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COMPUTATIONAL ASPECTS OF GRID FREE HYBRID METHODS IN FREE AND FORCED CONVECTION

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RESUMEN

Las dificultades encontradas en la resolución numérica de problemas de transmisión de calor y materia en la circulación de fluidos con baja viscosidad es bien conocida. La presencia de discontinuidades fuertes y débiles tales como ondas de choque, ondas de rarefacción y capas límite hace que los métodos dependientes de la malla (por ejemplo diferencias finitas o elementos finitos) produzcan resultados no siempre satisfactorios. Para dar una respuesta a este problema se desarrollaron los denominados métodos independientes de la malla que son una combinación de soluciones analíticas y técnicas de muestreo estadístico.

En este trabajo se introducen el método de la elección aleatoria y el método de la placa aleatoria conjuntamente con un método híbrido para la solución numérica de problemas de recirculación en régimen forzado y libre en cavidades cerradas.

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El método híbrido consiste en la aproximación de la circulación viscosa de un fluído incompresible en una cavidad rectangular por la circulación en capa límite en la zona adyacente a los bordes, de un fluido incompresible, y, por una circulación de un fluido compresible inviscido en la zona alejada de los bordes.

Las ventajas de esta metodología son:
a) El método de la elección aleatoria tiene resolución casi infinita, b) el método de la elección aleatoria y el método de la placa aleatoria son incondicionalmente estables e independientes de la malla y c) la aproximación en capa límite se tradúce en una redúcción considerable del volúmen de cálculo.

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INTRODUCTION AND GENERAL CONSIDERATIONS

The difficulties encountered in the numerical solution of recirculating flows at high Reynolds numbers when mesh dependent methods are used are well documented. The main difficulty appears in the simulation of the boundary layer whose thickness is of O $(R^{-1/2})$. Since it is presumably necessary that few grid points must fall within its thickness this implies that for higher Reynolds numbers the number of grid points as well as the amount of computational work required to obtain a solution would be prohibitive. The situation is more critical, in forced convection in liquid flows, in the presence of thermal boundary layers whose thickness is of order O $(R^{-1/2}Pr^{-1/2})$. Numerical viscosity and computational instabilities give rise to more difficulties on the matter, see for instance Chorin (1973), Marshall et al. (1976) or Shestakov (1976).
In an attempt to overcome this drawbacks the Random Vortex Method (RVM), was introduced by Chorin (1973), for solving the Navier-Stokes equations for rapid flows consisting in the approximation of the Euler equations by analyzing the interaction of point vortices and then introducing the effects of viscosity by addition of an appropriate random component. The RVM is a grid free method.
An hybrid method was introduced by Shestakov (1976), in which the Navier-Stokes equations were approximated, near the boundaries by the RVM method, and in the interior by finite differences. The matching of both solutions was done by a careful spline interpolation.
Independently, the Random Choice Method (RCM), was introduced by Chorin (1976), for the solution of hyperbolic systems of conservation laws in gas dynamics and extended to shallow water equations by Marshall et al. (1978). By this method the

solution is constructed as a superposition of locally theoretical solutions and sampling techniques. The RCM is a grid free method.
More recently the Random Vortex Sheet method (RVSM), was introduced by Chorin (1978), for the solution of the Prandtl incompressible velocity boundary layer equations. This method which is a simplified version of the RVM was applied, coupled with the RCM method, for the simulation of a flow problem in a model cylinder-piston assembly.
In this work we introduce a grid free hybrid method for the solution of rapid recirculating incompressible flows in free and forced convection in rectangulat domains. This hybrid method consists in the utilization of different physical hypotesis and thus different equations near the solid boundaries and on the interior of the domain and a proper matching technique. Hence, the interior flow is approximated by an inviscid compressible isoentropic gas flow, and the flow near the boundaries, by an incompressible velocity boundary layer flow. This is to say that in the laminar flow, in free and forced convection, of a viscous incompressible fluid with the Boussinesq approximation, in two dimensions and in cartesian coordinates

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = - \frac{1}{\rho_o} \frac{\partial p}{\partial x_i} + \frac{\rho}{\rho_o} X_i + \nu \, \Delta \, u_i \tag{1}$$

$$\frac{\partial u_j}{\partial x_j} = 0 \tag{2}$$

$$\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} = \frac{k}{\rho_o \, c_p} = \Delta \, T \tag{3}$$

$$\rho = \rho(T) \quad , \quad X_i = -g \, \varepsilon_i \qquad \begin{cases} \varepsilon_i = 1 & \text{for } i = 2 \\ \varepsilon_i = 0 & \text{for } i = 1 \end{cases}$$

equations (1) and (2) are replaced by the Prandtl velocity boundary layer and by the Euler equations in the boundary and interior, respectively. Both equations are solved by the RVSM and RCM methods, respectively. In forced convection (the free convection case will be treated elsewhere), the energy equation behaves as a passive scalar quantity transported by the fluid and thus it is approximated in a different way: for the whole domain the convective terms are solved by the RCM method and the dissipative terms by a random walk procedure. The approximation of a liquid flow by a gas flow is consistent provided the Mach number is small for, in this case, the density variations are small and the gas flow can be considered incompressible.

Of paramount importance is the matching of boundary and interior calculations. Corners produce the separation of the boundary layer flow greatly influencing the interior flow, in consequence, some interaction is needed to counteract the separation singularity. The following mechanism was introduced by Chorin (1978): the tangential velocity at the wall of the interior calculation is taken as the velocity at infinite for the boundary layer calculation and conversely, the velocity normal to the wall induced by the boundary layer is impressed as normal velocity at the wall of the interior calculation. In what follows we present a summary of the RVSM and RCM methods and some preliminary results for the square cavity.

THE RANDOM VORTEX SHEET METHOD

In the boundary layer the Navier-Stokes equations (1) - (2) (omitting the buoyancy term for simplicity), can be approximated by the Prandtl equations which can be written

$$\varepsilon_t + u\,\varepsilon_x + v\,\varepsilon_y = \nu\,\varepsilon_{yy} \tag{4}$$

$$\varepsilon = -u_y \tag{5}$$

$$u_x + v_y = 0 \tag{6}$$

where ε is the vorticity, u and v are de velocities (u tangential and v normal to the boundary, respectively), x is the coordinate along the boundary and y is the coordinate normal to the boundary. The boundary conditions are

$$u\,(x,o) = v\,(x,o) = o \tag{7}$$

$$u\,(x,\infty) = U_\infty(x) \tag{8}$$

From equation (6) we obtain

$$v\,(y) = -\frac{\partial}{\partial x}\int_o^y u(x,y)\;dy\;, \tag{9}$$

from equation (5) we obtain

$$u(x,y) = U_\infty - \int_\infty^y \varepsilon(x,y)\;dy \tag{10}$$

If ε is known u and v can be obtained by equations (10) and (9). In what follows we suppose $\nu = 0$ in equation (4). Consider a collection of N segments S_i of vortex sheets, of intensities ε_i, $i = 1,\ldots,N$. S_i is parallel to the x axis, of length h and center (x_i, y_i). The x component of the velocity of S_i due to the presence of the other segments can be found approximately from equation (10)

$$u_i = U_\infty (x_i) - \sum_j \varepsilon_j d_j \tag{11}$$

where $$d_j = 1 - |x_i - x_j|/h \tag{12}$$

and the Σ is over all S_j such that $y_j > y_i$ and $|x_i - x_j| < h$. From equation (9) the vertical velocity v_i of S_i can be approximated by

$$v_i = -1/2 \int_o^y u(x + h/2, y)\,dy + 1/2 \int_o^y u(x - h/2, y)\,dy, \tag{13}$$

and the first integral, for instance, is taken as

$$U_\infty (x_i + h/2)\, y_i - \sum_{j^+} \varepsilon_j d_j^+ y_j^* \tag{14}$$

where $d_j^+ = |x_i + h/2 - x_j|/h$ and $y_j^* = \min (y_i, y_j)$. The Σ_+ is over all S_j such that $o \le d_j^+ \le 1$.

At this stage it is possible to describe the motion of the vortices given by the Prandtl equations in the absence of viscosity. The effect of viscosity can be included by adding a random variable η_i drawn from a gaussian distribution with mean zero and variance $\sqrt{4\nu k}$. The complete algorithm approaching the Prandtl equations reads

$$x_i^{n+1} = x_i + k\, u_i \tag{15}$$

$$y_i^{n+1} = y_i + k\, v_i + \eta_i \tag{16}$$

The boundary conditions $u(x,\infty) = U_\infty$ and $v(x,o) = o$ are automatically satisfied, $u(x,o) = o$ must be satisfied through a vorticity creation mechanism at the boundary (see details in Chorin (1978)).

THE RANDOM CHOICE METHOD

We shall introduce the RCM method with a very simple one dimensional example.
Consider the equation

$$u_t - a\, u_x = o \,, \tag{17}$$

with $a = ct. > o$ and $u(x,o) = u(x)$. The RCM method provides with an approximation to $u(x,t)$ at grid points $u_i^n = u(ih,nk)$, where h and k are space and time steps, respectively. The RCM constructs the solution advancing the initial state forward in time by means of two fractional time steps of equal length. We shall describe the first one (the second one is identical). At time $t_n = nk$ the method approximates $u(x,t_n)$ by a succesion of constant states. Next, using as initial data the piecewise constant function, the theoretical solution of equation (17) is constructed, between t_n and $t_{n+1/2}$, by means of the solution of a succesion of Riemann problems (one for each point $x_i = ih$). A Riemann problem is an initial value problem of the form

$$v_t - a\, v_x = o \tag{18}$$

with the initial data $v\,(x,t_n) = \begin{cases} u_i^n & \text{for } x < (i+1/2)\,h \\ u_{i+1}^n & \text{for } x \geq (i+1/2)\,h \end{cases}$

In general there is a discontinuity in the initial data of equation (18) which propagates along the characteristic line with slope a^{-1}. If there is a discontinuity for every grid point they will not interact provided the CFL condition is satisfied. In this case the solution of equation (17) at $t_{n+1/2}$, with piecewise constant initial data at t_n, can be considered to be formed by the superposition of the solutions of each Riemann problem given by equation (18).
Next the solution is sampled by means of a random choice procedure, in the interval $[-h/2, h/2]$, for each point $(i+1/2)h$, to obtain the value of the solution to be adscribed to each $u_{i+1/2}^{n+1/2}$. If $v\,(x,t)$ is the solution of equation (18), then

$$u_{i+1/2}^{n+1/2} = v\,(P_i) \tag{19}$$

where P_i is a randomly chosen point with coordinates $[(i+1/2 + 1/2\ \theta_{i+1/2})h,\ (n+1/2)k]$, where $\theta_{i+1/2}$ is sampled at random on the interval $[-1,1]$. The whole process is repeated for increasing times. The success of the method depends on the possibility of solving Riemann problems efficiently.
The solution of the Riemann problem associated to equation (1) is obtained through a Godunov type iteration (see Chorin(1976)). The Riemann problem associated to equation (3) is straightforward since in forced convection the temperature T is transported by the fluid.
The solution of the Euler equations (convective terms in equations (1) - (3)), in two dimensions for rectangular domains is a straightforward generalization from the one dimensional case through the use of splitting techniques.

NUMERICAL RESULTS

We present some applications of the grid free methods discussed in the precedent sections.

In figure 1 it is shown a typical vorticity configuration for a flat plate problem solved with the RVSM. A star corresponds to the center of a vortex sheet, and o correspond to a negative vorticity. The following parameters were used $k = 0.2$, $h = 0.2$, $\varepsilon_{max} = 0.1$ and $\nu = 10^{-6}$. The picture is at $t = 7.4$. With the value of ε_{max} chosen the statistical error dominated all others (see detailed discussion in Chorin (1978)), as can be seen from the figure. A substantial improvement is obtained with averaged results. The results shown were obtained with a PDP 11/40 with 64 K.

In figure 2 we present a typical configuration of a one dimensional isoentropic gas flow calculation solved with the RCM. This model was used for simulating the problem of the breaking of a dam. The following parameters were used: $h = 1/50$, $k = 0.001$ and the picture is taken after 16 time steps. The shock front and the rarefaction wave are clearly seen in the picture (see details in Marshall et al.(1978)).

Finally we present some numerical results for the square cavity flow solved with the grid free hybrid method described earlier. The cavity has unit length and is filled with gas initially at rest, $p = 1$ and $\rho = 1$. The gas is an ideal γ law gas with $\gamma = 1.4$; care is taken to ensure that the Mach number is much smaller than unity. The following parameters were used: in the interior $\Delta = 1/7$, $k = 0.6\ \Delta$; in the boundary layer $h = 2\ \Delta$ and $\varepsilon_{max} = 0.1$. The viscosity is $\nu = 10^{-3}$. In tables I and II we

Table I

Horizontal velocities

0.01	0.25	0.63	1.07	0.80	1.01	0.45
0.03	0.04	0.24	0.28	0.22	0.19	0.10
0.02	-0.01	-0.25	-0.30	-0.17	-0.28	-0.07
-0.02	-0.08	-0.20	-0.31	-0.36	-0.29	-0.04
-0.09	-0.20	-0.24	-0.22	-0.21	-0.10	0.07
0.00	-0.07	-0.16	-0.22	-0.14	-0.16	-0.07
-0.01	-0.09	-0.16	0.08	-0.22	-0.23	-0.14

Table II

Vertical velocities

0.34	0.11	0.12	0.00	0.00	0.00	-0.01
0.25	0.15	0.28	-0.15	-0.07	-0.40	-0.43
0.37	0.38	0.12	-0.18	-0.11	-0.51	-0.54
0.29	0.14	0.10	-0.03	-0.12	-0.33	-0.36
0.17	0.06	0.05	-0.03	0.05	-0.02	-0.21
0.02	0.03	0.00	0.05	0.00	-0.04	-0.21
-0.02	0.03	0.02	0.01	-0.08	0.05	0.01

present the results obtained after 15 time steps. Table I show the horizontal velocity field and Table II the vertical velocity field. This results were obtained, with the above mentioned digital equipment in 28 minutes (compilation and printing included). By the distribution of velocities at the center lines it is possible to observe the pattern of the main vortex, although it is clear that the steady state is far from being reached. In the analysis of these results one must take into account the fluctuations of the random choice method, the extremely coarseness of the grid utilized and the fact that the edge of the calculations does not coincide with the boundary of the domain.

CONCLUSIONS

We have presented a grid free method for the solution of viscous incompressible fluids in free and forced convection. The main advantages of the method are: infinite resolution in the interior of the flow and no numerical viscosity, unconditional stability and a substantial saving of computer time due to boundary layer approximations. Obviously more numerical experiments are needed to prove conclusively the excellence of the method.

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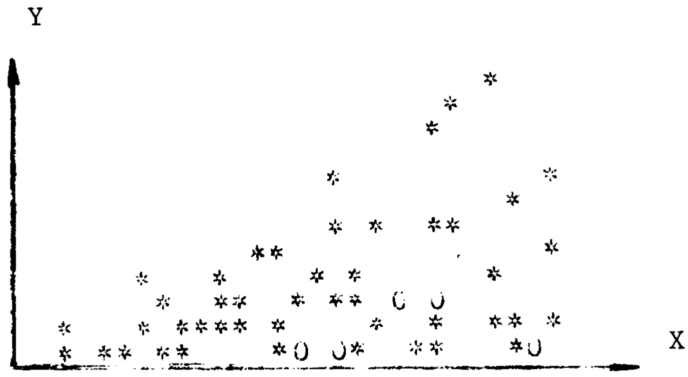

Figure 1. Vorticity configuration for the flat plate problem solved by the RVSM

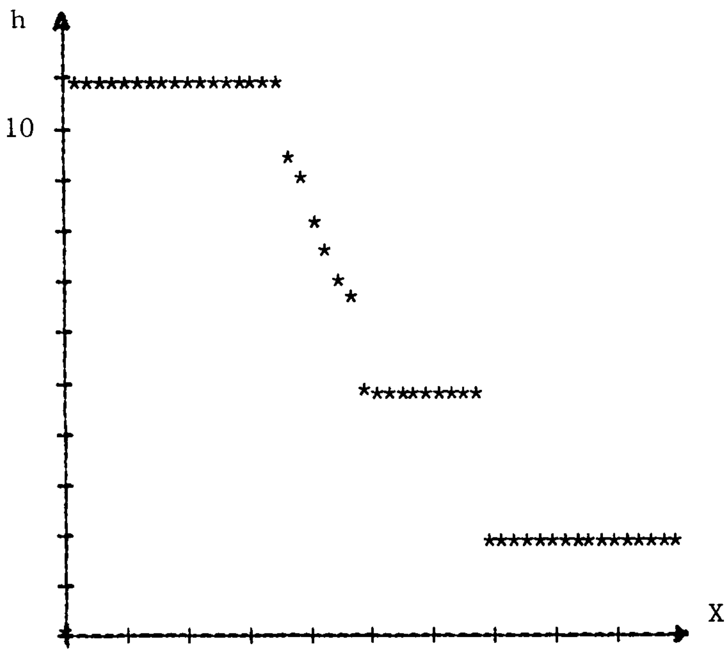

Figure 2. Height of water for the problem of the breaking of a dam solved by the RCM