partial\eta), \tag{20d}$$

to which similar integrability conditions can be applied. We obtain, after some tedious but straightforward algebra, the condition that

$$\frac{2\left[\dfrac{\dfrac{\partial^2 B}{\partial\xi\partial\eta}-\dfrac{\partial B}{\partial\xi}\dfrac{\partial \ln A}{\partial\eta}}{(\partial^2 \ln A/\partial\xi\partial\eta)}\cdot\dfrac{\partial \ln A}{\partial\eta}-\dfrac{\partial}{\partial\eta}\left\{\dfrac{\dfrac{\partial^2 B}{\partial\xi\partial\eta}-\dfrac{\partial B}{\partial\xi}\dfrac{\partial \ln A}{\partial\eta}}{(\partial^2 \ln A/\partial\xi\partial\eta)}\right\}\right]}{\dfrac{\partial}{\partial\eta}\left[\dfrac{3\left(\dfrac{\partial B}{\partial\eta}-B\dfrac{\partial \ln A}{\partial\eta}\right)+2A\dfrac{\partial^3 \ln A}{\partial\xi^2\partial\eta}}{(\partial^2 \ln A/\partial\xi\partial\eta)}\right]-\dfrac{3\left(\dfrac{\partial B}{\partial\eta}-B\dfrac{\partial \ln A}{\partial\eta}\right)+2A\dfrac{\partial^3 \ln A}{\partial\xi^2\partial\eta}}{(\partial^2 \ln A/\partial\xi\partial\eta)}\cdot\dfrac{\partial \ln A}{\partial\eta}+4A\dfrac{\partial^2 \ln A}{\partial\xi\partial\eta}}=F(\xi) \tag{21}$$

must be a function of the only variable ξ. In such a case Φ satisfies the differential equation

$$\Phi'/\Phi=F(\xi) \tag{22}$$

with the general solution

$$\Phi=\Phi_0 e^{\int F(\xi)d\xi}, \tag{23}$$

where Φ_0 is an arbitrary constant.

CHILD'S LAW

The dependence of current density on the three-halves power of the potential difference between electrodes is one of the well-established facts of space-charge flow of any type. It was derived theoretically by Child[8] for rectilinear flow between parallel plane

[8] C. D. Child, Phys. Rev. 23, 492 (1911).

electrodes. It will be shown how the expressions obtained indicate its general validity, and how the proportionality coefficient depends upon the particular shape of the beam. In fact, replacement of (23) in (12) gives

$$K=\Phi_0^{\frac{3}{2}}e^{\frac{3}{2}\int F(\xi)d\xi}\left[A(F'+F^2)+F\frac{\partial A}{\partial \xi}+B\right]. \quad (24)$$

Equation (24) together with (6) and the aforementioned meaning of K, gives for the current density in the three-dimensional case

$$J=\epsilon_0(2q/m)^{\frac{1}{2}}(h_1^3/h_2h_3)[A(F'+F^2)+F(\partial A/\partial\xi)+B]\varphi^{\frac{3}{2}}. \quad (25)$$

The metric coefficients, as well as the whole expression in square brackets appearing in Eq. (25), depend only on the geometry of the coordinate system in use, that is, on the *shape* of the beam. If φ is the potential difference between electrodes, Eq. (25) allows the determination of the proportionality coefficient between the current density and the three-halves power of the accelerating potential.

Unfortunately, the validity of Eq. (25) cannot be checked against the already known solutions of rectilinear motion between parallel plane electrodes, concentric cylinders, and concentric spheres. In fact, Eq. (16) holds identically for such cases, and the function $F(\xi)$ cannot be calculated. On the other hand, one obtains from Eq. (12) the already known equations for the potential.

For the foregoing reasons, the theory was applied to a particular case with the view of its experimental reproduction.

POTENTIAL DISTRIBUTION FOR A PARTICULAR BEAM

Let us consider an electron beam with hyperbolic paths. The beam is supposed unbounded in the z direction. The equation for the trajectories is

$$xy=\eta, \quad (26)$$

where η is a constant for each trajectory. The family of orthogonal trajectories is given by the equation

$$\tfrac{1}{2}(x^2-y^2)=\xi=\text{const}. \quad (27)$$

In the coordinate system ξ, η, the metric coefficients are

$$h_1^2=h_2^2=(x^2+y^2)^{-1}=\tfrac{1}{2}(\xi^2+\eta^2)^{-\frac{1}{2}} \quad \text{and} \quad h_3=1.$$

Formulas (13) now give

$$A=4(\xi^2+\eta^2); \qquad B=4$$

and Eq. (21) shows immediately that $F(\xi)=0$. Then, by Eq. (23),

$$\Phi=\Phi_0=\text{const}.$$

For the potential distribution inside the beam, we have from Eq. (6)

$$\varphi=-\Phi_0(x^2+y^2)=-\Phi_0 r^2. \quad (28)$$

The constant Φ_0 is negative for electrons. In Eq. (28), r is the distance from the z axis to the point under under consideration. The equipotentials are concentric cylinders.

According to Eq. (25), the current density is

$$J=4\epsilon_0(2q/m)^{\frac{1}{2}}(\varphi^{\frac{3}{2}}/r^2). \quad (29)$$

To obtain it in vector form we first calculate the charge density ρ from Eq. (28) by application of the Laplacian operator

$$\rho=4\epsilon_0\Phi_0. \quad (30)$$

The action function W follows from the definition of Φ by integration

$$W=(2qm\Phi_0)^{\frac{1}{2}}\xi+\text{const},$$

and because of Eqs. (8a) and (27) the velocity is given by

$$\mathbf{v}=(2q\Phi_0/m)^{\frac{1}{2}}(x\mathbf{i}-y\mathbf{j}). \quad (31)$$

Multiplication of (30) and (31) gives

$$\mathbf{J}=\rho\mathbf{v}=4\epsilon_0\Phi_0^{\frac{3}{2}}(2q/m)^{\frac{1}{2}}(x\mathbf{i}-y\mathbf{j}). \quad (32)$$

Now suppose our beam is not unbounded in the x, y plane, but is limited by the hyperbolic cylinders $xy=\lambda$ and $xy=-\lambda$ and the plane $x=0$. This is shown by the dotted curves of Fig. 1. In such a case the total current (per unit length in the z direction) is

$$I=\int_{-\lambda/x}^{\lambda/x} J_x dy=8\epsilon_0\Phi_0^{\frac{3}{2}}(2q/m)^{\frac{1}{2}}\lambda. \quad (33)$$

In the case illustrated in Fig. 2, the lower limit of integration must be 0, and we have

$$I=\int_{0}^{\lambda/x} J_x dy=4\epsilon_0\Phi_0^{\frac{3}{2}}(2q/m)^{\frac{1}{2}}\lambda. \quad (34)$$

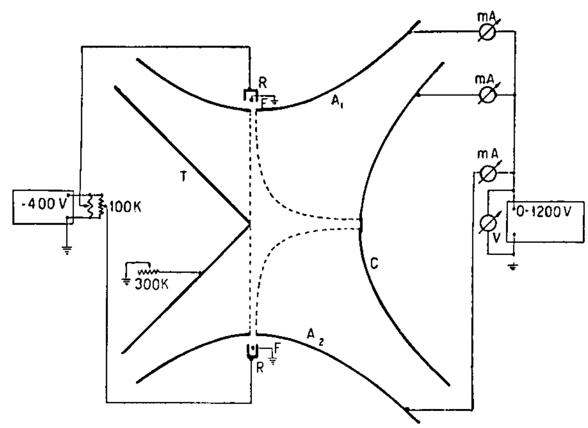


FIG. 1. Beam shape and electrical connections in the first experiment performed. The dotted curves show predicted beam limits. A_1, A_2, and C are high-tension electrodes. R, electron reflectors. F, filaments. T, grounded electrode.

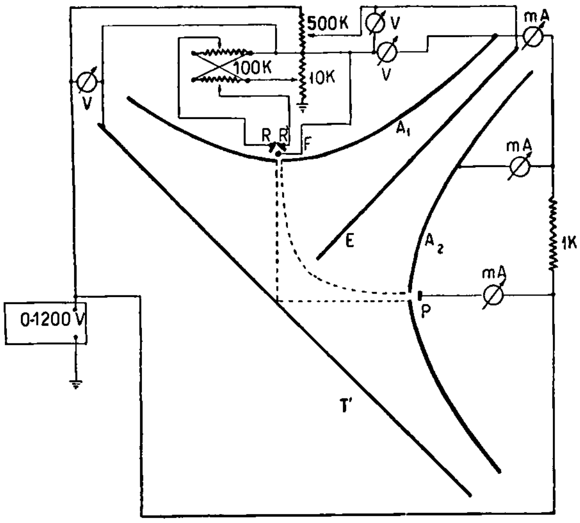


FIG. 2. Beam shape and electrical connections in the second experiment performed. The dotted curves show predicted beam limits. A_1 and A_2 are high-tension electrodes. P, collector plate. R and R', electron reflectors. F, filament. T', low-tension electrode. E, intermediate-tension electrode.

ELECTRODE ARRANGEMENT

The electrodes producing our particular beam must be equipotential surfaces in the region outside the beam. The mathematical problem involved is then the solution of Laplace's equation for the potential. The boundary conditions result from continuity of the potential and its normal derivative; this last requirement is a consequence of the fact that the centrifugal force on an electron depends only upon the velocity and the curvature of the path, irrespective of whether it is a boundary or an internal one.

Let us define now

$$u=\xi \qquad v=\eta-\lambda$$
$$w=u+iv \qquad w^*=u-iv.$$

The region devoid of charge corresponds to the half-plane $v>0$ in the complex plane w. The solution of Laplace's equation is then[9]

$$\varphi(u,v)=\mathrm{Re}\,\varphi_0(w)+\mathrm{Im}\int_0^w N_0(\zeta)d\zeta \tag{35}$$

with the following boundary conditions:

$$\varphi_0(u)=\varphi(u,0)=-2\Phi_0(u^2+\lambda^2)^{\frac{1}{2}} \tag{36a}$$

$$N_0(u)=\partial\varphi/\partial v]_{v=0}=-2\Phi_0\lambda/(u^2+\lambda^2)^{\frac{1}{2}}. \tag{36b}$$

Expressions (35) and (36) allow the calculation of the function φ.

$$\varphi=-2\Phi_0\left\{1/\sqrt{2}\left[\{(u^2+\lambda^2-v^2)^2+4u^2v^2\}^{\frac{1}{2}}+(u^2+\lambda^2-v^2)\right]^{\frac{1}{2}}\right.$$
$$\left.+\lambda\,\mathrm{tg}^{-1}\frac{v+(1/\sqrt{2})\left[\{(u^2+\lambda^2-v^2)^2+4u^2v^2\}^{\frac{1}{2}}-(u^2+\lambda^2-v^2)\right]^{\frac{1}{2}}}{u+(1/\sqrt{2})\left[\{(u^2+\lambda^2-v^2)^2+4u^2v^2\}^{\frac{1}{2}}+(u^2+\lambda^2-v^2)\right]^{\frac{1}{2}}}\right\}. \tag{37}$$

We have thus obtained the potential distribution for the region devoid of charge of the first quadrant in the x, y plane. For a beam like the one illustrated in Fig. 1, it is evident from symmetry arguments that the same distribution holds in the fourth quadrant. The boundary conditions for the other two quadrants are

$$\varphi_0(y)=-\Phi_0 y^2$$
$$N_0(y)=\partial\varphi/\partial x]_{x=0}=0,$$

and we obtain for the potential

$$\varphi(x,y)=\mathrm{Re}\,\varphi_0(y+ix)=-\Phi_0(y^2-x^2).$$

The equipotentials are equilateral hyperbolas, whose asymptotes correspond to the equipotential $\varphi=0$.

A similar argument holds for the fourth quadrant in the case of Fig. 2. The potential is

$$\varphi=-\Phi_0(x^2-y^2).$$

We have still to plot the equipotential surfaces corresponding to (37). We shall obtain a parametric representation of them. The parameter will be

$$\alpha=\mathrm{tg}^{-1}\frac{v+(1/\sqrt{2})\{[(u^2+\lambda^2-v^2)^2+4u^2v^2]^{\frac{1}{2}}-(u^2+\lambda^2-v^2)\}^{\frac{1}{2}}}{u+(1/\sqrt{2})\{[(u^2+\lambda^2-v^2)^2+4u^2v^2]^{\frac{1}{2}}+(u^2+\lambda^2-v^2)\}^{\frac{1}{2}}}. \tag{38}$$

If we express the potential as

$$\varphi=-2\Phi_0\lambda\psi,$$

with ψ a dimensionless variable, and then multiply (37) and $\mathrm{tg}\alpha$ [given by (38)], we obtain the parametric equation for v

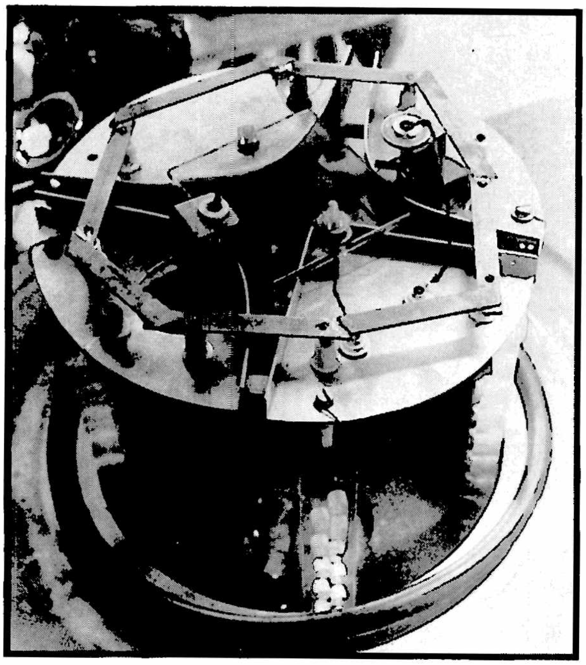
$$v=\lambda(\psi-\alpha)\mathrm{tg}\alpha. \tag{39}$$

Similarly, the parametric expression for u is given by

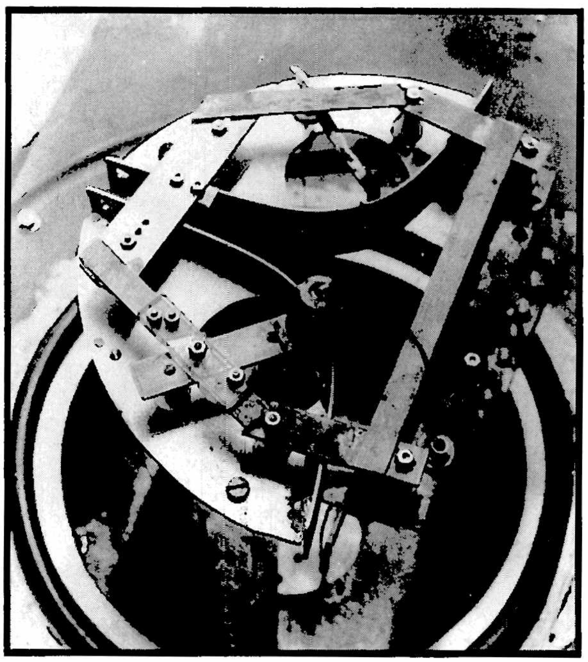
$$u=\pm\lambda[(\psi-\alpha)^2-\cos^2\alpha]^{\frac{1}{2}}. \tag{40}$$

It is easily seen that $0\leqslant\alpha\leqslant\psi$ if $\psi\leqslant\pi/2$ and $0\leqslant\alpha\leqslant\pi/2$ if $\psi>\pi/2$. Moreover $\psi\geqslant 1$. The justification for the

[9] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 1, p. 689.



(a)



(b)

FIG. 3. Electrode arrangements. (a) First experiment, corresponding to Fig. 1. (b) Second experiment, corresponding to Fig. 2. In case (b), the crystal plates were taken off before photographing.

additional electrode E of Fig. 2, corresponding to $\psi=\pi/2$, arises from the fact that when α varies from 0 to $\pi/2$, v varies from 0 to ∞, unless $\psi=\pi/2$. In that case $u\rightarrow 0$ when $\alpha\rightarrow\pi/2$, and v can be made to tend to any value from λ to infinity when ψ approaches $\pi/2$ from the right. The electrode E corresponds then to the conditions $u=0$, $\lambda\leqslant v<\infty$, $\psi=\pi/2$.

EXPERIMENTAL CHECK OF THE THEORY

Two experiments were performed corresponding to predicted beam limits as shown by the dotted curves of Figs. 1 and 2. The equipotentials were plotted from the above formulas. The electrodes A_1, A_2 and C correspond to the distance $r=5$ cm from the origin. The particular choice $\lambda=1$ cm^2 allows the determination of ψ according to its definition and Eq. (28)

$$\psi=r^2/(2\lambda)=12.5.$$

Through slits 3.2 cm high and 2 mm wide, a beam 3.2 cm high (in the z direction) was produced by collimating the electrons from thoriated-tungsten filaments F in Figs. 1 and 2. In order to reduce edge effects the beam was made to flow between two parallel crystal plates; their job was to become electrically charged until they were able to repel any electron tending to deviate from the plane path. These plates as well as the electrode arrangements of both experiments can be seen in the photographs of Fig. 3.

In the first experiment U-shaped, negative electrodes (R in Fig. 1) surrounded the filaments, and were used as electron reflectors. By properly adjusting the potential of the electrodes, it was possible (1) to reduce to a minimum the currents to A_1 and A_2 and (2) to obtain a close agreement between experimental and theoretical values of current in C for *one* value of the potential in electrodes A_1, A_2 and C. From then on the tension in R remained fixed and the accordance was conserved for other values of tension in A_1, A_2, C. Variation of the 300 kΩ resistance connected to the grounded electrode T produced no change in the collected current, showing that no measurable current was flowing through it. Of course, a complete proof of the theory could only be achieved if it could be known that the electrons escaping from the slits in A_1 and A_2 had not only the correct magnitude of the velocity but also the proper *direction*. Nevertheless, the very fact that the expected current was obtained is a qualitative verification of the theory. Moreover, Eq. (25) is verified by the calculation of a least-square value of $I/V^{\frac{3}{2}}$

$$(I/V^{\frac{3}{2}})_{\text{exp}}=(11.2\pm0.1)\times10^{-7}\text{amp}/(\text{v})^{\frac{3}{2}}.$$

This can be compared with the theoretical value obtained from Eq. (33),

$$(I/V^{\frac{3}{2}})_{\text{theor}}=10.76\times10^{-7}\text{ amp}/(\text{v})^{\frac{3}{2}}.$$

The deviation observed can be ascribed to: (1) edge effects not completely removed by the crystal plates; (2) unavoidable mechanical misalignments; (3) the fixed tension of R producing a different over-all effect for different values of tension in A_1 and A_2.

The second experiment, schematically shown in Fig. 2, was designed to obtain information also about the *shape* of the beam. This was done by collecting a considerable fraction of the beam in a plate (P on Fig. 2) after the slit of A_2. To obtain a differential effect on the beam, variable with the total current emitted by the filament, the reflectors R were replaced by two plates R and R′, connected as shown in Fig. 2. From the experimental results the following value has been obtained:

$$(I/V^{\frac{3}{2}})_{\mathrm{exp}} = (5.5 \pm 0.2) \times 10^{-7}\ \mathrm{amp}/(\mathrm{v})^{\frac{3}{2}}.$$

This is in close agreement with

$$(I/V^{\frac{3}{2}})_{\mathrm{theor}} = 5.38 \times 10^{-7}\ \mathrm{amp}/(\mathrm{v})^{\frac{3}{2}}.$$

A further evidence of the correctness of the theory is derived from the large fraction (30%) of the total current emitted from the slit in A_1 collected in the electrode P; moreover, less than 0.2% of that current reached the electrode E. It was, therefore, concluded that the shape of the beam was very close to the predicted one.

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