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# ON THE MEASUREMENT OF YOUNG'S MODULUS OF TUBES BY PROPAGATION OF LONGITUDINAL WAVES

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The equation that describes the propagation of elastic waves has been solved numerically for longitudinal vibrations of tubes and the results are used to calculate Young's modulus for Zircaloy-4 fuel sheathings. The values are compared with the results obtained by using the approximate equations proposed in the literature. The differences observed are less than 0.5% for the fundamental frequency. For the harmonics, however, the numerical solution leads to resonant frequencies that are within 1% of the experimental values and the approximate equations give errors higher than 10%.

## 1. Introduction

Resonance methods are widely used to determine the elastic constants of solids. In the resonant bar techniques the specimen is excited in different vibration modes and the appropriate elastic constant is evaluated from the measured resonant frequencies [1]. For the case of longitudinal vibrations the simplest solution of the wave equation for a cylindrical rod leads to [1]

$$f_k = \frac{k}{2L}(E/\rho)^{1/2}, \tag{1}$$

where $f_k$ is the resonant frequency, $E$ is Young's modulus, $\rho$ is the density of the material, $L$ is the length of the specimen and $k = 1,2,3,\ldots$ gives the order of the harmonics. Eq. (1) can be used to obtain $E$ from the measured frequencies.

The propagation of free harmonic waves in an infinitely long cylindrical rod has been discussed on the basis of the linear theory of elasticity by Pochhammer [2] and Chree [3]. According to these authors the solution of the equations of motion for an isotropic elastic medium leads, in a second approximation, to the known contribution of lateral inertia on the propagation of longitudinal waves, already considered by Lord Rayleigh [4]. This contribution leads to [4,5]

$$f_k = \frac{k}{2L}(E/\rho)^{1/2}\left(1 - \frac{k^2\pi^2\nu^2r^2}{4L^2}\right), \tag{2}$$

where $\nu$ is Poisson's ratio and $r$ is the radius of the rod. Rayleigh's correction factor can be extended to several geometries by using the equation

$$f_k = \frac{k}{2L}(E/\rho)^{1/2}\left(1 - \frac{k^2\pi^2\nu^2}{4}\frac{2\theta}{L^2M}\right), \tag{3}$$

where $\theta$ is the moment of inertia with respect to the longitudinal axis and $M$ is the mass of the specimen. For the case of a hollow cylinder with inner radius $a$ and outer radius $b$, eq. (3) gives

$$f_k = \frac{k}{2L}(E/\rho)^{1/2}\left(1 - \frac{k^2\pi^2\nu^2}{4}\frac{a^2+b^2}{2}\right). \tag{4}$$

Eqs. (2) and (4) introduce only small corrections to the results given by eq. (1) in the case of long thin rods or long hollow cylinders with thin walls.

Eqs. (1) to (4) are only approximate and do not result from the complete solution of the equations of motion for an isotropic elastic solid. Gazis [6], however, has studied the propagation of free harmonic waves along a hollow circular cylinder of infinite extent within framework of the linear theory of elasticity. A characteristic equation appropriate to the circular hollow cylinder was obtained by the use of Helmholtz potentials for arbitrary values of the physical parameters involved.

It is the purpose of this paper to obtain a general relationship between Young's modulus and the resonant frequencies for longitudinal vibrations, by solving numerically the equations of motion. The results will be compared with actual experimental data obtained in Zircaloy-4, for rods and tubes, and the validity of the approximate equations (1) to (4) will be discussed.

## 2. Theory

The equations of motion for an isotropic elastic medium are [7], in invariant form

$$\mu\nabla^2 \boldsymbol{u} + (\lambda + \mu)\nabla\nabla\cdot\boldsymbol{u} = \rho\frac{\partial^2 \boldsymbol{u}}{\partial t^2}, \tag{5}$$

where $\boldsymbol{u}$ is the displacement vector, $\rho$ is the density, $\lambda$ and $\mu$ are Lamé's constants and $\nabla^2$ is the three-dimensional Laplace operator. The detailed solution of eq. (5) for a hollow circular cylinder is described in [6] and only the procedure will be outlined. The vector $\boldsymbol{u}$ is expressed in terms of a dilatational scalar potential and an equi-voluminal vector potential. The general solution depends on the characteristic equation, formed by the determinant

$$|c_{ij}| = 0, \quad (i, j = 1 \text{ to } 6) \tag{6}$$

where $i$ identifies the row and $j$ the column of the determinant. The elements of the determinant are given in terms of different kinds of Bessel functions of the inner and outer radii of the cylinder and the parameters $\alpha$ and $\beta$ given by

$$\alpha^2 = \frac{\omega^2}{v_1^2} - \xi^2 \quad \text{and} \quad \beta^2 = \frac{\omega^2}{v_2^2} - \xi^2,$$

$$v_1^2 = (\lambda + 2\mu)/\rho; \qquad v_2^2 = \mu/\rho.$$

where $\xi$ is the longitudinal wave number and $\omega$ is the angular frequency of the waves; $v_1$ and $v_2$ are the velocities for propagation of longitudinal and transverse waves, respectively. For the case considered in this paper, where only longitudinal vibrations are considered and the motion is independent of the angular coordinate, eq. (6) breaks into the product of two subdeterminants and the frequency equation

$$D = \begin{Vmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{Vmatrix} = 0 \tag{7}$$

corresponds to longitudinal waves with displacements independent from the angular coordinate. The type of Bessel functions to be used for the evaluation of the elements of the determinant of eq. (7) depends on the parameters $\lambda_1$ and $\lambda_2$ which take into account the differences in the recursion and differentiation formulas between the different kinds of Bessel functions. $\lambda_i$ vary as follows

$$\begin{aligned} &v_1\xi < \omega && \lambda_1 = 1, \quad \lambda_2 = 1; \\ &v_2\xi < \omega < v_1\xi && \lambda_1 = 1 \quad \lambda_2 = 1; \\ &\omega < v_2\xi && \lambda_1 = 1 \quad \lambda_2 = -1. \end{aligned}$$

On taking appropriate values for the elastic constants and the density of Zircaloy-4 it can be easily seen that $\lambda_1 = 1$ and $\lambda_2 = -1$.

In these conditions, the elements of the characteristic determinant of eq. (7) are

$$\begin{aligned} c_{11} &= -(\beta_1^2 - \xi^2) a I_0(\alpha_1 a) - 2\alpha_1 I_1(\alpha_1 a), \\ c_{12} &= 2\beta_1\xi a J_0(\beta_1 a) - 2\xi J_1(\beta_1 a), \\ c_{13} &= -(\beta_1^2 - \xi^2) a K_0(\alpha_1 a) + 2\alpha_1 K_1(\alpha_1 a), \\ c_{14} &= 2\beta_1\xi a Y_0(\beta_1 a) - 2\xi Y_1(\beta_1 a), \\ c_{21} &= 2\xi\alpha_1 I_1(\alpha_1 a), \\ c_{22} &= -(\beta_1^2 - \xi^2) J_1(\beta_1 a), \\ c_{23} &= -2\xi\alpha_1 K_1(\alpha_1 a), \\ c_{24} &= -(\beta_1^2 - \xi^2) Y_1(\beta_1 a), \end{aligned} \tag{8}$$

where

$$\alpha_1^2 = \left[4\pi^2 f_k^2 \rho(1 + \nu)(1 - 2\nu)/E(1 - \nu)\right] - \xi^2,$$

$$\beta_1^2 = \left[8\pi^2 f_k^2 \rho(1 + \nu)/E\right] - \xi^2,$$

$$\xi = k\pi/L.$$

$J_n(x)$ and $Y_n(x)$ are Bessel functions of the first and the second kind, respectively, and $I_n(x)$, $K_n(x)$ are modified Bessel functions of the first and the second kind, respectively [8]. The remaining two rows of the determinant, i.e., $c_{3i}$ and $c_{4i}$ are obtained by substitution of $a$ for $b$.

On substituting the elements of the determinant given by eq. (8) into eq. (7) an implicit transcendental function

$$\phi = \phi(E, \nu, \rho, L, f_k, a, b) = 0 \tag{9}$$

is obtained, where $f_k$ cannot be solved algebraically. In this implicit function $E$, $\nu$ and $\rho$ characterize the material and $L$, $a$, $b$ the dimensions of the specimen. By fixing $E$, $\nu$, $\rho$, $L$ and $b$ eq. (9) is reduced to an equation of two variables which, for a given $a$, determines the resonant frequencies $f_k$.

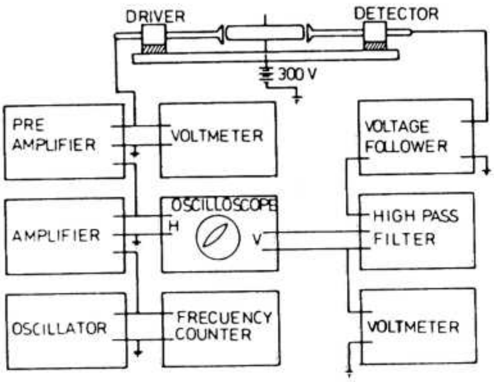


Fig. 1. Block diagram of the basic electronic instrumentation.

## 3. Experimental procedure

The longitudinal resonant frequencies were measured by using the "free–free" or floating beam resonant method described by Spinner and Tefft [9] and Sorrentino [10]. In particular, the electrostatic drive and detection method with a polarizing voltage (condenser microphone arrangement) [11,12,13] was used. The apparatus was constructed in the laboratory and the supporting device was made of Zircaloy-4 to avoid the influence of dilatations on the interelectrode capacity. The basic electronic instrumentation is shown as a block diagram in fig. 1 and the details are described elsewhere [14,15].
The equipment utilizes the capacitive coupling between the specimen and fixed electrodes positioned close to vibration antinodes. Displacement sensitivities of 1–2 nm have been obtained at the strain amplitudes of the order of $10^{-8}$ used for the measurements. Specimens of Zircaloy-4 in the form of cylindrical rods approximately 150 mm long and 12 mm in diameter and tubes, normally used as fuel sheathings, approximately 150 mm long and 12 mm outer diameter were used. The densities of all the specimens, determined with a picnometer at 299 K, were

$6.680 \pm 0.003$ g cm$^{-3}$ fuel sheathings

$6.587 \pm 0.003$ g cm$^{-3}$ rods.

The detailed dimensions of the specimens used are given in table 1. Finally, the frequency can be determined with a relative error of $2 \times 10^{-3}$, introduced mainly by the suspension point. In fact, the temperature was controlled within $\pm 2$ K which gave a relative error of $3 \times 10^{-5}$ in the determination of the resonant frequencies. All the measurements were performed at room temperature.

Table 1
Dimensions of the tubes and rods used

| Specimen | Length (mm) | $2a$ (mm) | $2b$ (mm) |
|---|---|---|---|
| P–1–V | 149.941 | 10.78 | 11.90 |
| P–5–V | 149.745 | 10.78 | 11.90 |
| P–1–M | 149.883 | – | 12.37 |
| P–2–M | 149.959 | – | 12.37 |

## 4. Results

Eq. (9) was evaluated numerically by using a computer program to obtain the resonant frequencies in terms of the elastic constants and the dimensions of the specimens. The details of the computer program are given elsewhere [14].

Fig. 2 shows the numerical results obtained for specimen P–2–M, where the inner radius has been changed from zero (rod) to values near the outer radius. The ratio between the fundamental frequency for the rod, $f_m$, where $a = 0$, and the fundamental frequency for the tube, $f$, has been plotted as a function of the ratio between the inner and the outer radius. A value of $\nu = 0.33$, the measured dimensions and density and $E =$ 98 GPa, as obtained from the measurement of the

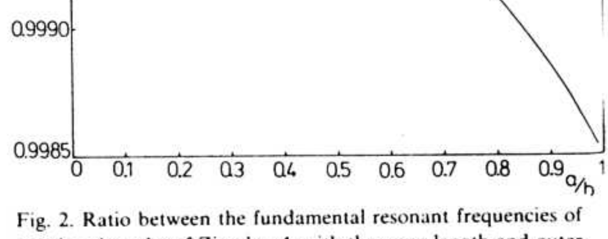


Fig. 2. Ratio between the fundamental resonant frequencies of a rod and a tube of Zircaloy-4, with the same length and outer diameter, as a function of the ratio between the inner and outer radii. Specimen P–2–M: $E = 98.0$ GPa and $\nu = 0.33$.

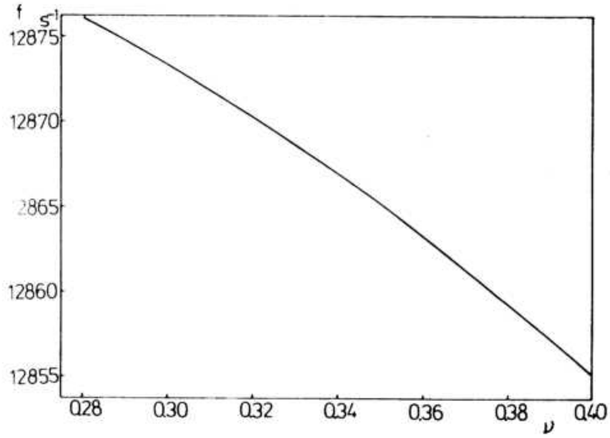


Fig. 3. Fundamental resonant frequency as a function of Poisson's ratio for a Zircaloy-4 tube. Specimen P-5-V: $E = 99.7$ GPa.

resonant frequency and eq. (1), have been used for the numerical calculations. Fig. 3 shows the influence of variations in Poisson's ratio on the numerical results, for tube P-5-V with $E = 99.7$ GPa.

Fig. 4 shows the influence of variations in Young's modulus on the fundamental resonant frequency, plotted as a function of the inner radius, for the tube P-5-V. $\nu = 0.33$ was used for this figure. Plots similar to figs. 2 to 4 can be made for the frequencies of the overtones.

The results given in fig. 2 were verified experimentally by drilling holes of different diameters on rod P-2-M and measuring the fundamental resonant frequencies. These results were compared with the numerical estimate given by eq. (9). As shown in table 2, the measured frequencies, $f_{exp}$, show a relative difference with the numerical results, $f_{theor}$, of the order of $3 \times 10^{-3}$, which is close to the experimental error. As shown in fig. 2 the resonant frequency changes by a maximum of 0.15 % on going from a rod to a tube with a very thin wall.

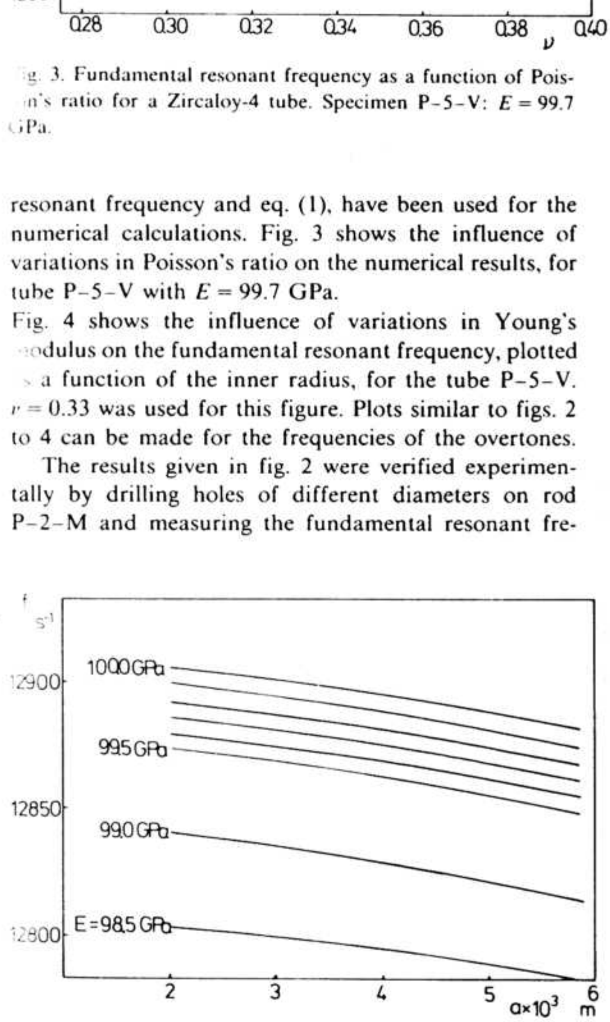


Fig. 4. Fundamental resonant frequency versus inner radius for different values of Young's modulus. Specimen P-5-V: $\nu = 0.33$.

Table 2
Comparison between calculated and measured frequencies for specimen P-2-M and different inner radii

| $a$ (mm) | $f_{exp}(s^{-1})$ | $f_{theor}(s^{-1})$ |
|---|---|---|
| 0 | 12863.1 | 12844.7 |
| 2.09 | 12829.7 | 12842.6 |
| 2.12 | 12813.2 | 12842.5 |
| 2.25 | 12838.5 | 12842.3 |

## 5. Discussion and conclusions

As shown by figs. 2 to 4, the numerical evaluation of the resonant frequencies depends mainly on the value of Young's modulus and only slightly from Poisson's ratio and the ratio between the inner and the outer radius. Fig. 4 was used to determine the correct value for Young's modulus for tubes specimens, from the measured resonant frequencies. The values obtained are indicated by $E(9)$ in table 3. These results are compared with those obtained by using the approximate equations in the same table. $E(1)$ indicates the values obtained with eq. (4). It is seen that the results do not differ more than 0.5 % indicating that, for practical purposes, the approximate equations can be used to obtain Young's modulus from the measured fundamental frequencies.

Fig. 5 shows that differences were encountered, however, for the overtones. In fact, on increasing the order of the harmonic, stronger differences were found between the resonant frequencies for tubes, calculated with the approximate eqs. (1) and (4) and the numerical

Table 3
Comparison between the values obtained for Young's modulus of Zircaloy-4 tubes by using the approximate equations and the numerical solution

| Specimen | $E(1)$ GPa | $E(4)$ GPa | $E(9)$ GPa |
|---|---|---|---|
| P-1-V | 99.4 | 99.5 | 99.8 |
| P-5-V | 99.3 | 99.5 | 99.7 |

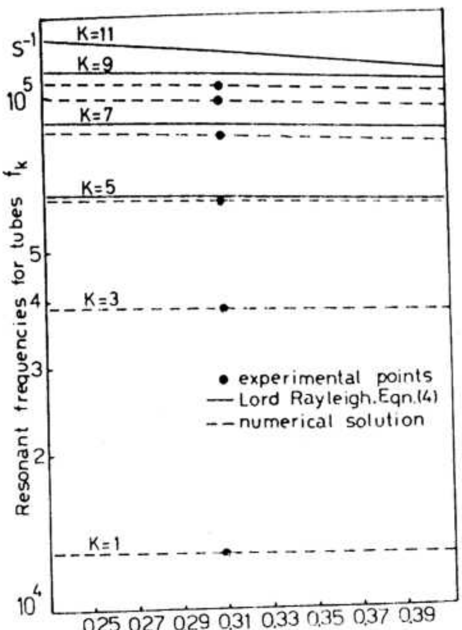


Fig. 5. Resonant frequencies as a function of Poisson's ratio for tubes. Full circles are the measured values. Specimen P–1–V.: $E = 99.8$ GPa.

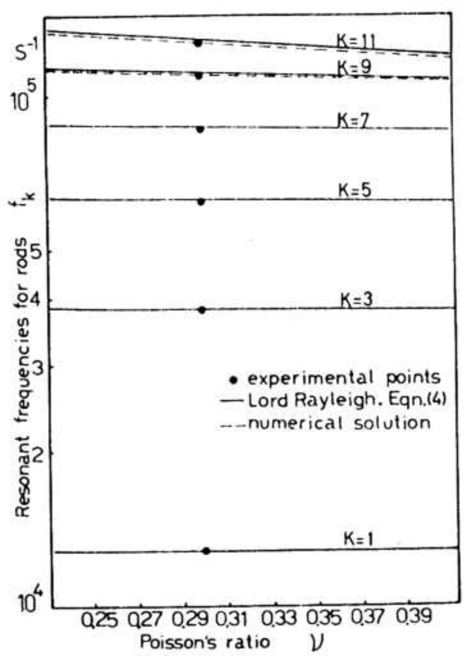


Fig. 6. Resonant frequencies as a function of Poisson's ratio for rods. Full circles are the measured values. Specimen P–1–M.: $E = 98.0$ GPa.

evaluation through eq. (9). This difference is higher than 10 % for $k = 7$. The measured $f_k$ are within 1 % of the values obtained by the numerical solution, even for $k = 9$, 11. For $k = 1$, 3 the numerical solutions are superimposed on the approximate eq. (4). Other approximate equations cannot be indicated in the diagram due to the coincidence of the numerical values.
For rods, on the contrary, the differences between the resonant frequencies calculated either with the approximate equations or numerically or the measured values differ only slightly. As we can see in fig. 6 the experimental results practically coincide with the eqn. (4) and the numerical evaluation are superimposed with that equation.
In conclusion, no substantial differences were found between the values for Young's modulus calculated, from the measured resonant frequencies, by using the approximate equations proposed in the literature and those obtained by using the complete solution of the equations of motion.
No practical differences were found also if the approximate equations are used also for tubes, except for the higher harmonics.

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