a reprint from

Fibre Science and Technology

an International Journal

Edited by

G. HOLISTER

Published by

APPLIED SCIENCE PUBLISHERS LTD

Ripple Road, Barking, Essex, England

APPROXIMATE, ANALYTICAL DETERMINATION OF STEADY STATE TEMPERATURE DISTRIBUTION IN THERMALLY ORTHOTROPIC RODS

P. A. A. LAURA, R. H. GUTIÉRREZ

Institute of Applied Mechanics, Base Naval Puerto Belgrano, 8111 (*Argentina*)

and

G. SÁNCHEZ SARMIENTO

Centro Atómico Bariloche, CNEA, Rio Negro, 8400 (*Argentina*)

SUMMARY

A conformal mapping–variational approach which allows solution of a type of steady state temperature distribution in bars made of composite materials is developed.

INTRODUCTION

Orthotropic elements find wide application in modern technology. On the other hand a survey of the literature reveals that a limited amount of technical papers and reports dealing with heat conduction problems in orthotropic solids is available.

The present paper deals with the determination of the temperature field in an orthotropic prismatic bar of arbitrary cross section where the temperature has a uniform value on one end and is equal to zero on the other faces (Fig. 1).

It is assumed that the problem is governed by the well-known equation[1]

$$k_x \frac{\partial^2 T}{\partial x^2} + k_y \frac{\partial^2 T}{\partial y^2} + k_z \frac{\partial^2 T}{\partial z^2} = 0 \tag{1}$$

where k_x, k_y and k_z are the heat conduction coefficients in the x, y and z directions, respectively and they are assumed to be independent of the temperature.

Fibre Science and Technology 0015-0568/79/0012-0471/$02·25

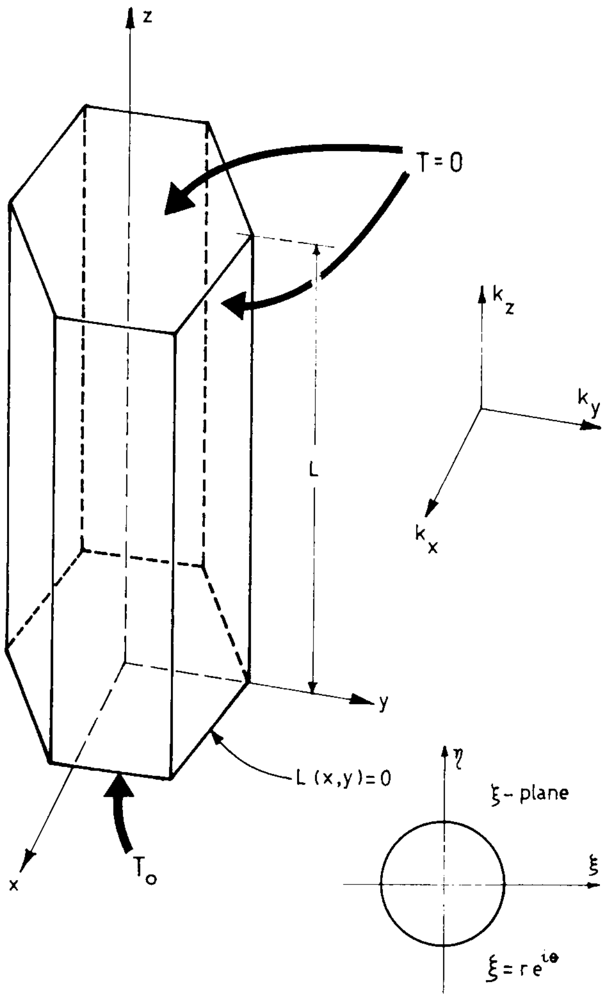

Fig. 1. Thermally orthotropic, prismatic rod of complicated cross section.

APPROXIMATE SOLUTION

Making

$$T(x, y, t) = T_1(x, y)Z(z) \tag{2}$$

and substituting in (1) one obtains

$$k_x \frac{\partial^2 T_1}{\partial x^2} + k_y \frac{\partial^2 T_1}{\partial y^2} + \beta^2 T_1 = 0 \tag{3(a)}$$

$$\frac{d^2Z}{dz^2} - \frac{\beta^2 Z}{k_z} = 0 \tag{3(b)}$$

where β^2 is the separation constant.

Since the temperature is zero for $z = L$, the solution of (3(b)) is simply

$$Z(z) = \mathrm{A} \sinh \frac{\beta}{\sqrt{k_z}}(L - z) \tag{4}$$

Finding the solution of the partial differential equation (3(a)) is equivalent to performing the minimisation of the functional

$$J[T_1] = \iint_D \left[k_x \left(\frac{\partial T_1}{\partial x} \right)^2 + k_y \left(\frac{\partial T_1}{\partial y} \right)^2 - \beta^2 T_1^2 \right] dx\, dy \tag{5}$$

subject to the boundary condition

$$T_1[L(x, y) = 0] = 0 \tag{6}$$

Let

$$w = x + iy = f(\zeta); \qquad \zeta = \xi + i\eta \tag{7}$$

be the functional relation which transforms the given, complicated shape in the w-plane, onto a unit circle in the ζ-plane (Fig. 1).

Admittedly finding $w = f(\zeta)$ is not an easy task, but it is known for many simply and doubly connected shapes of practical significance.[2]

Substituting (7) in (5) one obtains

$$\begin{aligned} J[T_1] = k_x & \iint_C \left[2\,\mathrm{Re}\left[\left(\frac{\partial T_1}{\partial \zeta} \right)^2 \frac{1}{f'^2(\zeta)} \right] + 2 \frac{\partial T_1}{\partial \zeta} \cdot \frac{\partial T_1}{\partial \bar{\zeta}} \frac{1}{\| f'(\zeta) \|} \right] \| f'(\zeta) \| \, d\xi \, d\eta \\ - k_y & \iint_C \left[2\,\mathrm{Re}\left[\left(\frac{\partial T_1}{\partial \zeta} \right) \frac{1}{f'^2(\zeta)} \right] - 2 \frac{\partial T_1}{\partial \zeta} \frac{\partial T_1}{\partial \bar{\zeta}} \frac{1}{\| f'(\zeta) \|} \right] \| f'(\zeta) \| \, d\xi \, d\eta \\ & - \beta^2 \iint_C T_1^2 \cdot \| f'(\zeta) \| \, d\xi \, d\eta \end{aligned} \tag{8}$$

The simplest approximation which satisfies the transformed boundary condition

$$T_1(\zeta, \bar{\zeta})|_{|\zeta|=1} = 0 \tag{9}$$

is probably the expression[3]

$$T_1 \simeq T_{1a}(\zeta, \bar{\zeta}) = \sum_{n=1}^{N} \mathrm{A}_n [1 - (\zeta\bar{\zeta})^n] \tag{10}$$

Substituting (10) in (8) and using the minimisation condition

$$\frac{\partial J[T_{1a}]}{\partial \mathrm{A}_n} = 0 \qquad (n = 1, 2, \ldots, N) \tag{11}$$

one obtains a linear system of equations in the A_n's.

From the non-triviality condition one obtains a secular determinant whose roots are the desired eigenvalues β_n^2's.

It should be clear at this point that expression (10) makes the algorithmic procedure quite simple from the point of view of calculating the separation constants. When expressing the temperature field in its final form, it is considerably more expedient to use the Fourier–Bessel expansion[3]

$$T(\zeta, \bar{\zeta}, z) \simeq \sum_{n=1}^{N} B_n J_0[\alpha_n(\zeta \cdot \bar{\zeta})^{1/2}] \sinh \frac{\beta_n}{\sqrt{k_z}}(L - z) \tag{12}$$

where J_0 is the Bessel function of the first kind and order zero and the α_n's are the roots of $J_0(x)$.

Since

$$T(\zeta, \bar{\zeta}, z)|_{z=L} = T_0 \tag{13}$$

one finally obtains from (12) and (13) and making use of the well-known expressions for the Fourier–Bessel coefficients

$$B_n = \frac{2T_0}{\alpha_n \cdot J_1(\alpha_n) \cdot \sinh \dfrac{\beta_n}{\sqrt{k_z}} L} \tag{14}$$

Substituting (14) in (12) results in the expression

$$\frac{T}{T_0}(\zeta, \bar{\zeta}, z) \simeq 2 \sum_{n=1}^{N} \frac{1}{\alpha_n \cdot J_1(\alpha_n) \sinh \dfrac{\beta_n L}{\sqrt{k_z}}} J_0(\alpha_n r) \sinh \frac{\beta_n}{\sqrt{k_z}}(L - z) \tag{15}$$

In the case of an infinitely long rod ($L \to \infty$), eqn. (15) yields

$$\frac{T}{T_0} \simeq 2 \sum_{n=1}^{N} \frac{1}{\alpha_n J_1(\alpha_n)} J_0(\alpha_n r) \exp\left[(\beta_n/\sqrt{k_z})z\right] \tag{16}$$

NUMERICAL RESULTS

Figure 2 depicts the dimensionless temperature variation along the axis for rods of square and octogonal cross sections,† respectively. The parameter N has been taken equal to 2 in all cases.‡

† The mapping function is available in reference 4.

‡ Obviously the accuracy of the analytical determinations will improve increasing the number of terms of the approximate solution. However, from the point of view of some practical applications it will suffice to take a small number of terms.

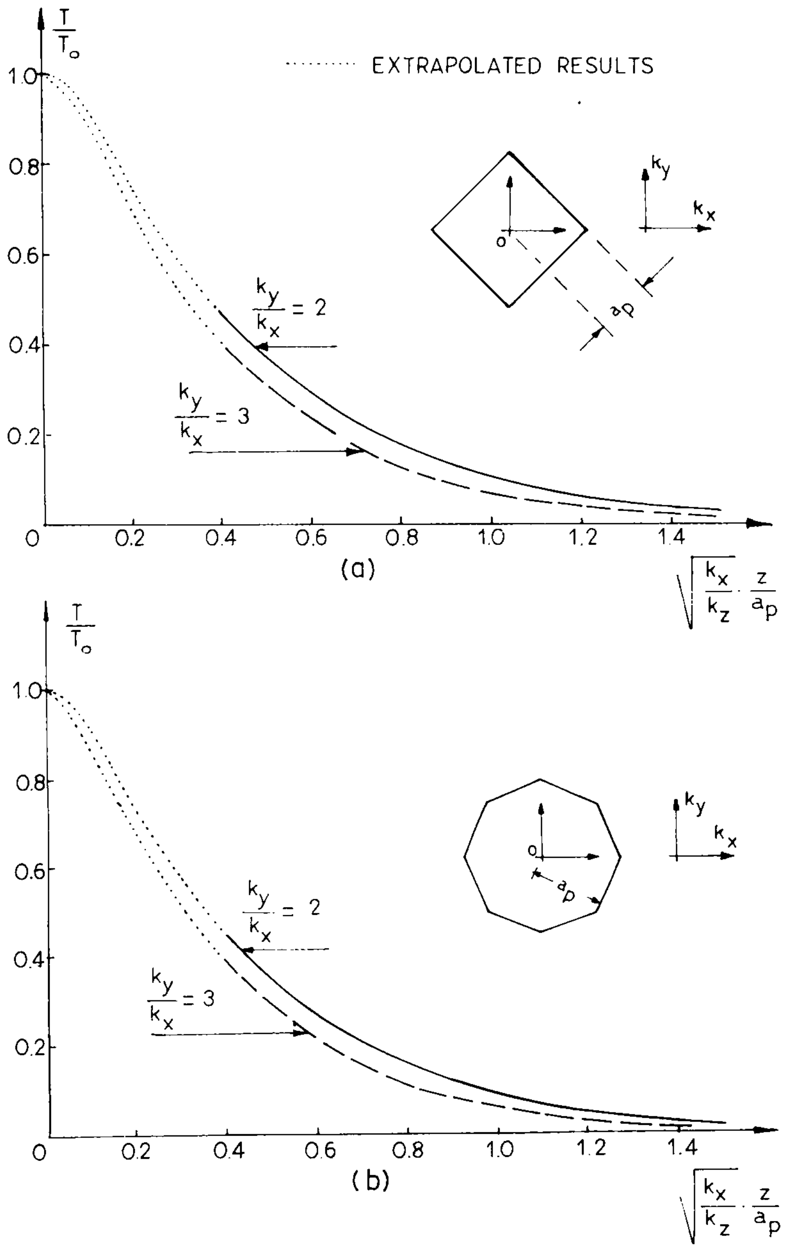

Fig. 2. Temperature variation along the axis of an infinitely long prismatic rod: (a) Square cross section and (b) octagonal cross section.

It is important to point out that very good agreement has been obtained between results obtained by means of the finite element method and conformal mapping–variational techniques predictions in the case of unsteady state situations in orthotropic plates.[4]

REFERENCES

1. H. S. CARSLAW and J. C. JAEGER, *Conduction of Heat in Solids*, Oxford Clarendon Press (2nd Edition), 1959.
2. L. V. KANTOROVICH and V. I. KRYLOV, *Approximate Methods of Higher Analysis*, Interscience Publishers, New York, 1958.
3. P. A. A. LAURA and R. ERCOLI, A solution of the unsteady diffusion equation in an arbitrary, doubly connected region, *Nuclear Engineering and Design*, **23** (1972) pp. 1–9.
4. P. A. A. LAURA, R. H. GUTIERREZ, G. SANCHEZ SARMIENTO and F. BASOMBRIO, 'Transient Temperature Distribution in Thermally Orthotropic Plates of Complicated Boundary Shape', Institute of Applied Mechanics, Publication No. 78-18, June 1978 (submitted for publication).