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VARIATIONAL APPROACH FOR THE N-STATE SPIN AND GAUGE POTTS MODEL

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A hamiltonian variational treatment is applied both to the spin Potts model and to its gauge version for any number of states N and spatial dimensions $d \geqslant 2$. Regarding the former we reproduce the correct critical coupling and latent heat for not too low N and d. For the latter, our approach turns the gauge theory into an equivalent d-dimensional classical spin model, which evaluated for $d+1=4$ gives results in agreement with $1/N$ expansions.

1. Introduction

Recently the Potts model, both in its global symmetric spin formulation and in the locally gauge-invariant version, has received much attention as a theoretical laboratory as well as for its practical applications. In the present work we wish to study it according to a hamiltonian variational approach which is of easy use and turns out to give good results.

The spin Potts model [1] has been extensively studied beginning with the Landau theory [2] and continuing with the perturbative series [3], cluster variational methods [4], renormalization group techniques [5], Monte Carlo simulations [6] and $1/N$ expansions [7]. There is now strong evidence [6, 7] that for $N \geqslant 3$ and at least $d+1=3$ dimensions, the model experiences a first-order transition between disordered and ordered phases. For the $d+1=2$ case the transition is of second order for $N \leqslant 4$ and of first order otherwise [8]. All the above references consider the lagrangian version. For the hamiltonian formalism, which has the virtue of treating

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exactly one dimension, leaving the approximations for the other d ones, a $1/N$ expansion [9] has shown that the phase transition is of first order for $d+1=3$ and $N \gtrsim 2.6-2.8$. On the contrary, block-spin methods [10] always give second-order transitions in agreement with the general tendency of the renormalization group, also shared by the Migdal-Kadanoff lagrangian version [11], of giving continuous transitions unless dilution is introduced [12].

Regarding the gauge Potts theory, whereas for $d+1=3$ it is dual to the spin model, for $d+1=4$ it is self-dual and strong coupling and $1/N$ expansions [13] have shown that there is a first-order transition for $N \gtrsim 1$ from a confined to a free phase.

One of the purposes of this work is to indicate that a simple variational treatment is easily able to produce results in fair agreement with those of ref. [9] for the spin model and generalized to any number of dimensions. Because our method is of the mean-field type, it cannot be expected to give good results either for very small N – it predicts first-order transitions for N strictly larger than 2 – or for too small d – the transition turns out to be of first order even for $d+1=2$. Therefore, it must be taken as a reasonable approximation for not too low N and d. In a certain sense it gives results similar to those of the self-consistent mean-field approach for the lagrangian formalism [14]. However, it is apparent that our method is much simpler than the self-consistent calculation one might attempt for the hamiltonian formalism, giving, moreover, an intuitive insight into the ground state.

Moreover, we also apply the hamiltonian variational method to the gauge model for the physically relevant $d+1=4$ case, though our expressions are valid for any dimension. It is seen that we obtain good agreement with ref. [13] already for $N=2$, showing that for this dimensionality the mean-field ideas are accurate enough. The gauge-invariant variational approach which we use [15, 16] turns the $d+1=4$ gauge theory into an equivalent 3-dimensional classical spin model. Instead of applying high and low temperature expansions which would be cumbersome for general N, we calculate the necessary statistical averages in the mean-field approximation [14] which one expects to be accurate in three dimensions.

The plan of the paper is the following. In sect. 2 we study the spin Potts model. In sect. 3 we apply our new version of the variational approach to the Z(2) gauge model, equivalent to the $N=2$ Potts case, to show the remarkable agreement with the previous more elaborate treatment [16]. Sect. 4 describes the method for the N-state gauge Potts model, and a few conclusions are included in sect. 5.

2. Spin Potts model

The hamiltonian for the N-state Potts model is [9, 17]

$$H = -\sum_{ss'} \frac{1}{N} \sum_{k=1}^{N-1} \left(Q_s Q_{s'}^{\dagger}\right)^k - T \sum_{s} \frac{1}{N} \sum_{k=1}^{N-1} \left(P_s^k - 1\right), \tag{1}$$

where s and s' are neighbouring sites in a d-dimensional lattice, T a temperature-like coupling constant and the matrices Q and P satisfy

$$Q_s P_s = e^{-i2\pi/N} P_s Q_s. \tag{2}$$

In the representation where Q is diagonal the hamiltonian takes the form

$$H_Q = -\sum_{ss'} \left(\delta_{m_s m_{s'}} - \frac{1}{N} \right) - T \sum_s \frac{1}{N} \sum_{k=1}^{N-1} \left(P_s^k - 1 \right), \tag{3}$$

with $m_s = 0, 1 \ldots N-1$, whereas in that for diagonal P

$$H_P = -\sum_{ss'} \frac{1}{N} \sum_{k=1}^{N-1} \left(Q_s Q_{s'}^{\dagger} \right)^k - T \sum_s \left(\delta_{m_s 0} - 1 \right). \tag{4}$$

The variational approach, as applied to other spin and gauge models [15, 16], consists in assuming a form for the ground state which depends on one or more parameters introduced in a physically reasonable way and which reproduces the known low and high temperature situations. The energy is minimized under variations of the parameters: the appearance of a discontinuity in one of the derivatives of this minimum with respect to the coupling constant indicates a phase transition of the corresponding order.

In the Q-representation we write the variational state as

$$|\varepsilon\rangle = \prod_s |\varepsilon\rangle_s = \prod_s \frac{1}{\sqrt{1 + (N-1)\varepsilon^2}} \begin{pmatrix} 1 \\ \varepsilon \\ \vdots \\ \varepsilon \end{pmatrix}_s, \tag{5}$$

where ε must be zero for $T = 0$ (ordered state) and 1 for $T \to \infty$ (disordered state). The fact that all the Q-eigenstates, apart from the one chosen at $T = 0$, have the same weight reflects the type of disordering processes induced by the Potts interaction.

The variational energy is easily calculated giving

$$E(\varepsilon) = \langle \varepsilon | H_Q | \varepsilon \rangle = d\left[-A(N, \varepsilon) + \tau B(N, \varepsilon) \right], \tag{6}$$

where

$$A(N, \varepsilon) = \frac{N-1}{N} \frac{(\varepsilon^2 - 1)^2}{\left[1 + (N-1)\varepsilon^2\right]^2}, \qquad B(N, \varepsilon) = \frac{(\varepsilon - 1)^2}{1 + (N-1)\varepsilon^2}, \qquad \tau = \frac{T}{d} \frac{N-1}{N}.$$

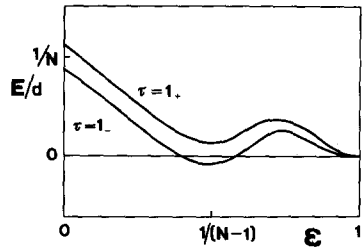

Fig. 1. Variational energy for spin Potts model, eq. (6). Curves for $\tau = 1_-$ and 1_+ indicate the first-order transition for $N > 2$.

Eq. (6) can be analysed numerically but its properties are seen by simple inspection. For $\varepsilon = 1$ there is always a double zero, and for $\tau = 1$ another double zero appears at $\varepsilon_c = 1/(N-1)$, where

$$A(\varepsilon_c) = B(\varepsilon_c) = \frac{(N-2)^2}{N(N-1)}. \tag{7}$$

Therefore, as suggested by fig. 1, for $\tau = 1_+$ the absolute minimum is at $\varepsilon = 1$ (disordered phase) and for $\tau = 1_-$ is at ε_c (partially ordered phase). The critical temperature

$$T_c = d\frac{N}{N-1} \tag{8}$$

can be compared with the $1/N$ expansion result [9] for $d+1=3$:

$$T_c = 2 + 4/(3N) + 14/(9N^2) - 1/(270N^3) + \cdots.$$

The characterization of a first-order transition is given by the latent heat

$$C = \frac{\partial E_{\min}}{\partial T}\bigg|_{\tau=1_-} - \frac{\partial E_{\min}}{\partial T}\bigg|_{\tau=1_+} = \left(\frac{N-2}{N}\right)^2 \tag{9}$$

which compared with the $1/N$ result,

$$C = (1-1/N)\left[1 - 31/(9N) + 257/(162N^2) + 59668/(145800N^3) + \cdots\right],$$

predicts the same large N values, whereas for $N=3$ it gives $C = \frac{1}{9}$ instead of 0.029.

If one prefers to use the P-representation the result is the same since

$$\begin{pmatrix} 1 \\ \varepsilon \\ \vdots \\ \varepsilon \end{pmatrix} = \begin{pmatrix} \varepsilon \\ \varepsilon \\ \vdots \\ \varepsilon \end{pmatrix} + \begin{pmatrix} 1-\varepsilon \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \sqrt{N} \begin{bmatrix} \varepsilon \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \frac{1}{\sqrt{N}} \begin{bmatrix} 1-\varepsilon \\ 1-\varepsilon \\ \vdots \\ 1-\varepsilon \end{bmatrix} = \frac{\varepsilon(N-1)+1}{\sqrt{N}} \begin{bmatrix} 1 \\ \mu \\ \vdots \\ \mu \end{bmatrix},$$

where square brackets indicate the P-representation and $\mu = (1-\varepsilon)/[\varepsilon(N-1)+1]$. Using a state $|\mu\rangle$ and eq. (4) we obviously get $E(\mu) = E(\varepsilon)$.

It is clear that a self-consistent mean-field method for the hamiltonian formalism, analogous to the lagrangian one of ref. [14], is considerably more complicated since it would imply the diagonalization of the $N \times N$ equation (4) with the replacement $Q_{s'} = \rho e^{i\varphi}$ and the solution of the consistency equation $\langle E_0|Q|E_0\rangle = \rho e^{i\varphi}$, where E_0 is the lowest eigenvalue. Therefore, the variational approach seems to be the simplest of the mean-field type.

We also remark that the hamiltonian result $T_c^{(H)} \to$ constant for $N \to \infty$ is consistent with the lagrangian behaviour $T_c^{(L)} \to 2/\ln N$ in $1+1$ dimensions. This is due to the fact that the former corresponds to a very anisotropic situation between spatial and temporal couplings β_x and β_t, whereas the latter refers to the isotropic case. Both results come out from the self-duality condition for $d+1=2$ in the lagrangian formalism,

$$(e^{\beta_x} - 1)(e^{\beta_t} - 1) = N.$$

For the isotropic case $\beta_x = \beta_t$ this gives $\beta_c^{-1} \simeq 2/\ln N$ for large N. On the other hand, the hamiltonian version is obtained performing the continuous time limit [17] $\beta_x \simeq t$ and $e^{-\beta t} \simeq \lambda t$, with t small, so that $\lambda \simeq N^{-1}$. Since, according to the transfer matrix formalism [18], one may rewrite the partition function as

$$Z \propto \prod \exp\left[\ln\left(1 + \frac{N}{e^{\beta_t} - 1}\right)\frac{1}{N}\sum_k P_s^k + \beta_x \frac{1}{N}\sum_k \left(Q_s Q_{s+x}^{\dagger}\right)^k\right],$$

the continuous time limit provides the hamiltonian equation (1) with $T = N\lambda$. Therefore, it follows that $T_c = 1$, as it must be.

3. Z(2) gauge theory in 3 + 1 dimensions

We wish to show that the combined use of the hamiltonian variational approach and of the mean-field approximation for the equivalent Ising model, which is the particular case of the Potts model for $N = 2$, leads to a quite satisfactory description of this theory. This will be useful for the application of the method to the general N-state gauge Potts model.

For $N=2$ the operators P and Q which appear in the Potts hamiltonian correspond to the Pauli matrices σ_1 and σ_3, respectively. The gauge-invariant hamiltonian is given by [19;17]

$$H = -\sum_{\text{links}} \sigma_{1_\ell} - \lambda \sum_{\text{plaquettes}} (\sigma_3\sigma_3\sigma_3\sigma_3)_{\text{p}}. \tag{10}$$

The local invariance of this hamiltonian corresponds to $G_s H G_s = H$, where $G_s = \Pi_{\ell\in s}\sigma_{1_\ell}$ is the product of all σ_1 matrices of links belonging to the site s.

As variational state we use the same assumption made for the spin case. But due to gauge invariance we symmetrize it, performing all possible local transformations

$$|0\rangle = \prod_{\text{sites}} (1+G_s) \prod_{\text{links}} \begin{pmatrix} \cos\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta \end{pmatrix}_\ell. \tag{11}$$

The norm of this state is proportional to the partition function of a classical 3-dimensional Ising model [15,16]

$$\langle 0|0\rangle \propto \sum_{\{\mu\}} (\sin\theta)^{-1/2\Sigma_{ss'}\mu_s\mu_{s'}} = \sum_{\{\mu\}} e^{\beta\Sigma_{ss'}\mu_s\mu_{s'}}, \tag{12}$$

where $\mu_s = \mp 1$ indicates whether a gauge transformation has been performed on site s or not, and s, s' denote neighbouring sites.

The expectation value of the hamiltonian, eq. (10), in the variational state, eq. (11), gives the energy per link:

$$E(\beta) = -\frac{\langle e^{-2\beta\mu_s\mu_{s+\ell}}\rangle}{\langle 1\rangle} - \lambda(1-e^{-4\beta})^2 \frac{\langle \Pi_{\ell\in \text{p}}\delta_{\mu_s\mu_{s+\ell}}\rangle}{\langle 1\rangle}, \tag{13}$$

where

$$\langle \mathcal{O}\rangle = \sum_{\{\mu\}} \mathcal{O}(\mu) e^{\beta\Sigma\mu_s\mu_{s'}}.$$

Eq. (13) was analysed in ref. [16] with low and high temperature expansions giving the result that the minimum of $E(\beta)$ defines a first-order transition, i.e. $dE_{\min}(\lambda)/d\lambda$ exhibits a discontinuity for $\lambda = 0.94$. We wish to show that the same feature may be obtained in a much easier way using a mean-field approximation for the classical statistical averages of eq. (13).

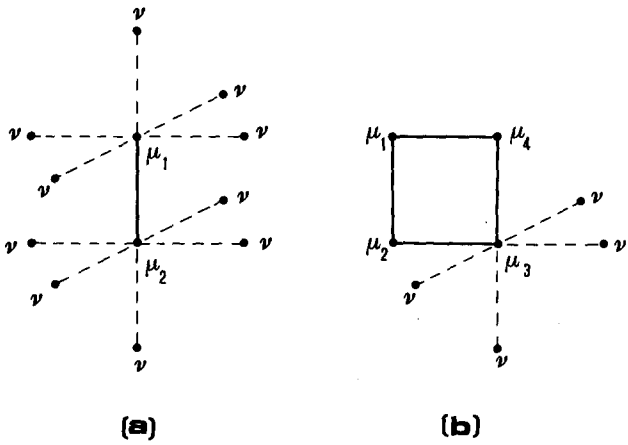

Fig. 2. Mean-field calculation for the equivalent classical system eqs. (13), (20) and (23). (a) Link terms are evaluated summing over configurations of sites 1 and 2 and replacing all other variables by the mean field ν. (b) Plaquette terms consider all possible configurations of sites 1,2,3 and 4.

For this purpose we evaluate the link term (see fig. 2a) as

$$\frac{\langle e^{-2\beta\mu_1\mu_2}\rangle}{\langle 1\rangle} = \frac{\Sigma_{\{\mu_1\mu_2\}} e^{-\beta\mu_1\mu_2} e^{\beta(2d-1)\nu(\mu_1+\mu_2)}}{\Sigma_{\{\mu_1\mu_2\}} e^{\beta\mu_1\mu_2} e^{\beta(2d-1)\nu(\mu_1+\mu_2)}}$$

$$= \frac{e^{-\beta}\cosh[2\beta(2d-1)\nu] + e^{\beta}}{e^{\beta}\cosh[2\beta(2d-1)\nu] + e^{-\beta}}, \tag{14}$$

and the plaquette contribution (fig. 2b) as

$$\frac{\langle \delta_{\mu_1\mu_2}\delta_{\mu_2\mu_3}\delta_{\mu_3\mu_4}\delta_{\mu_4\mu_1}\rangle}{\langle 1\rangle} = \frac{\Sigma_{\{\mu_1\cdots\mu_4\}} \prod_{i=1}^{4} \left(\delta_{\mu_i\mu_{i+1}} e^{\beta\mu_i\mu_{i+1}} e^{2\beta(d-1)\nu\mu_i}\right)}{\Sigma_{\{\mu_1\cdots\mu_4\}} \prod_{i=1}^{4} (e^{\beta\mu_i\mu_{i+1}} e^{2\beta(d-1)\nu\mu_i})}$$

$$= \frac{e^{4\beta}\cosh[8(d-1)\beta\nu]}{e^{4\beta}\cosh[8(d-1)\beta\nu] + 4\cosh[4(d-1)\beta\nu] + e^{-4\beta} + 2}, \tag{15}$$

where $2d$ is the coordination number, and the mean field ν is determined by the self-consistent equation $\langle\mu\rangle = \nu = \mathrm{tgh}(2\beta d\nu)$. In eqs. (14) and (15) we have correctly treated only the sites involved in the link and plaquette respectively, considering the others in a mean field way. We have checked that in enlarging the number of the exactly considered sites one does not significantly change the result.

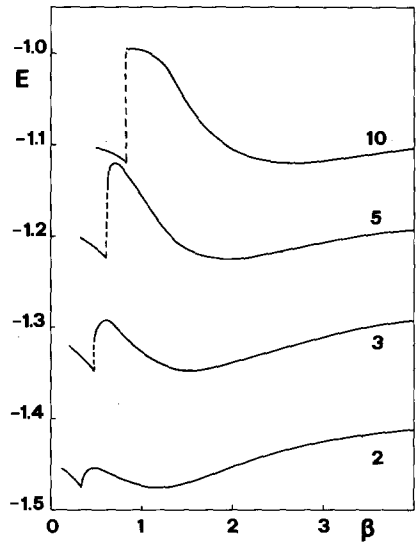

Fig. 3. Variational energy for Potts gauge theory eqs. (20), (23), (27), (28) and (29) for λ_c. The jump from the left to the right minimum indicates the first-order transition. Numbers on the curves denote the value of N. The curve for $N=2$ coincides with the Z(2) energy, eq. (13), replacing β by 2β since $(\beta\mu\mu')_{\text{Ising}} = (2\beta\delta_{\mu\mu'}-\beta)_{\text{Potts}}$.

It is clear that for $2d\beta<1$ the solution of the self-consistent equation is $v=0$, but for larger β a non-vanishing mean field appears continuously. This produces, for $d=3$, a second minimum of $E(\beta)$ which, as shown in fig. 3, becomes the absolute minimum for $\lambda>0.9$. The discontinuous change in the variational state corresponds to a first-order transition, and the quantitative agreement of fig. 3 with that obtained with low and high temperature expansions is remarkable.

4. Gauge Potts model in 3 + 1 dimensions

The N-state Potts model is described by the hamiltonian

$$H=-\sum_{\ell}\frac{1}{N}\sum_{k=0}^{N-1}(P_\ell)^k-\lambda\sum_{\mathrm{p}}\frac{1}{N}\sum_{k=0}^{N-1}\left(\prod_{\ell\in\mathrm{p}}Q_{s,s+\ell}\right)^k, \tag{16}$$

where the plaquette term in the example of fig. 2b is $Q_{12}Q_{23}Q_{34}Q_{41}$. This hamiltonian has the local symmetry $H=G_s^kHG_s^{-k}$, with

$$G_s=\prod_{\ell\in s}P_\ell^\eta,$$

where η is a sign, positive for links in the positive directions and negative otherwise.

We wish to apply to eq. (16) the already discussed variational approach. Thus we build the gauge-invariant state

$$|0\rangle = \prod_s \sum_{\mu_s=0}^{N-1} (G_s)^{\mu_s} \prod_\ell |\varepsilon\rangle_\ell, \tag{17}$$

where $|\varepsilon\rangle_\ell$ is the state introduced in eq. (5) but referred to link variables.

The norm of $|0\rangle$ now corresponds to the partition function of a classical d-dimensional Potts spin model since

$$\langle 0|0\rangle = \chi^{N_\ell} \sum_{\{\mu\}} \prod_\ell e^{\beta\delta_{\mu_s,\mu_{s'}}}, \tag{18}$$

with ss' determining the link ℓ. N_ℓ is the number of links and

$$\chi = \frac{2\varepsilon + (N-2)\varepsilon^2}{1 + (N-1)\varepsilon^2} = e^{-\beta} \tag{19}$$

defines the equivalent temperature.

The link and plaquette terms of eq. (16) take the form of statistical averages for the above classical d-dimensional Potts model. Thus

$$\begin{aligned} -E_\ell &= \frac{\langle 0|(1/N)\sum_{k=0}^{N-1}(P_\ell)^k|0\rangle}{\langle 0|0\rangle} \\ &= \frac{1}{N}\frac{\Sigma_k \Sigma_{\{\mu\}} \chi^{1-\delta_{\mu_s,\mu_{s+\ell}+k}} \prod_{\ell'\neq\ell} \chi^{1-\delta_{\mu_{s'},\mu_{s'+\ell'}}}}{\Sigma_{\{\mu\}} e^{\beta\Sigma_\ell \delta_{\mu_s,\mu_{s+\ell}}}} \\ &= \frac{e^\beta + N - 1}{N} \frac{\langle e^{-\beta\delta_{\mu_s,\mu_{s+\ell}}}\rangle}{\langle 1\rangle}. \end{aligned} \tag{20}$$

We turn now to the plaquette term. Considering the average of an operator $(Q_{s,s+\ell})^k$ we obtain

$$\begin{aligned} -E_p &= \frac{1}{N}\frac{\Sigma_{k=0}^{N-1}\langle 0|\Pi_{\ell\in p}(Q_{s,s+\ell})^k|0\rangle}{\langle 0|0\rangle} \\ &= \frac{1}{\langle 1\rangle}\frac{1}{N}\sum_{k=0}^{N-1}\prod_{\ell\in p}\left\{a\delta_{\mu_s,\mu_{s+\ell}} + b[1 + e^{i(2\pi/N)k(\mu_{s+\ell}-\mu_s)}] + \left(c - a\delta_{\mu_s,\mu_{s+\ell}}\right)\delta_{k0}\right\}, \end{aligned} \tag{21}$$

where, taking eq. (19) into account,

$$a=\frac{N\varepsilon(1-\varepsilon)^2}{\left[1+(N-1)\varepsilon^2\right]\left[2+(N-2)\varepsilon\right]},$$

$$b=\frac{1-\varepsilon}{2+(N-2)\varepsilon}, \qquad c=\frac{N\varepsilon}{2+(N-2)\varepsilon}. \tag{22}$$

Performing the summation over k of the product of the four links we obtain according to the notation of fig. 2b, and calling $\delta_{\mu_s,\mu_{s+\ell}}\equiv\delta_{s,s+\ell}$,

$$\begin{aligned}-E_{\mathrm{p}}=\frac{1}{\langle 1\rangle}\langle\alpha_0 1+\alpha_1(\delta_{12}+\delta_{23}+\delta_{34}+\delta_{41})+\alpha_2(\delta_{13}+\delta_{24})+\alpha_3(\delta_{12}\delta_{34}+\delta_{23}\delta_{41})\\ +\alpha_4(\delta_{12}\delta_{23}+\delta_{23}\delta_{34}+\delta_{34}\delta_{41}+\delta_{41}\delta_{12})\\ +\alpha_5\delta_{12}\delta_{23}\delta_{34}\delta_{41}+\alpha_6\delta_{1+3,2+4}\rangle,\end{aligned} \tag{23}$$

where the coefficients α_i are given in the appendix. Even though eq. (23) involves statistical averages of several δ-functions, it turns out that for $N=2$ it coincides with the plaquette term of eq. (13) because of elementary identities among δ-functions on links and plaquette.

To evaluate the statistical averages of eqs. (20) and (23) we use the same mean field approximation described in sect. 3, calculating exactly the sites μ which form the solid links of fig. 2 and putting an average field for the other sites μ'. Following the approach of ref. [14] we write

$$\delta_{\mu\mu'}=\frac{1}{N}\sum_{k=0}^{N-1}\mathrm{e}^{i(2\pi/N)k\mu}\mathrm{e}^{-i(2\pi/N)k\mu'}.$$

Assuming a probability distribution for μ' of the type

$$P(\mu')=p\delta_{\mu'0}+(1-p)(1-\delta_{\mu'0}), \tag{24}$$

which corresponds to a spin pointing with preference along the state 0 and treats all the others on the same footing, we obtain

$$\delta_{\mu\mu'}=\frac{Np-1}{N-1}\frac{1}{N}\sum_{k=0}^{N-1}\mathrm{e}^{i(2\pi/N)k\mu}=\nu\delta_{\mu0}. \tag{25}$$

The mean field ν is determined by the self-consistent equation

$$\nu=\frac{\langle\mathrm{e}^{i(2\pi/N)\mu}\rangle}{\langle 1\rangle}=\frac{\mathrm{e}^{2d\beta\nu}-1}{\mathrm{e}^{2d\beta\nu}+N-1}. \tag{26}$$

We thus obtain for the link average (fig. 2a)

$$\frac{\langle e^{-\beta\delta_{\mu_1\mu_2}}\rangle}{\langle 1\rangle}=\frac{e^{2\varphi}+2(N-1)e^{\varphi}+(N-1)^2}{e^{\beta+2\varphi}+2(N-1)e^{\varphi}+(N-1)e^{\beta}+(N-1)(N-2)}, \tag{27}$$

with $\varphi=\beta(2d-1)v$.

The statistical averages of eq. (23), calculated in a similar way according to fig. 2b, are of the form

$$\langle\ \rangle_i=\sum_{m,n=0}^{4} f_{mn}^{(i)}e^{m\beta+n\gamma}, \tag{28}$$

where $\gamma=2\beta(d-1)v$ and the index i is in correspondence with that of α_i appearing in eq. (23) (e.g. $\langle\ \rangle_1=\langle\delta_{12}\rangle$). The coefficients $f_{mn}^{(i)}$ are reported in the appendix.

Calculating for $d=3$

$$E(\beta)=\frac{1}{N_l}\frac{\langle 0|H|0\rangle}{\langle 0|0\rangle}=E_l+\lambda E_p \tag{29}$$

for different values of N, one obtains a first-order phase transition for λ close to 1, the value required by self-duality, as shown in fig. 3. The critical values λ_c are reproduced in fig. 4. One must remark that for $N\geqslant 3$ the mean field v jumps discontinuously from zero to a finite value where β passes through a critical point which corresponds to the left minimum of fig. 3. However for $N=2$, equivalent to the Z(2) gauge theory, though v starts increasing without discontinuity, the phase transition is still of first order. In fact at the critical value of λ the two minima are

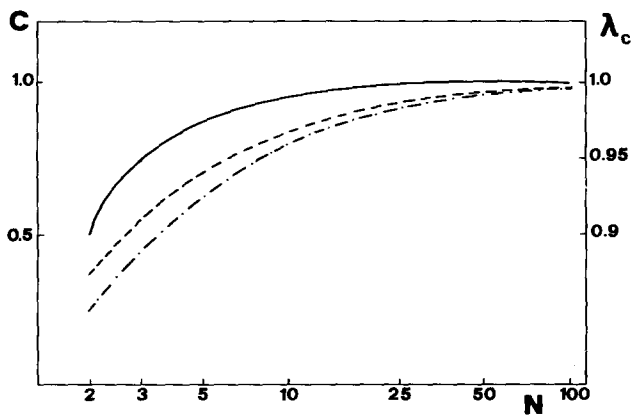

Fig. 4. Critical coupling λ_c (solid curve) and latent heat C, eq. (30), (dashed curve) from the variational approach to gauge Potts theory. Dashed-dotted curve represents the latent heat from $1/N$ expansion [13].

equal and the transition corresponds to a jump of the vacuum from one minimum to the other. The vacuum energy is therefore always continuous. Only the first derivative of the minimum energy exhibits a discontinuity defined as the latent heat

$$C = \left.\frac{\partial E}{\partial \lambda}\right|_{\lambda=\lambda_{c_+}} - \left.\frac{\partial E}{\partial \lambda}\right|_{\lambda=\lambda_{c^-}} = E_p^{(+)} - E_p^{(-)}, \tag{30}$$

where $E_p^{(\pm)}$ is the plaquette contribution for the (right, left) minimum corresponding to $(\lambda = \lambda_c, \lambda = \lambda_c)$. The numerical values of C are shown in fig. 4 from which one sees that they compare favourably with those obtained from $1/N$ expansions [13].

These results for critical coupling and latent heat indicate therefore that this variational approximation, combined with the mean field approach, works very well for $3+1$ dimensions and $N \geqslant 2$ leaving open the possibility for further applications to other gauge theories.

5. Conclusions

We have applied the variational hamiltonian approach to both the spin and gauge Potts models as useful laboratories to check approximate methods. The results are quite encouraging since, apart from very small dimensions and number of states for the spin case, the comparison with other methods is quite successful. In particular for the gauge theory in $3+1$ dimensions, already for the 2-state model the first-order transition is correctly reproduced.

The virtues of our approach rely on the simplicity of the calculations and the physical insight which one obtains into the ground state. For the gauge theory our variational method leads to statistical averages over the classical spin model in one less dimension. The application to it of the mean field theory gives, with much less effort, results quite close to those obtained with low and high-temperature expansions.

We may therefore envisage further applications to slightly more complicated models such as general $Z(N)$ systems and gauge theories in interaction with matter fields. Could one get a sensible extrapolation to $N = 1$ state Potts model, one might also explore the percolation limit.

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Appendix

We include here some formulae relevant for the gauge Potts model.

The sum over the powers k of the plaquette terms gives the statistical averages appearing in eq. (23) whose coefficients are

$$\alpha_0 = 2b^4 + \frac{c}{N}(32ab^3 + 24ab^2c + 8abc^2 + ac^3),$$

$$\alpha_1 = 2b^4 + 3ab^3 - \frac{8}{N}ab^3,$$

$$\alpha_2 = 2b^4,$$

$$\alpha_3 = 2ab^3 + 2a^2b^2 - \frac{1}{N}a^2b^2,$$

$$\alpha_4 = 4ab^3 + 2a^2b^2 - \frac{4}{N}a^2b^2,$$

$$\alpha_5 = 12a^2b^2 + 8a^3b + a^4 - \frac{1}{N}(a^4 + 8a^3b),$$

$$\alpha_6 = 2b^4,$$

where a, b and c are defined in eq. (22).

TABLE 1
Coefficients $f_{mn}^{(i)}$ corresponding to the mean-field evaluation of statistical averages eq. (23)

m,n \ i	0	1	2	3	4	5	6
4,4	1	1	1	1	1	1	1
4,0	N_1	N_1	N_1	N_1	N_1	N_1	N_1
2,3	$4N_1$	$2N_1$	$2N_1$		N_1		
2,2	$4N_1$	$2N_1$		$2N_1$			$4N_1$
2,1	$4N_1$	$2N_1$	$2N_1$		N_1		
2,0	$6N_{12}$	$3N_{12}$	$2N_{12}$	N_{12}	N_{12}		$2N_{12}$
1,2	$4N_{12}$	N_{12}					
1,1	$8N_{12}$	$2N_{12}$					
1,0	$4N_{123}$	N_{123}					
0,2	$2(N_1 + N_{12})$		$2N_1 + N_{12}$				$2N_1$
0,1	$4(N_{12} + N_{123})$		$2N_{12}$				$3N_{12} + N_1\theta(N \geqslant 3) + L$
0,0	$N_{12} + 2N_{123} + N_{1234}$		$N_{12} + N_{123}$				$N_{12}\theta(N \geqslant 3) + N_{123} - L$

$$N_1 = N - 1, \qquad N_{12} = N_1(N-2), \qquad N_{123} = N_{12}(N-3),$$

$$N_{1234} = N_{123}(N-4), \qquad N_{13} = N_1(N-3),$$

$$L = N_{13} + \left[\cos\left(\tfrac{1}{2}\pi N\right)\right]^2.$$

The statistical averages of the plaquette term using the mean-field approximation give results of the form of eq. (28) whose non-vanishing coefficients $f_{mn}^{(i)}$ are reproduced in table 1.

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