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polarized electromagnetic radiation

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POLARIZED ELECTROMAGNETIC RADIATION

The subject of Electromagnetic Polarized radiation has become important in the last few years, especially because there are available sources of polarized radiation, such as Co^{60} at low temperatures, bremsstrahlung radiation from electron accelerators and γ - lines from charged particle reactions.

Starting with Maxwell's equations

$$\text{rot } H = \frac{1}{c}\frac{\partial \vec{D}}{\partial t} + \frac{4\pi}{c}\vec{j}$$

$$\text{rot } E = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\text{div } D = 4\pi\rho$$

$$\text{div } B = 0$$

We shall confine our attention to that part of the field which contains no charges or currents, i.e. where $\rho = 0$, $\vec{j} = 0$.

Combining Maxwell's equations one gets

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

$$\nabla^2 \vec{H} - \frac{1}{c^2}\frac{\partial^2 \vec{H}}{\partial t^2} = 0 \qquad (1)$$

These are standard equations of wave motion for the Electric and Magnetic field in free space propagated with velocity c.

The simplest solutions of these equations are spherical and plane waves. Here we shall consider only plane harmonic waves for the Electric vector, i.e., which have a constant amplitude and phase linearly variable with time. In this case we say that the waves are monochromatic. An analogous treatment can be used for the magnetic vector. It will be shown that a consequence of the monochromacity is the completely polarized character of this wave.

a) Elliptic polarization

Let us consider the z axis as the direction of propagation of the wave. Each component of E and H is given by

$$E = a \cos (\omega t - \vec{k} \cdot \vec{r} + \delta) = R\{a\, e^{i(\omega t - \vec{h} \cdot \vec{r} + \delta)}\}$$

ω = angular frequency of the wave

k = propagation vector $= 2\pi/\lambda$

δ = phase factor

The argument of the cosine is the phase.

Since the Electric and Magnetic fields are transversal we have $E_z = 0$ and

$$E_x = a_1 \cos (\tau + \delta_1)$$

$$E_y = a_2 \cos (\tau + \delta_2) \qquad (2)$$

where $\tau = \omega t - \vec{k} \cdot \vec{r}$

Let us consider now the nature of the curve described by the envelope of the extreme of the Electric vector. To do this we have to eliminate τ. Rewriting the above equations

$$\frac{E_x}{a_1} = \cos\tau \cos\delta_1 - \sin\tau \sin\delta_1$$

$$\frac{E_x}{a_2} = \cos\tau \cos\delta_2 - \sin\tau \sin\delta_2$$

$$\frac{E_x}{a_1} \sin\delta_2 = \cos\tau \cos\delta_1 \sin\delta_2 - \sin\tau \sin\delta_1 \sin\delta_2$$

$$\frac{E_x}{a_2} \sin\delta_1 = \cos\tau \cos\delta_2 \sin\delta_1 - \sin\tau \sin\delta_1 \sin\delta_2$$

$$\frac{E_x}{a_1} \sin\delta_2 - \frac{E_x}{a_2} \sin\delta_1 = \cos\tau \cos\delta_1 \sin\delta_2 -$$

$$- \cos\tau \cos\delta_2 \sin\delta_1 = \cos\tau \sin(\delta_2 - \delta_1)$$

Analogously

$$\frac{E_x}{a_1}\cos\delta_2 - \frac{E_x}{a_2}\cos\delta_1 = \sin\tau\sin(\delta_2-\delta_1) \tag{3}$$

Squaring and adding (3)

$$\left(\frac{E_x}{a_1}\right)^2\sin^2\delta_2 + \left(\frac{E_y}{a_2}\right)^2\sin^2\delta_1 - 2\frac{E_xE_y}{a_1a_2}\sin\delta_2\sin\delta_1 =$$

$$= \cos^2\tau\sin^2(\delta_2-\delta_1)$$

$$\left(\frac{E_x}{a_1}\right)^2\cos^2\delta_2 + \left(\frac{E_y}{a_2}\right)^2\cos^2\delta_1 - 2\frac{E_xE_y}{a_1a_2}\cos\delta_2\cos\delta_1 =$$

$$= \sin^2\tau\sin^2(\delta_2-\delta_1)$$

$$\left(\frac{E_x}{a_1}\right)^2 + \left(\frac{E_y}{a_2}\right)^2 - 2\frac{E_xE_y}{a_1a_2}\cos\delta = \sin^2\delta \quad \text{where } \delta = \delta_2 - \delta_1$$

This is an equation of an ellipse because the associated determinant is not negative:

$$\begin{vmatrix} \frac{1}{a_1^2} & -\frac{\cos\delta}{a_1a_2} \\ -\frac{\cos\delta}{a_1a_2} & \frac{1}{a_2^2} \end{vmatrix} = \frac{1}{a_1^2a_2^2}\left[1-\cos^2\delta\right] = \frac{\sin^2\delta}{a_1^2a_2^2} \geq 0$$

The ellipse is inscribed in a rectangle of sides $2a_1$ and $2a_2$, which are parallel to the axis x and y respectively. Graphical solutions for different values of δ are given in fig. 1

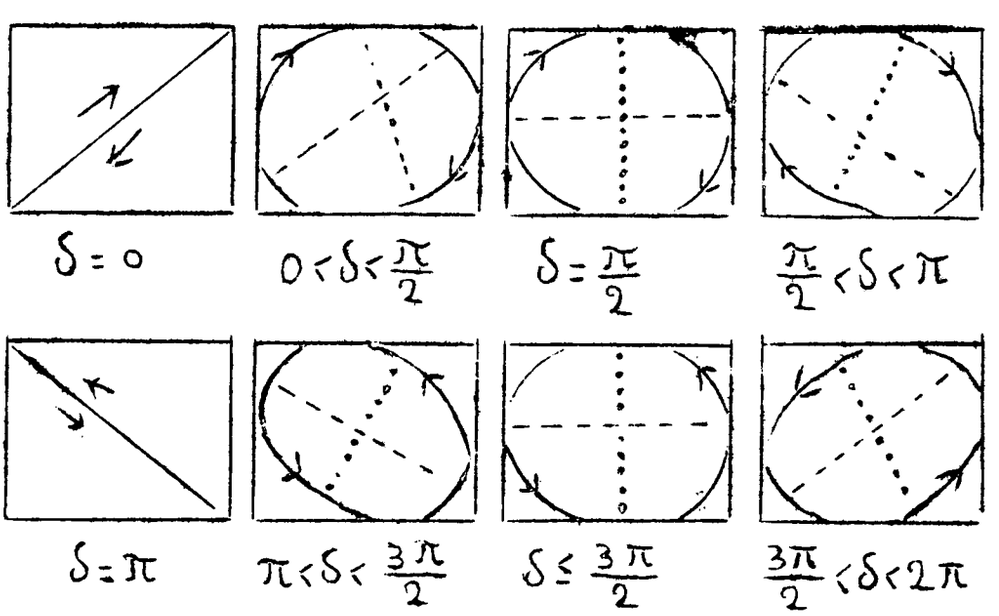


Fig.1

We distinguish two cases of polarization according to the sense in which the end point of the electric vector describes the ellipse, and this is determined by $\sin \delta \gtrless 0$.

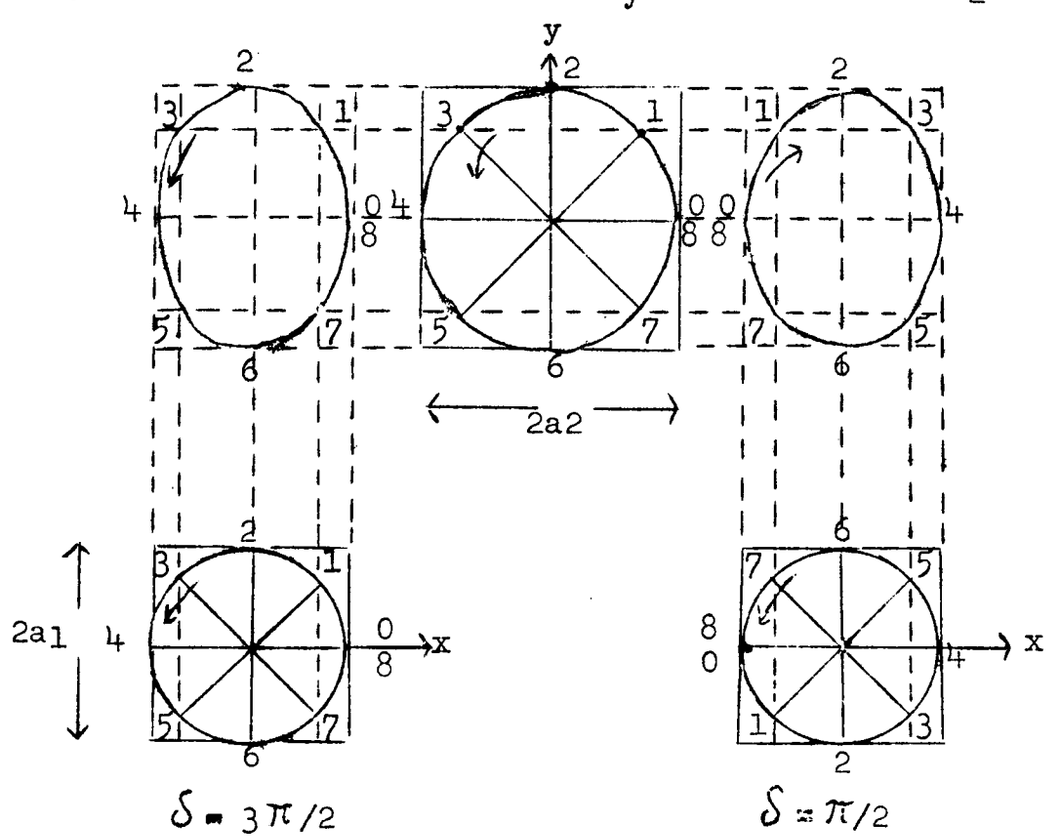
In the two cases the light is said to be elliptically polarized.

We shall define here the polarization as <u>right-handed</u> (R.H.) when to an observer looking in the direction from which the light is coming the end point of the electric vector describes the ellipse in the clockwise sense. In this case we have $\sin \delta > 0$. For <u>Left-handed</u> (L.H.) polarization, we have the counter clockwise sense. In this case $\sin \delta < 0$.

We can easily see that $\sin \delta \gtrless 0$ determines the two kinds of polarization. This is done by using a graphical method of determineing the resultant of two motions at right angles, using the projection of uniform circular motion for generating each of the two simple periodic motions, for different values of δ.

For example consider the two cases: a) $\delta = \frac{\pi}{2}$ and b) $\delta = \frac{3\pi}{2}$

a) $\begin{cases} E_x = a_1 \cos(\tau + \delta_1) \\ E_y = a_2 \cos(\tau + \delta_2) \end{cases}$ b) $\begin{cases} E'_x = a_1 \cos(\tau + \delta'_1) \\ E'_y = a_2 \cos(\tau + \delta'_2) \end{cases}$



Dividing a complete period into eights, the numbers 0 to 8 represent the position of the generating points. The phase difference determines the starting points. For example $\delta = \frac{3\pi}{2}$ when $y = 0$ we have $x = a_1$, and for $\delta = \frac{\pi}{2}$; $y = 0$, $x = -a_1$.

b) Linear Polarization.

If $\delta = \delta_2 - \delta_1 = m\pi$ $\qquad m = 0, \pm 1, \pm 2,$

then we have

$$\left(\frac{E_x}{a_1}\right)^2 + \left(\frac{E_y}{a_2}\right)^2 \pm \frac{2 E_x E_y}{a_1 a_2} = 0$$

$$\frac{E_x}{a_1} = (-1)^m \frac{E_y}{a_2}$$

The ellipse is transformed into a straight line, and the light is said to be plane polarized or linearly polarized. In this case there is only one kind of polarization.

c) Circular Polarization

If $\delta = \delta_2 - \delta_1 = m \frac{\pi}{2}$ $\qquad m = \pm 1, 3, 5$

and $a_1 = a_2 = a$

then the equation of the ellipse becomes

$$E_x^2 + E_y^2 = a^2$$

which is an equation of a circle, and we have two kinds of polarization given by the sign of $\sin \delta$.

$\sin \delta > 0$ right-handed circular polarization.

$$\delta = \frac{\pi}{2} + 2m\pi \qquad m = 0, \pm 1, \pm 2, \ldots$$

and

$$E_x = a \cos (\tau + \delta_1)$$

$$E_y = a \cos \left(\tau + \delta_1 + \frac{\pi}{2}\right) = - a \sin (\tau + \delta_1)$$

for $\sin\delta < 0$ left-handed circular polarization

we have $\delta = -\frac{\pi}{2} + 2m\pi \qquad m = 0, \pm 1, \pm 2$

$$E_x = a\cos(\tau + \delta_1)$$

$$E_y = a\cos(\tau + \delta_1 - \frac{\pi}{2}) = a\sin(\tau + \delta_1)$$

Characterization of the state of Polarization by Stokes parameters.

In order to characterize the elliptic polarization of a light wave, we used 3 independent parameters, the amplitudes a_1, and a_2, and the phase difference δ, or the two axes of the ellipse, and the angle of inclination ψ.

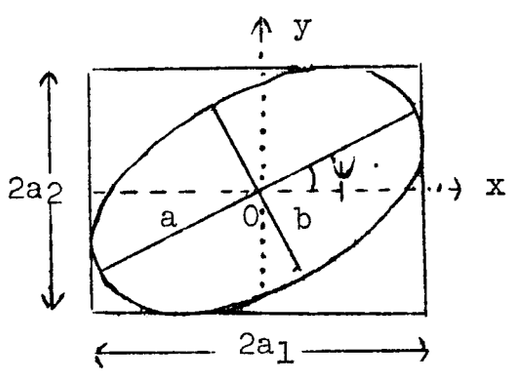


Fig. 2

Stokes in 1852, found a very pictorial way of describing the polarized light, by introducing four parameters associated with Poincare's sphere, and defined in the following way:

$$s_0 = a_1^2 + a_2^2$$

$$s_1 = a_1^2 - a_2^2$$

$$s_2 = 2a_1a_2\cos\delta$$

$$s_3 = 2a_1a_2\sin\delta$$

where only three of these parameters are independent since we have

$$s_0^2 = s_1^2 + s_2^2 + s_3^2$$

The parameter s_o is proportional to the intensity of the wave, while s_1, s_2, s_3 are associated with the inclination of the ellipse (ψ) with respect to the axis, and the direction in which the ellipse is described (χ).

If we describe a sphere with a radious s_o, then we see that s_1, s_2, s_3 are the projections of a point P on the carthesian axis x, y, z and we have

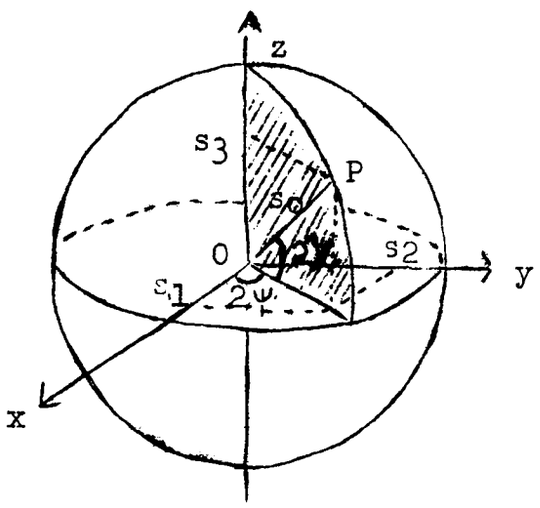


Fig. 3

$$s_1 = s_o \cos 2\chi \cos 2\psi$$

$$s_2 = s_o \cos 2\chi \sin 2\psi$$

$$s_3 = s_o \sin 2\chi$$

with

$$\tan \chi = \pm \frac{b}{a}$$

where

b = minor axis

a = major axis

The sign $\pm$ determine the direction of polarization

+	polarization R.H.
−	polarization L.H.

These equations show a simple geometric way of representing a beam of light by the point P, whose cartesian coordinates are s_1, s_2 and s_3.

Since χ is positive or negative depending on the kind of polarization we have

Elliptic polarization

$\chi > 0$	R.H. polarization	point P - North hemisphere
$\chi < 0$	L.H. polarization	point P - South hemisphere

Circular polarization

$\chi > 0$	R.H. polarization	$a_1 = a_2$	$s_1 = 0$
		$\delta = \pi/2$	$s_2 = 0$

The point P is situated on the North pole

$\chi < 0$ L.H. polarization $a_1 = a_2$ $s_1 = 0$

$\delta = -\pi/2$ $s_2 = 0$

The point P is located on the south pole

Linear polarization

$$\delta_2 - \delta_1 = m\pi \quad \therefore \quad s_3 = 0$$

The point P is located on the equator.

Thus we can see that to every polarized beam of light there corresponds a point on the Poincare's sphere, which describes completly the state of polarization. It will be shown that if the light is unpolarized then it is represented by the whole superficie, and with a variable radius.

QUASI-MONOCHROMATIC LIGHT

Let us consider a beam of light, which consist of sharp spectral lines, e.g. emitted by Cadmium vapour, and let one of them illuminated a Michelson interferometer, adjusted to give circular fringes, then if the two paths PM_1 and PM_2 are nearly equal, distinct interference fringes are seen by the observer. If, however, one of the paths becomes longer then the other one, then the fringes tend to blur and eventually dissapear. This can be explained by supposing that the spectral lines are not strictly monochromatic, but consist of a superposition of waves of different frequencies, which form a wave packet of duration t, and a large number of them pass at random time intervals during the observation time. Thus, when a wave packet enters the interferometer, it is divided into two which follow the paths PM_1 P0 and PM_2 P0 respectively. If the difference $PM_1 - PM_2 > \Delta t$ then one of the wave packets will arrive at 0 after the other one and there will be no interference.

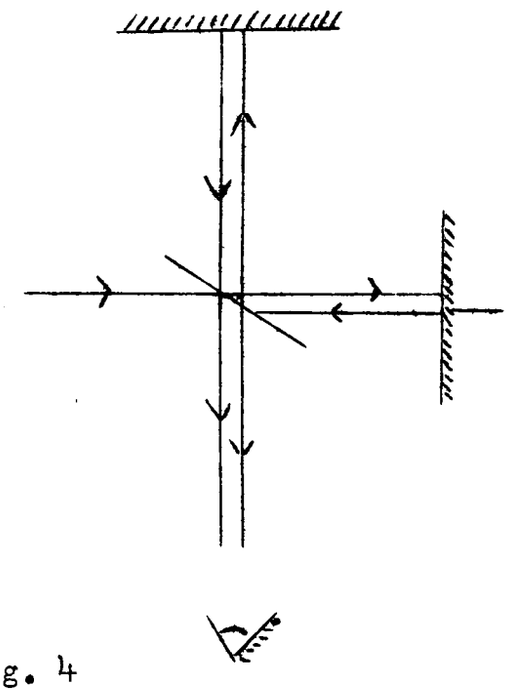


fig. 4

We define Δt, the duration of the wave packet, as the coherence time, since this is the maximum time interval during which interference can be made.

This can be illustrated mathematically by a simple example.

Consider a light disturbance $F(t)$ at a time t, due to a wave packet, and let us assume that $F(t) = 0$ when $|t| \geq t_0$.

Then we can express it as a Fourier integral

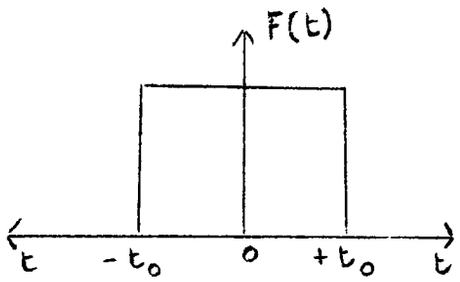


fig. 5

$$F(t) = \int_{-\infty}^{+\infty} f(\nu)\, e^{-2\pi i \nu t}\, d\nu \qquad (1)$$

Thus we see that in order to have a wave packet of this special form, we must make a sumation over an infinite number of frequencies.

Using the Fourier inversion theorem we have

$$F(\nu) = \int_{-\infty}^{+\infty} F(t)\, e^{2\pi i \nu t}\, dt \qquad (2)$$

Suppose now that all wave packets are of a duration Δt, during which F(t) is simply periodic with frequency

$$\begin{aligned} F(t) &= f_0\, e^{-2\pi i \nu_0 t} && \text{when } |t| \leq \Delta t/2 \\ &= 0 && \text{when } |t| > \Delta t/2 \end{aligned} \qquad (3)$$

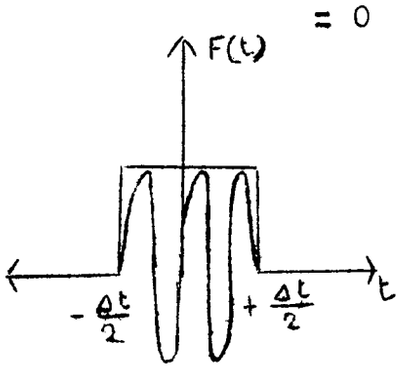


fig. 6

In this case the wave packet is composed only of sinusoidal waves of the same frequency ν_0. Here f_0 is a constant.

Using (2) and (3) we write

$$f(\nu) = f_0 \int_{-\frac{\Delta t}{2}}^{+\frac{\Delta t}{2}} e^{2\pi i (\nu - \nu_0) t}\, dt =$$

$$= f_0\, \Delta t \left[\frac{\sin\{\pi(\nu - \nu_0)\Delta t}{\pi(\nu - \nu_0)\Delta t} \right]$$

This is a function of a type sin x/x. The intensity distribution of the Fourier components is given by

$$\left(\frac{\sin\{\pi(\nu - \nu_0)\Delta t\}}{\pi(\nu - \nu_0)\Delta t} \right)^2$$

as shown in the fig. 7

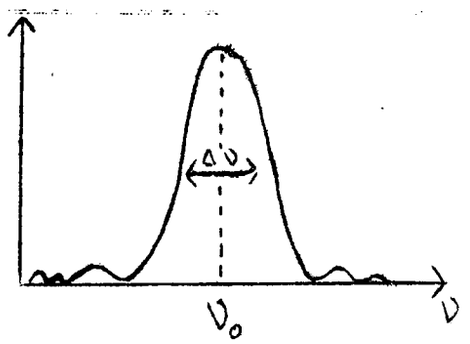


fig. 7

The first zero, which occurs when the argument of the sin is equal to π is

$$\nu - \nu_0 = \pm \frac{1}{\Delta t}$$

so we have $\Delta\nu \simeq 1/\Delta t$

Thus the reciprocal of the coherence time is equal to the effective spectral width of the wave packet.

In practice, wave packets are not all identical, and of this simple form, since according to the atomic thoery the loss of energy suffered by the atoms during the light emission, results in damping of the wave packets. Besides this the atoms move in a random thermal motion, so that the spectral line emitted is modified by the Doppler effect. All this influences the form of the wave packet. Therefore we can in a general way say that if we have a light disturbance F(t), whose Fourier inverse is f(ν), it is possible to define Δt and $\Delta\nu$, which are respectively the average duration and the average width of the wave packet and which satisfy the relation and

$$\Delta t \, \Delta\nu \geq \frac{1}{4}$$

and this inequality is analogous to the Heisenberg uncertainty principle.

Consider a simple sinusoidal wave of one frequency ν_0 then we have $\Delta\nu = 0$, $\Delta t = \infty$. Therefore we get an infinetly long wave packet, which is the monochromatic case, and the coherence time is infinite.

If on the other hand we have a very sharp light pulse, such that $\Delta t = 0$, then this pulse is composed of an infinite number of monochromatic waves, and $\Delta\nu = \infty$. This corresponds to a light emitted by an atom; the coherence time is zero, and one never gets interference effects.

There is an intermediate case, which is the quasi-monochromatic light. Light emitted from real sources is never strictly monochromatic because of the finite width of the spectral lines, and because the source is never a strictly point source, but has a finite extension due to a large number of elementary radiation (atoms). Thus the amplitude and phase of such a wave is no longer a constant, but undergo irregular

fluctuations, with a rapidity depending on the offective width $\Delta \nu$. The amplitude practically constant only during a time interval which is small compared to the coherence time, i.e. the reciprocal of the effective widht $\Delta\nu$, and the change of the relative phase of any two Fourier components is much less then 2π during the same time interval. Thus the addition of such component will represent a disturbance which in the time interval Δ t, behaves like a monochromatic wave of mean frequency $\bar{\nu}$. But this is not valid for times longer than

$$\Delta t \sim \frac{1}{\Delta \nu}$$

i.e. the coherence time.

Consider then the quasi-monochromatic light of mean frequency $\bar{\nu}$ given by

$$E_x(t) = a_1 (t)\, e^{i\left[\phi_1(t) - 2\pi \bar{\nu} t\right]}$$

$$E_y(t) = a_2 (t)\, e^{i\left[\phi_2(t) - 2\pi \bar{\nu} t\right]}$$

We can see that if we do not make any assumptions about the time variations of the amplitudes and phases, then the above equations represent natural light.

We shall show here that the quasi-monochromatic light is partially polarized and we shall define the degree of polarization.

Let us then take the two above component E_x and E_y, and introduce a retardation $e^{i\varepsilon}$ in the component E_y. This is done, for example, by using a compensator, which can in particular transform elliptically or circularly polarized light into linearly polarized light. In this case if the phase difference between the two components E_x and E_y is δ , then we introduce a retardation ε in one of the components such that the total phase difference is $\delta + \varepsilon = \pi$ or 0. In particular a quarter wave plate is used, whose retardation is $\varepsilon = \frac{\pi}{2}$.

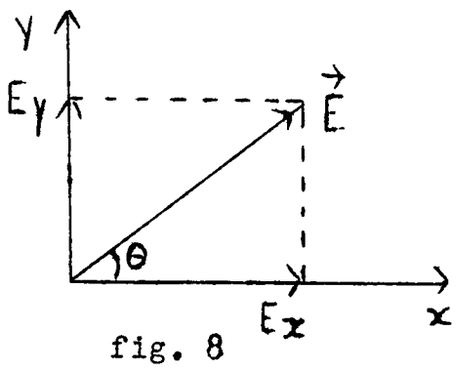


fig. 8

The component of the electric vector in the direction Θ is then given by

$$E(t,\Theta,\varepsilon) = E_x \cos\Theta + E_y e^{i\varepsilon} \sin\Theta$$

so that taking the time average of the intensity over a time long compared to the period

$$I(\Theta,\varepsilon) = \langle E_x E_y^* \rangle \cos^2\Theta + \langle E_y E_y^* \rangle \sin^2\Theta + \langle E_y E_x^* \rangle e^{i\varepsilon} \sin\Theta\cos\Theta + \langle E_x E_y^* \rangle e^{-i\varepsilon} \cos\Theta\sin\Theta$$

$$= J_{xx}\cos^2\Theta + J_{yy}\sin^2\Theta + J_{yx}\, e^{i\varepsilon} \sin\Theta\cos\Theta + J_{yx}\, e^{-i\varepsilon} \sin\Theta\cos\Theta$$

where the $J_{xx}, \ldots$ are the elements of the matrix

$$J = \begin{vmatrix} J_{xx} & J_{xy} \\ J_{yx} & J_{yy} \end{vmatrix} = \begin{vmatrix} \langle a_1^2 \rangle & a_1 a_2 e^{i(\phi_2 - \phi_2)} \\ \langle a_1 a_2\, e^{-i(\phi_2 - \phi_1)} \rangle & \langle a_2^2 \rangle \end{vmatrix}$$

This is the coherence matrix.

The diagonal elements of J represent the intensity of the components in the x and y direction. Therefore the trace of the matrix is equal to the total intensity of light.

$$\mathrm{Tr}\, J = J_{xx} + J_{yy} = \langle E_x E_x^* \rangle + \langle E_y E_y^* \rangle = \langle a_1^2 \rangle + \langle a_2^2 \rangle$$

The matrix is hermitian in general, so that we have

$$J_{xy} = J_{yx}^*$$

We shall define here

$$\mu_{xy} = |\mu_{xy}|\, e^{i\beta_{xy}} = \frac{J_{xy}}{\sqrt{J_{xx}\, J_{yy}}} .$$

The term $|\mu_{xy}| \leq 1$ represents the degree of coherence and β_{xy} the effective phase difference

We can write now

$$I(\Theta,\varepsilon) = J_{xx}\cos^2\Theta + J_{yy}\sin^2\Theta + 2\cos\Theta\sin\Theta\, R(J_{xx}\, e^{-i\varepsilon})$$

$$= J_{xx}\cos^2\Theta + J_{yy}\sin^2\Theta + 2\sqrt{J_{xx}J_{yy}}\,|\mu_{xy}|\cos\Theta\sin\Theta\cos(\beta_{xy} - \varepsilon)$$

The elements of the coherency matrix can be determined by a relatively simple experiment. Let us take a polarizer, i.e. a Nicol prism, and a compensator, and measure the intensity of light as the function of the angle of polariser Θ , and the retardation ε . Then we can solve the above equation for these angles. For a certain Θ we shall obtein a maximum of intensity, and by puting a quarter wave plate after the Nicol we obtein zero intensity for elliptically polarized light.

But for partially polarized light we shall obtein only maxima and minima in the light intensity, measured after the quarter wave plate, as we turn the Nicol prism, varying the angle Θ .

NATURAL LIGHT

Light most frequently encountered in nature has the property that the intensity of its components in any direction perpendicular to the direction of propagation is a constant, and it is not affected by any retardation introduced in one of the rectangular components. So that we have:

$$I(\Theta,\varepsilon) = \text{constant} \qquad \text{for all } \Theta \text{ and } \varepsilon .$$

This is called natural light.

In order to have $I(\Theta,\varepsilon)$ independent of Θ and ε , we must put

$$|\mu_{xy}| = 0 \quad \text{and} \quad J_{xx} = J_{yy}$$

The first condition implies that E_x and E_y are mutually incoherent, and since $J_{xy} = J^{*}_{yx}$, we can write the second condition as

$$J_{xy} = J_{yx} = 0 \quad , \quad J_{xx} = J_{yy}$$

and since

$$J_{xx} + J_{yy} = I$$

it follow that the coherency matrix for natural light is

$$J = \frac{1}{2} I \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

MONOCHROMATIC LIGHT

In this case the amplitudes a_1, a_2 and the phase ϕ_1, ϕ_2 are independent of time, so that the coherency matrix is

$$J = \begin{vmatrix} a_1^2 & a_1 a_2 e^{i\delta} \\ a_1 a_2 e^{-i\delta} & a_2^2 \end{vmatrix} \text{ with } \delta = \phi_1 - \phi_2$$

and in this case we have

$$|J| = J_{xx} J_{yy} - J_{xy} J_{yx} = 0$$

and the complex degree of coherence of E_x and E_y is given by

$$\mu_{xy} = \frac{J_{xy}}{\sqrt{J_{xx} J_{yy}}} = e^{i\delta} \quad \therefore |\mu_{xy}| = 1$$

and the phase is equal to the difference of phase of the two components i.e. $\beta_{xy} = \delta$.

In this case we have complete coherence or complete polarization.

LINEAR POLARIZATION

We have $\delta = m\pi \qquad m = 0, \pm 1, \pm 2, \pm \ldots$

and the coherence matrix becomes

$$\begin{vmatrix} a_1^2 & (-1)^m a_1 a_2 \\ (-1)^m a_1 a_2 & a_2^2 \end{vmatrix}$$

Therefore the electric vector vibrates in the direction given by the line $\frac{E_y}{E_x} = (-1)^m \frac{a_2}{a_1}$

and in particular we have

direction x - axis

$a_2 = 0$

$$J = I \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$$

direction y - axis

$a_1 = 0$

$$J = I \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

direction 45º with x - axis

$a_1 = a_2 \qquad m = 0$

$$J = \frac{1}{2} I \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$$

direction 135º with y - axis

$a_1 = a_2 \qquad m = 1$

$$J = \frac{1}{2} I \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}$$

CIRCULAR POLARIZATION

$$a_1 = a_2 \quad \delta = m \frac{\pi}{2} \qquad m = \pm 1, \pm 3, \pm \ldots$$

The coherence matrix is

$$J = \frac{1}{2} I \begin{vmatrix} 1 & \pm i \\ \pm i & 1 \end{vmatrix}$$

upper sign gives R.H. polarization
lower sign gives L.H. polarization

PARTIALLY POLARIZED LIGHT

In this discussion we have to use an important theorem.

If several <u>independent</u> light waves which are propagated in the same direction are superposed, the coherence matrix of the resulting wave is equal to the sum of the coherence matrices of the individual waves.

Let $E_x^{(n)}$, $E_y^{(n)}$ (n 1, 2,...,N) be the components of the individual waves. The components of the electric vector of the resulting wave are

$$E_x = \sum_{n=1}^{N} E_x^{(n)} \qquad E_y = \sum_{n=1}^{N} E_y^{(n)}$$

so that the element of the coherency matrix is given by

$$J_{k\ell} = \langle E_k E_\ell^* \rangle = \sum_{n=1}^{N} \sum_{m=1}^{N} \langle E_k^{(n)} E_\ell^{(m)*} \rangle = \sum_n \langle E_k^{(n)} E_\ell^{(n)*} \rangle + \sum_{n \neq m} \langle E_k^{(n)} E_\ell^{(m)*} \rangle$$

since the waves are assumed to be independent, then each term in the last summation is zero, so that we have

$$J_{k\ell} = \sum_n \langle E_k^{(n)} \; E_\ell^{(n)*} \rangle = \sum_n \; J_{k\ell}^{(n)}$$

where J_{ke}^{n} are the elements of the coherency matrix of the $n^{\underline{th}}$ wave.

Conversely any wave may be regarded as the sum of independent waves. For example:: suppose we have natural light

$$J = \frac{1}{2} I \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \frac{1}{2} I \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \frac{1}{2} I \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}.$$

two independent linearly polarized waves, with electric vectors at 90º, each of intensity I

$$J = \frac{1}{2} I \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \frac{1}{4} I \begin{vmatrix} +1 & +i \\ -i & +1 \end{vmatrix} + \frac{1}{4} I \begin{vmatrix} 1 & -i \\ +i & 1 \end{vmatrix}$$

two independent circularly polarized waves in the right and left handed direction respectively, each of intensity $\frac{1}{2}$ I.

Thus we can see that matural light can be considered as a superposition of linearly polarized light at 90º or circularly polarized in L.H. and R.H. directions.

We shall apply this to partially polarized light.

We will show that any quasi-monochromatic lifht wave may be regarded as the sum of natural light waves, and monochromatic wave, independent of the formes, and this representation is unique.

To establish this we must only show that the coherency matrix J can be uniquely expressed by $J = J^{(1)} + J^{(2)}$ (1) where

$$J = \begin{vmatrix} J_{xx} & J_{xy} \\ J_{yx} & J_{yy} \end{vmatrix}$$

and

$$J^{(1)} = \begin{vmatrix} A & 0 \\ 0 & A \end{vmatrix} \qquad J^{(2)} = \begin{vmatrix} B & D \\ D^* & C \end{vmatrix}$$

$J^{(1)}$ being the natural light, and $J^{(2)}$ the monochromatic wave, and we also have

$$A \geq 0 \quad , \quad B \geq 0 \quad , \quad C \geq 0$$

and

$$B C - D D^{*} = 0 \tag{2}$$

Therefore according to (1) we must have

$$\begin{aligned} A + B &= J_{xx} \qquad D = J_{xy} \\ D^{*} &= J_{yx} \qquad A + C = J_{yy} \end{aligned} \tag{3}$$

substituting (3) in (2) we have

$$(J_{xx} - A)\ (J_{yy} - A) - J_{xy}\ J_{yx} = 0 \tag{4}$$

This equation has two roots given by

$$A = \frac{1}{2}\ (J_{xx} + J_{yy}) \pm \frac{1}{2}\ \sqrt{(J_{xx} + J_{yy})^2 - 4\ |J|} \tag{5}$$

where

$$|J| = J_{xx}\ J_{yy} - J_{xy}\ J_{yx} \geq 0 \tag{6}$$

since

$$J_{yx} = J_{xy}^{*} \quad \text{then} \quad J_{xy}\ J_{yx} \geq 0$$

$$\therefore \quad |J| \leq J_{xx}\ J_{yy} \leq \frac{1}{4}\ (J_{xx} + J_{yy})^2 \tag{7}$$

Both roots are real and non negative, but the root of A with positive sign leads to negative values of B and C; so it has to be rejected. Let us then consider the other root with the negative sign. We then have

$$A = \frac{1}{2}\ (J_{xx} + J_{yy}) - \frac{1}{2}\sqrt{(J_{xx} + J_{yy})^2 - 4\ |J|}$$

$$B = \frac{1}{2}\ (J_{xx} - J_{yy}) + \frac{1}{2}\sqrt{(J_{xx} + J_{yy})^2 - 4\ |J|}$$

$$C = \frac{1}{2}\ (J_{yy} - J_{xx}) + \frac{1}{2}\sqrt{(J_{xx} + J_{yy})^2 - 4\ |J|}$$

(8)

$$D = J_{xy}$$

$$D^{*} = J_{yx}$$

We have thus obteined a unique decomposition of the required kind.

The total intensity of light is

$$I_{tot} = \mathrm{Tr}\, J = J_{xx} + J_{yy} \tag{9}$$

and the total intensity of polarized part is

$$I_{pol} = \mathrm{Tr}\, J^{(2)} = B + C = \sqrt{(J_{xx} + J_{yy})^2 - 4\,|J|} \tag{10}$$

The ratio of the polarized intensity to the total intensity is called the <u>degree of polarization P.</u>

$$P = \frac{I_{pol}}{I_{tot}} = \sqrt{1 - \frac{4\,|J|}{(J_{xx} + J_{yy})^2}} \tag{11}$$

from (11) and (7) we have

$$0 \leq P \leq 1$$

when $P = 1$ the light is monochromatic and $|J| = 0$ and we have completly polarized light.

when $P = 0$ we have completly unpolarized light and $4|J| = (J_{xx}+J_{yy})^2$

$$\therefore\ (J_{xx} - J_{yy})^2 + 4\, J_{xy}\, J_{yx} = 0$$

and since $J_{yx} = J^{*}_{xy}$

$$\therefore\ J_{xx} = J_{yy} \qquad J_{xy} = J_{yx} = 0$$

which is the case of natural light.

If we have $0 < P < 1$, then the light is partially polarized, and when E_x and E_y are mutually incoherent (but the light not necessarily natural) then $J_{xy} = J$ and $|J| = J_{xx}\, J_{yy}$ and we have

$$P = \left| \frac{J_{xx} - J_{yy}}{J_{xx} - J_{yy}} \right| = \frac{\text{difference of component intesities}}{\text{total intensity}}$$

We can also write

$$\pi = \frac{I + P}{I - P} = \frac{J_{xx}}{J_{yy}} = \text{ratio of the axis of the ellipsoid of intensities.}$$

ANGULAR MOMENTUM OF POLARIZED LIGHT

On the basis of either the wave thoery or the quantum theory, a beam of circularly polarized light exerts a torque on a doubling refracting plate which changes the state of polarization of the light beam.

The results are also true for elliptically polarized light.

Classical Theory: According to the electromagntic theory, a light wave passing through a plate exerts a torque given by

$$\bar{\ell} = \bar{P} \times \bar{E}$$

the torque per unit volume produced by the action of the electric field E on the polarization P of the medium; since χ (dielectric constant) is a tensor P and E are not parallel.-

Let the electric componets of a plane light wave propagated in the z direction be given by:

$$E_x = A \cos \theta \cos (\tau_1 + \Delta)$$

$$E_y = A \sin \varphi \sin (\tau_2 - \Delta)$$

$$E_z = 0$$

We may calculate the torque for a simple case (assuming a doubling refracting medium of permeability unity; and take as x, y, z axes the principal axes of the tensor χ, for the light frquency in question). One gets:

$$L = \left(\frac{A}{4\pi}\right)^2 n \lambda \sin 2\theta (\sin 2\Delta_2 - \sin 2\Delta_1)$$

for the torque per unit area on the crystal between z_1 and z_2. (The derivation of this formula is given by R. Beth (P.R.50,115/936))

Quantum Theory: Here one has to assume explicitly that the angular momentum of a circularly polarized photon is $\pm \hbar$ (right and left circ. pol.)

Consider an elliptically polarized light wave, propagated in the z direction in a vacuum, whose components along an arbi-

trary pair of perpendicular x and y axes in the plane of the ellipse are:

$$E_x = a_1 \cos(\zeta + \delta_1)$$
$$E_y = a_2 \cos(\zeta + \delta_2) \qquad (1)$$

Supose we have now the same wave whose components are given as functions of the principal axes of the ellipse 0η, 0ξ. Then the components E_ξ and E_η are related to E_x and E_y by

$$E_\xi = E_x \cos\psi + E_y \sin\psi$$
$$E\eta = -E_x \sin\psi + E_y \cos\psi \qquad (2)$$

The equation of the ellipse referred to 0ξ, 0η is:

$$E_\xi = a \cos(\zeta + \delta_o)$$
$$E_\eta = \pm b \cos(\zeta + \delta_o) \qquad (3)$$

The two signs distinguish the two possibles senses in which the end point of the electric vector may describa the ellipse. To determine a and b, we compare (2) and (3) and use (1), and one gets:

$$E_\xi = a(\cos\zeta\cos\delta_o - \sin\zeta\sin\delta_o) = a_1(\cos\zeta\cos\delta_1 - \sin\zeta\sin\delta_1)\cos\psi + a_2(\cos\zeta\cos\delta_2 - \sin\zeta\sin\delta_2)\sin\psi$$

$$E_\eta = b(\sin\zeta\cos\delta_o - \cos\zeta\sin\delta_o) = -a_1(\cos\zeta\cos\delta_1 - \sin\zeta\sin\delta_1)\sin\psi + a_2(\cos\zeta\cos\delta_2 - \sin\zeta\sin\delta_2)\cos\psi$$

Compare the coefficients of $\cos\zeta$ and $\sin\zeta$

$$a\cos\delta_o = a_1\cos\delta_1\cos\psi + a_2\cos\delta_2\sin\psi \qquad (a)$$

$$a\sin\delta_o = a_1\sin\delta_1\cos\psi + a_2\sin\delta_2\sin\psi \qquad (b)$$

$$b\cos\delta_o = a_1\sin\delta_1\sin\psi - a_2\sin\delta_2\cos\psi \qquad (c)$$

$$b\sin\delta_o = -a_1\cos\delta_1\sin\psi + a_2\cos\delta_2\cos\psi \qquad (d)$$

Squarring an adding (a), (b) and (c), (d); we have

$$a^2 = a_1^2 \cos^2\psi + a_2^2 \sin^2\psi + 2\, a_1\, a_2 \cos\psi \sin\psi \cos\delta$$

$$b^2 = a_1^2 \sin^2\psi + a_2^2 \cos^2\psi - 2\, a_1\, a_2 \cos\psi \sin\psi \cos\delta$$

Hence $$a^2 + b^2 = a_1^2 + a_2^2 \qquad (4)$$

Multipliyng (a) x (c) and (b) x (d) and adding:

$$a \cdot b = a_1\, a_2 \sin\delta \qquad (5)$$

(4) and (5) are the natural invariancs of the wave. We can easely regard the wave (1) as coherence superposition of left and right circularly polarized components of amplitudes L and R respectivly

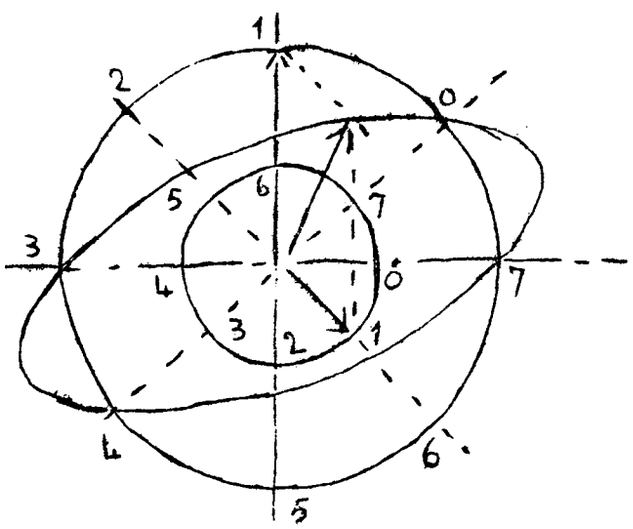


fig. 9

Fig. 9 shows by graphical construction that this is possible. One can see by calculation that:

$$L^2 - R^2 = a_1\, a_2 \sin\delta$$
$$L^2 + R^2 = \tfrac{1}{2}(a_1^2 + a_2^2) \qquad (6)$$

The number of left circularly polarized photons transmitted per unit area per second is the Poynting vector intensity $cL^2/4\pi$ divided by the energy per photon $h\nu$ or

$$cL^2/4\pi\, h\nu = L^2\, \lambda /4\pi\, h$$

The angular momentum of the light beam left circularly polarized is then $\frac{L^2 \lambda \nu}{4\pi h} = \frac{L^2 \lambda}{8\pi^2 2}$

according to the assumption stated above.

Considering the similar expression for the right circularly polarized component, the angular momentum of the bean is

$$M = \frac{\lambda}{8\pi^2} (L^2 - R^2) \qquad (7)$$

If now the state of polarization of the light is changed by transmission through a double refracting plate, M will change, which means that momentum is transferred to the plate.

In the Quantum Thoery (expression (6) L or R change and in the classical theory λ changes. To obtain the torque on a section of the crystal from expression (7) for a wave in a vacuum, we place the face of crystal at $z = z_1$ (vacuum for $z < z_1$) and ussing the conservation of angular momentum and Fresnel's relations for the amplitudes of the reflected and transmitted waves we obtain an expression that is the excess of angular momentum per unit area, per second flowing into a section of the crystal at $z = z_1$ over that flowing out at $z = z_2$

$$M_1 = (\frac{A}{4\pi})^2 \; n \lambda \; \sin 2\theta \; \sin 2 A_1$$

Hence both the field and quantum theories predict the same value for this effect which has been meassured with a double refracting plate arrangement, fig.10 .-

M is a quartz circular half wave plate, hanging at the bottom of a quartz fiber forming a torssional pendulum.-

T is a fixed plate above M with a relfecting layer of aluminum on the top.

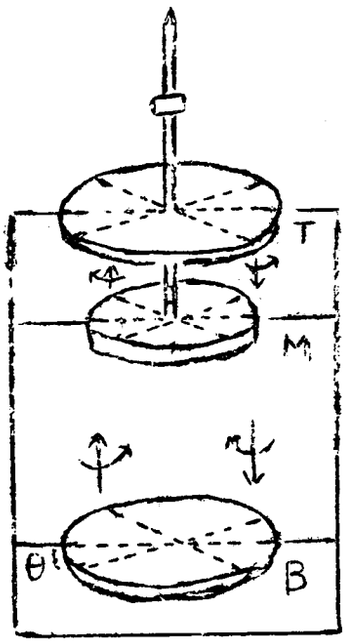


Fig. 10

The ideas underlying fig. 10 may be qualitativly understood as follows. The light coming from a Nickol prism upward through the plates is reflected by the aluminum layer on the top side of T passing downward again through the plates.

For a certain wave length λ , B is a quarter wave plate. If light of wave length λ from the Nickol, linearly polarized at $\theta = 45^{\circ}$ enters the bottom plate B, both, direct and reflected beams will be circularly polarized in each space between the plates in the directions indicated by the curved arrows and force the plate M (double refracting) to turn.

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POLARIZATION IN QUANTUM MECHANICS

To describe polarized electromagnetic radiation in quantum mechanics one needs two basic quantum states corresponds to the two possible polarizations i.e.

a) two states of linear polarization with their planes of vibration of the electric vector perpendicular to each other.

b) two states of circular polarized light, one left and other right.

As we have shown in the study of light

it is possible to obtain any beam of light by a combination of these two basic states.

A definite state of polarization is described by a wave function

$$\Psi = c_1 \Psi_1 + c_2 \Psi_2$$

this corresponds to a totally polarized beam (elliptically pol.); if one chooses conveniently the system of axis it is possible to transform this wave function in order to have

$$\Psi = c_1' \Psi \qquad \text{or} \qquad \Psi = c_2' \Psi_2$$

We call such a state a pure state.-

States that are not pure (natural or partially polarized beam of light) have to be described by incoherent superposition of pure states (with adequate weights). Incoherent superposition means that in order to calculate the probability of finding a certain experimental results, one must first calculate the probability for each of the pure states and then take an average (atributing weights to each of the pure states). For example: if one scatters a beam of unpolarized protons by He^4, the resulting beam has more protons in one direction than in the other. To find out the amounts in each state, one has to perform another scattering of the polarized protons in He^4.

A pure state is identified by the coefficients c_1 and c_2 of the expansion of the wave function Ψ in the eigenfunctions Ψ_1 and Ψ_2

The expectation value of an operator Q is

$$\langle Q \rangle = \int \Psi^* Q \Psi \, d\tau$$

Taking the complex conjugate for Ψ this expression becomes:

$$\langle Q \rangle = \int (c_1 \Psi_1 + c_2 \Psi_2)^* Q (c_1 \Psi_1 + c_2 \Psi_2) \, d\tau =$$

$$= c_1 c_1^* \int \Psi_1^* Q \Psi_1 \, d\tau + c_1^* c_2 \int \Psi_1^* Q \Psi_2 \, d\tau + \ldots = \sum_{n'n} Q_{nn'} c_n^* c_{n'}$$

where $Q_{n'n} = \int \psi^*_{n'} \, Q \, \psi_n \, d\tau$

Because of the ortogonality of and the two cross term vanish and we have:

$$\langle Q \rangle = c_1^* c_1 \int \psi_1^* \, Q \, \psi_1 \, d\tau + c_2^* c_2 \int \psi_2^* \, Q \, \psi_2 \, d\tau$$

One way for representing Q is to define the matrices:

$$||\rho|| = \begin{pmatrix} c_1^2 & c_1^* c_2 \\ c_1 c_2^* & c_2 \end{pmatrix} \quad \text{and} \quad ||Q|| = \begin{vmatrix} \int \psi_1^* Q \psi_1 d\tau & \int \psi_2^* Q \psi_1 d\tau \\ \int \psi_1^* Q \psi_2 d\tau & \int \psi_2 Q \psi_2 d\tau \end{vmatrix}$$

then:

$$\langle Q \rangle = \quad T_r \left\{ ||\rho|| \cdot ||Q|| \right\}$$

The matrix $||\rho||$ is call density matrix. For the case of linearly polarized light, we have:

$$||\rho|| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{or} \qquad ||\rho|| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

If the state is not pure one can always decompose its coherence matrix, after diagonalizing it

$$||\rho|| = \begin{pmatrix} \rho' & 0 \\ 0 & \rho'' \end{pmatrix} = \rho'' \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (\rho' - \rho'') \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (1-P) \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} + P \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

This is the incoherent supperposition of the unpolarized beam $\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$ (natural light) and the totally polarized beam

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

P is called <u>degree of polarization</u>. A different way to characterize the state of polarization is by giving the Stoke's parameters of the light beam:

$$\xi_1 = |c_1|^2 - |c_2|^2$$
$$\xi_2 = c_1 c_2^* + c_1^* c_2$$
$$\xi_3 = i \, (c_1 c_2^* - c_2 c_1^*)$$

That this are really the Stoke's parameters can be prooved by rememberign their definition:

$$\begin{cases} s_1 = a_1^2 - a_2^2 \\ s_2 = 2\, a_1\, a_2 \cos \delta \\ s_3 = 2\, a_1\, a_2 \sin \delta \end{cases}$$

where a_1 and a_2 are the amplitudes of the wave (corresponds to the c' s).-

It can be shown then that:

$$||\rho|| = \begin{pmatrix} c_1\, c_1^* & c_1\, c_2^* \\ c_2\, c_1^* & c_2\, c_2^* \end{pmatrix} = \frac{1}{2}\,(1 + \vec{\xi} \cdot \vec{\sigma})$$

where $\vec{\xi}$ is a vector with the three components given above, and $\vec{\sigma}$ has the Pauli matrices as components. One can also show that

$$P = |\vec{\xi}|$$

If one has a certain ideal polarization detector (for example one which detect only linear polarization in a certain plane, equivalent to Nickol prism), one can characterize it by the wave function of the photons to which it is sensible.-

$$\psi^{det} = c_1^{det}\, \psi_1 + c_2^{det}\, \psi_2$$

The probability of a response of the detector to a quantum given by

$$\psi = c_1\, \psi_1 + c_2\, \psi_2$$

is:

$$W = \left| \int \psi^{*det}\, \psi\, d\tau \right|^2 = \left| c_1^{*det}\, c_1 + c_2^{*}\, det\ \ c_2 \right|^2$$

One can show castly that in the density matrix formalism

$$\rho^{det} = \frac{1}{2}\,(1 + \vec{\xi}^{\,det} \cdot \vec{\sigma})$$

The probability of response of the detector is then

$$W = T_r\,(\rho\, \rho^{det}) = \frac{1}{2}\,(1 + \vec{\xi}\ \vec{\xi}^{\,det})$$

INTERPRETATION OF STOKES PARAMETERS IN QUANTUM MECHANICS

In the transition between two states of definites quantum numbers (j;m) and (j';m') completely polarized light is emited. In general this will be elliptically polarized.

Let us consider the representative point in the Poincaré sphere and choose a convenient system of coordinates with axis k (direction of propagation), θ and ϕ (directions of latitude and longitude); in these axis the electric field has components E_θ and E_ϕ, then we have:

$$\frac{E_\varphi}{E_\theta} = \frac{C_2}{C_1}$$

Moreover, the difference in phase between the two components is $\delta = \frac{\pi}{2}$ (a particular case we are considering).

The Stokes parameters are: $S_1^2 = a_1^2 - a_2^2 = C_1^2 - C_2^2$

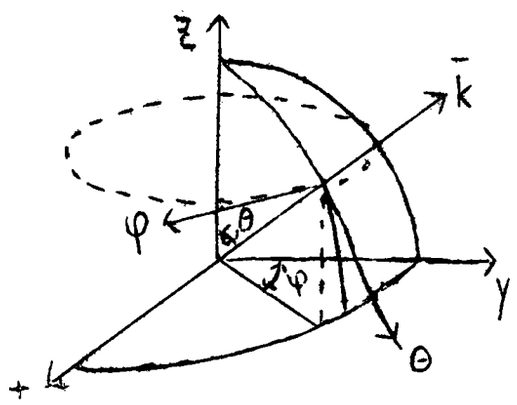

fig. 11

$$S_2^2 = 2\, a_1 a_2 \cos\delta = C_1 C_2^* - C_2 C_1^* \ldots C$$

$$S_3^2 = 2\, a_1 a_2 \operatorname{sen}\delta = i(C_1 C_2^* - C_2 C_1^*)$$

but we have: $\frac{I_\varphi}{I_\theta} = \left(\frac{E_\varphi}{E_\theta}\right)^2 = \frac{C_2^2}{C_1^2}$

where $\begin{cases} I_\varphi = E_\varphi^2 \\ I_\theta = E_\theta^2 \end{cases}$

then $\begin{cases} \frac{I_\theta - I_\varphi}{I_\theta} = \frac{C_1^2 - C_2^2}{C_1^2} \\ \frac{I_\theta + I_\varphi}{I_\theta} = \frac{C_1^2 + C_2^2}{C_2^2} \end{cases}$

Finally we have:

$$S_1^2 = C_1^2 - C_2^2 = \frac{I_\theta - I_\varphi}{I_\theta + I_\varphi} \qquad S_3^2 = \frac{2(I_\theta I_\varphi)^{1/2}}{I_\theta + I_\varphi}$$

The meanning of $S_2 = 0$ is obvious; the points in the Poincaré sphere are along a single meridian.

S_3^2 corresponds to the fraction of circular polarized light (to see this remember that when $S_1 = 0$ the corresponding points are on the poles of the sphere).

S_1^2 corresponds to points on the equator (linear polarization).-

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II - THE PRODUCTION OF PLANE POLARIZED RADIATION

Radiation from an accelerated charge.-

Field of an oscilating dipole.-

So far we have not considered the sources which emit the waves, those sources can be either antennas which emit waves, or radiating atoms. Hertz has found a matematical solution of the field equations, which gives a unified representation of the sources and radiation field in the whole space.

Consider then Maxwell's equations in free space:

$$\left.\begin{array}{ll} \text{I)}\ \ \text{rot}\ \vec{H} = \frac{1}{c}\ \frac{\partial \vec{E}}{\partial t} & \text{div}\ \vec{E} = 0 \\ \text{II)}\ \ \text{rot}\ \vec{E} = -\frac{1}{c}\ \frac{\partial \vec{H}}{\partial t} & \text{div}\ \vec{H} = 0 \end{array}\right\} \quad \text{III}$$

Since $\text{div}\ \vec{H} = 0$ we can express $\vec{H}$ as a curl of an auxiliary vector potencial. Hertz introduced this vector potencial as a time derivative of a vector $\vec{Z}$, therefore we have

$$(1) \qquad \vec{H} = \frac{1}{c}\ \text{rot}\ \frac{\partial \vec{Z}}{\partial t}$$

Since we can reverse the order of diferentiation

$$(3) \qquad \vec{H} = \frac{1}{c}\ \frac{\partial}{\partial t}\ (\text{rot}\ \vec{Z})$$

And substituting in I : (3) $\frac{1}{c}\ \text{rot}\ \frac{\partial}{\partial t}\ (\text{rot}\ \vec{Z}) = \frac{1}{c}\ \frac{\partial}{\partial t}\ \vec{E}$

Integrating with respect to I : (4) $\text{curl}\ \vec{Z} = \vec{E} = \text{grad div}\ \vec{Z} - \triangle \vec{Z}$

In order to satisfy II we must differentiate (1) with respect to t.

$$(5) \quad \text{rot}\ \vec{E} = -\frac{1}{c^2}\ \text{rot}\ \frac{\partial^2 \vec{Z}}{\partial t} \qquad \text{and integrating gives}$$

$$(6) \quad \vec{E} = -\frac{1}{c^2}\ \frac{\partial^2 \vec{Z}}{\partial t^2} + \text{grad}\ \phi$$

where the integration constant is given by grad ϕ, because since curls of the two vectors are equal it means that the

vectors differ only by an irrotational vector: grad ϕ .
And if we take ϕ = div $\vec{Z}$, then the equations (4) and (6) are equal, and we have: (7) $\Delta \vec{Z} = \frac{1}{c^2} \frac{\partial^2 Z}{\partial t^2}$

This is a wave equation for $\vec{Z}$, and assuming that $\vec{Z}$ is a function of $\underline{r}$ only we can write (7) in spherical coordinates:

$$(8) \quad \frac{\partial^2 \vec{Z}}{\partial r^2} + \frac{2}{r} \frac{\partial \vec{Z}}{\partial r} = \frac{1}{c^2} \frac{\partial^2 \vec{Z}}{\partial t^2}$$

where the solution for $\bar{Z}$ will be a spherical wave of simple periodic variations with time: (9) $\vec{Z} = \vec{F}(r)\, e^{i\omega t}$

then we have :(10) $\frac{\partial^2 \vec{F}}{\partial r^2} + \frac{2}{r} \frac{\partial \vec{F}}{\partial r} + \frac{\omega^2}{c^2} \vec{F} = 0$

And the solution is of this type:(11) $\vec{F} = \frac{\vec{P}_o}{r}\, e^{-i\omega \frac{r}{c}}$

where the negative sign means that the wave is outgoing from a center r = 0.- Thus we have for $\bar{Z}$

$$(12) \quad \vec{Z} = \frac{\vec{P}_o}{r}\, e^{i\omega (t - r/c)} = \frac{\vec{P}(t)}{r}\, e^{-i\omega r/c}$$

where $\vec{P}(t) = \vec{P}_o\, e^{i\omega t}$

We can obtain $\bar{H}$ and $\bar{E}$ by simple diferentiation of $\bar{Z}$, by putting

$$f(r) = \frac{e^{-i\omega r/c}}{r}$$

then we have:

$$\text{rot } \vec{Z} = \text{rot}\left[\vec{P}(t)\, f(r)\right] = \text{grad } f \times \vec{P} + f \text{ rot } \vec{P}$$

but p(t) is a function only of $\underline{t}$, therefore curl $\vec{p}(t) = 0$

$$\text{rot } \vec{Z} = \text{grad } f \times \vec{P} = \frac{df}{dr} (\vec{r}_o \times \vec{P}) = \frac{df}{dr}\, e^{i\omega t}\, (\vec{r}_o \times \vec{P}_o) \; ; \; \vec{r}_o = \vec{r}/r$$

but $$\frac{df}{dr} = \frac{d}{dr} \frac{e^{-i\omega r/c}}{r} = -\left(\frac{i\omega}{rc} + \frac{1}{r^2}\right) e^{-i\omega r/c}$$

$$\text{rot } \vec{Z} = -\left(\frac{i\omega}{rc} + \frac{1}{r^2}\right) e^{-i\omega (t - r/c)} (\vec{r}_o \times \vec{P}_o)$$

Magnetic field.

$$\vec{H} = \frac{1}{c} \frac{\partial}{\partial t} \text{ rot } \vec{Z}$$

$$(14) \qquad \vec{H} = \frac{i\omega}{c} \left(\frac{1}{r^2} + \frac{i\omega}{r\,c}\right) e^{i\omega(t-r/c)} \quad (\vec{P}_o \times \vec{r}_o)$$

Thus we see that the magnetic field vector is composed of two parts, with the same direction, but of different intensities, one part decreasing with $1/r^2$ and the other with $1/r$. We can see this better if we rewrite it in terms of wave lenght: (15)

$$\frac{1}{r^2} + \frac{i\omega}{rc} = \frac{1}{r}\left(\frac{1}{r} + \frac{2\pi i}{\lambda}\right)$$

so that in the near region $r \ll \lambda$, the first term predominates, while in the far region $r \gg \lambda$, on the radiation field, the second term is more important.

Let us examine the near region first

$$\vec{H}_n = \frac{i\omega}{c} \frac{1}{r^2} e^{i\omega(t-r/c)} \cdot (\vec{P}_o \times \vec{r}_o)$$

$$(16) \qquad \vec{H}_n = \frac{1}{cr^2} \left(\frac{\partial \vec{P}(t)}{\partial t} \times \vec{r}_o\right) e^{-i\omega r/c}$$

From the study of electrodynamics one has the Biot-Savart's law:

$$(17) \qquad \vec{H} = \frac{I}{c} \cdot \frac{\vec{d}_s \times \vec{r}_o}{r^2}$$

where I is the current, and $\vec{d}_s$ a segment of the wire where the current flows.

Comparing (16) and (17) we see that except for the term $e^{-i\omega r/c}$ the field in the near region is due to a current element:

$$(18) \qquad I\,\vec{d}s = \frac{\partial \vec{P}(t)}{\partial t} = i\omega \vec{P}_o \, e^{i\omega t}$$

$\vec{p}(t)$ can be written as the moment of the dipole of lenght ℓ and periodically oscilating charges $\pm q_o e^{i\omega t} = q(t)$

$$\therefore \vec{p}(t) = q(t)\vec{\ell} \longrightarrow \frac{\partial \vec{p}(t)}{\partial t} = \frac{\partial q}{\partial t} \cdot \vec{\ell} \qquad (19)$$

where $\frac{\partial q}{\partial t}$ is the current flowing as produced by the charge. Thus we have that the solution we obtained gives a magnetic field due to a dipole of fixed axis and oscilating moment p near the dipole region.

The term $e^{-i\omega r/c}$ in (16) corresponds to the velocity of propagation and gives the "Retarded Field" as r becames larger. The kind of dipole, as shown above, can be represented by a short linear antenna in which high frequency oscilatory currents are induced. The law of Biot Savart is valid only at short distances from the antenna, because at larger distances we have seen that the second term in (14) is more predominant. We can see that this second term corresponds to the rate of change of the current by writing:

$$(20)\quad \vec{H}_R = -\frac{\omega^2}{c^2}\ \frac{1}{r}\ \ell\ i\omega\left(t-\frac{r}{c}\right)\ (\vec{p}_o \times \vec{r}_o) = \frac{1}{cr^2}\left(\frac{\partial^2 p(t)}{\partial t^2} \times \vec{r}_o\right) e^{-i\omega r/c}$$

where the second derivative of $\bar{p}(t)$ gives the rate of change of the current, and corresponds to an acceleration of the oscilating charge.

Electric field.

In a similar manner we can calculate $\vec{E}$ by combining equations (4) and (7):

$$(21)\qquad \vec{E} = \text{grad div } \vec{Z} - \frac{1}{c^2}\ \frac{\partial^2 \vec{Z}}{\partial t^2}$$

and from (12):

$$(22)\qquad \text{div } \vec{Z} = e^{i\omega t}\ \text{div}\left(\vec{p}_o\ \frac{e^{-i\omega r/c}}{r}\right)$$

and in a similar manner as before we can dedude that:

$$(23)\qquad \text{div } \vec{Z} = e^{i\omega t}\ \frac{1}{r}\ \frac{df}{dr}\ \vec{p}_o \cdot \vec{r}$$

whence

$$(24)\quad \text{grad div } \vec{Z} = e^{i\omega t}\left[\vec{p}_o\ \vec{r}_o\ \left(\frac{\partial^2 f}{\partial r^2} - \frac{1}{r}\ \frac{df}{dr}\right) r_o + \frac{1}{r}\ \frac{df}{dr}\ \vec{p}_o\right]$$

and we also have

$$(25) \qquad \frac{df}{dr} = -\left(\frac{i\omega}{rc} - \frac{1}{r^2}\right) e^{-i\omega r/c}$$

$$(26) \qquad \frac{d^2f}{dr^2} = \left(-\frac{\omega^2}{c^2 r} + \frac{2i\omega}{c\, r^2} + \frac{2}{r^3}\right) e^{-i\omega r/c}$$

$$(27) \qquad \frac{1}{c^2}\frac{\partial^2 \vec{z}}{\partial t^2} = -\frac{\omega^2}{c^2}\cdot\frac{1}{r}\,\vec{p}_o\, e^{i\omega(t - r/c)}$$

We can see that in the near region which is governed by the highest power of $1/r$, i.e. $1/r^3$, the electric field as derivable from a scalar potential given by:

$$(28) \qquad {}_n = -\operatorname{div}\vec{Z}$$

The term

$$\frac{1}{c^2}\frac{\partial^2 \vec{Z}}{\partial t^2}$$

has no effect on the scalar potential because it does not contain the term $1/r^3$.

Combining the equations (28), (23) and (25) and neglecting lower powers of $1/r$ we have:

$$(29) \qquad \varphi_n = \frac{\vec{p}_o \cdot \vec{r}}{r^3}\, e^{i\omega(t - r/c)} = \frac{\vec{p}_o\,(t)\cdot\vec{r}_o}{r^2}\, e^{-i\omega r/c}$$

where $\bar{r} = \bar{r}_o r$.- We see that this is a potential due to a dipole moment $\bar{p}(t)$, taking into account the variations of phase with increasing distance due to the finite velocity of propagation. This is then called the "Retarded Potential".

Let us consider now the far region. In this case let us neglet the higher powers of $1/r$, and consider only the terms which contain $1/r$ in the equation (25), (26) and (27), then we get

$$\vec{E}_R = \frac{\omega^2}{c^2 r}\, e^{i\omega(t-r/c}\left[\vec{p}_o - (\vec{p}_o \cdot \vec{r}_o)\,\vec{r}_o\right]$$

or since $r_o^2 = 1$ and $\vec{r}_o \cdot \vec{r}_o\,\vec{p}_o = \vec{p}_o\,\vec{r}_o \cdot \vec{r}_o$

$$\vec{E}_R = \frac{\omega^2}{c^2 r}\, e^{i\omega(t-r/c)}\left[\vec{p}_o \cdot \vec{r}_o^{\,2} - \vec{r}_o \cdot \vec{r}_o\,\vec{p}_o\right]$$

$$(30) \qquad \vec{E}_r = \frac{\omega^2}{c^2 r}\, e^{i\omega(t-r/c)}\left[\vec{r}_o \times (\vec{p}_o \times \vec{r}_o)\right]$$

Comparing (30) and (20) we can see that the electric and magnetic field in the far regions are equal in magnitude, and only differ in direction, being at right angles to each other. Taking the axis of a dipole as the polar axis, the magnetic field has the direction of the latitude circles, while the electric field is along the meridians. Thus the vectors $\vec{E}$, $\vec{H}$, $\vec{r_o}$ form a right handed system. The amplitudes of $\vec{E}$ and $\vec{H}$ are not constant along the surface of the sphere, but as can be seen from

$$|\vec{p}_o \times \vec{r}_o| = p_o r_o \sin\theta$$

they decrease towards the pole of the sphere, ($\theta = 0$).

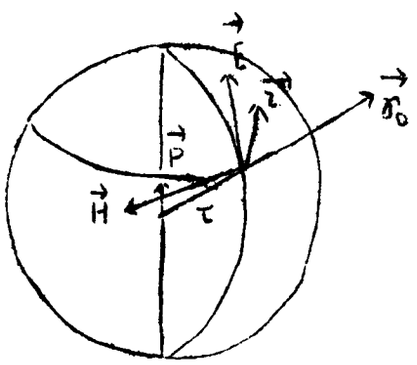


Fig. 12

Thus the linear oscilator does not radiate in the direction of its axis. As can be seen from fig. 12, the radiation is always plane polarized and in particular, on the equator, the direction of polarization is parallel to the axis of the dipole, and the amplitude is maximum.

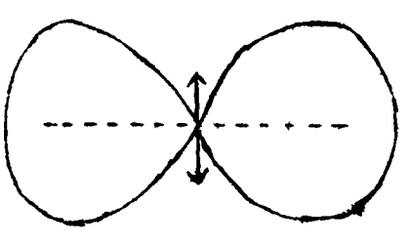


Fig. 13

Let us calculate the energy radiated by the dipole oscilator. This is given by the Poynting vector:

$$\vec{S} = \frac{c}{4\pi} (\bar{E} \times \bar{H})$$

Substituting the real parts of $\bar{E}$ and $\bar{H}$, and taking the modulus we have:

$$|\vec{S}| = \frac{c}{4\pi} \frac{\omega^4}{c^4 r^2} p_o^2 \sin^2\theta \cos^2\omega (t-r/c)$$

and taking the time average, since $\overline{\cos^2 \zeta} = \frac{1}{2}$ we have

$$\overline{|\vec{S}|} = \frac{c}{8\pi} \frac{\omega^4}{c^4 r^2} p_o^2 \sin^2\theta$$

we see that the angular distribution of the radiated energy has the from shown on fig. 13.-

Field of an accelerated charge.- We shall now consider radiation produced by an accelerated charge (we will study a charge which is deaccelerated; this corresponds to the brems-strahlung radiation).

Let us first consider a charge moving with a slow uniform motion, i.e. $v \ll c$, which at the instant $t = 0$ is at a point O.

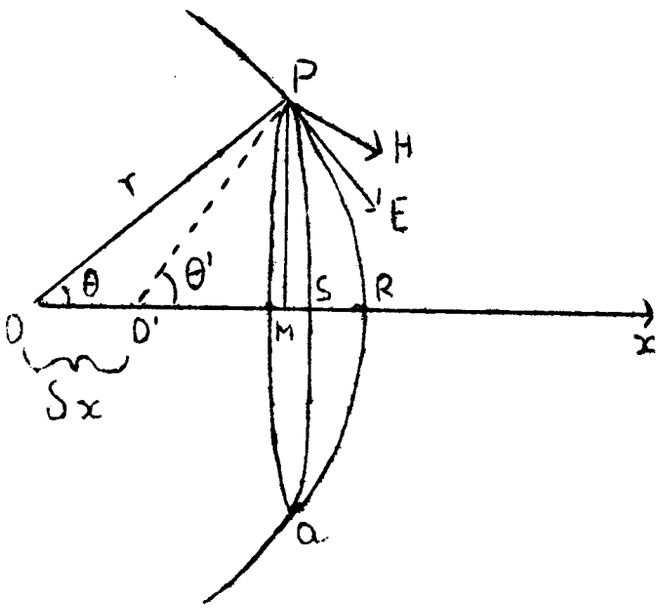


fig. 14

We wish to determine the field at a point P. The number of force lines trough a sphere with center in O, and radius OP is equal to the flux of the displacement across the sphere which is equal to the charge at O, i.e. charge e (Gauss theorem)

$$\int \vec{D} \cdot d\vec{s} = 4\pi e$$

if we consider the circle PSQ, then the displacement through this circle is:

$$D = e \frac{\text{Area P R Q}}{\text{Area sphere}} = e \frac{2\pi R}{2\ 0\ R}$$

$$(1) \qquad D = \frac{e}{2}(1 - \cos\theta)$$

Then the displacement current through PSQ is $i_D = \frac{dD}{dt}$ whic is the same as if $\frac{dD}{dt}$ was the rate at which the charge traverse, the circle. So that the work done in carrying the unit charge about the circle is

$$\int H \cdot ds = \frac{4\pi}{c} i_D = \frac{4\pi}{c} \frac{d\,D}{d\,t}$$

$$(2) \quad H\,2\pi\,r\,\sin\theta = \frac{4\pi}{c} i_D = \frac{4\pi}{c} \frac{d}{d\,t}\left[e/_2 (1-\cos\theta) \right] =$$

$$= \frac{2\pi}{c} e \sin\theta \frac{d\theta}{dt}$$

$$(3) \qquad H = e/_{rc} \frac{d\,\theta}{dt}$$

Suppose now that the charge is slowing down, and eventually comes to rest at a point O!- The observer at P will not see anything before a time $t = r/c$, and if δt is the time necessary for traversing OO', then the observer will see the charge at O' only after a time $t = \frac{r}{c} + \delta t$. The displacement through the circle PSQ due to the charge at O' will be:

$$D' = \frac{e}{2}(1 - \cos\theta')$$

so that the change in displacement due to the stopping of the charge during a time δt, is:

$$\frac{\delta D}{\delta t} = \frac{D - D'}{\delta t} = \frac{e}{2}(\cos\theta' - \cos\theta) =$$

$$= \frac{1}{2} e \frac{\delta\cos\theta}{\delta t} = -\frac{1}{2} e \sin\theta \frac{\delta\theta}{\delta t}$$

Therefore:

(4) $$i_D = -\frac{1}{2} e \sin\theta \, \delta\theta/\delta t$$

But

$$\delta\theta = \frac{\delta x}{r}\sin\theta = \frac{\delta v}{c}\sin\theta ; \frac{\delta\theta}{\delta t} = \frac{1}{c}\sin\theta\frac{\delta v}{\delta t} = -\frac{a}{c}\sin\theta$$

where $a = -\delta v/\delta t$ is the deaccelleration of the charge. Substituting in (4): $i_D = \frac{1}{2}\frac{ae}{c^2}\sin^2\theta$ and substituting in (2):

$$H = \frac{1}{2}\frac{ae}{c}\sin^2\theta\frac{1}{2\pi r\sin\theta}\frac{4\pi}{c} ; H = \frac{a e}{c^2 r}\sin\theta$$

Suppose now the charge is moving with a velocity comparable to that of light $v \sim c$ and is deaccelerated. In this case we calculate H in a system of coordinates which is moving with the electron at a constant velocity $v \sim c$. Then, by using Lorentz' transformation we obtain in the laboratory system:

$$H = \frac{ea}{rc^2}\frac{\sin\theta}{(1 - \beta\cos\theta)^3} \qquad \beta = v/c$$

and since $|E| = |H|$ the Poynting vector is

$$I = |\bar{S}| = \frac{c}{4\pi}|\bar{H}|^2 = \frac{a^2 e^2}{4\pi r^2 c^3}\frac{\sin^2\theta}{(1 - \beta\cos\theta)^6}$$

A polar diagram of intensity using β as a parameter show us that a slowly moving charge $\beta \sim 0$, it irradiates at 90º to the direction of motion, while as the velocity of the charge grows larger, $\beta \sim 1$, the maximum intensity of the radiation is in the forward direction.

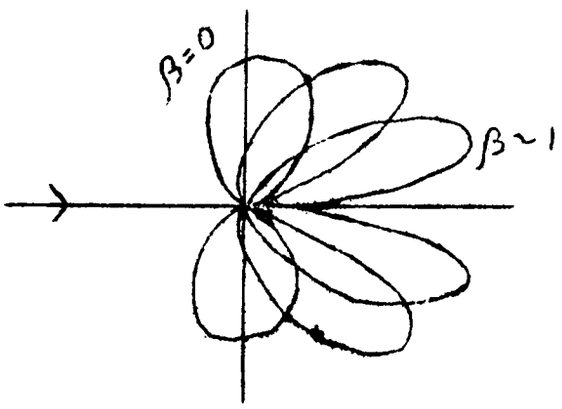


fig. 15

References:

1.- "Theoretical Physics" Joos

2.- "X-Rays in theory and experiment" - Compton and Allison Samuel King.

POLARIZATION IN THE COMPTON EFFECT.

The earliest experiments performed on the inteaction of X-rays with matter were the ones on the reflection of X-rays from the surface of a mirror; they were not succesful because the X-rays were diffusely scattered in all directions by the mirror.

The same was true in the reflection by paraffine or any other object on which X-rays impinged. J.J. Thompson interpreted this scattering as probably due to the interaction of the X-rays with the electrons. Treating the X-rays as a classical electromagnetic wave Thompson derived an expression for the scattering which would be produced by each electron regarded as free (Binding energy of the electron $\ll$ energy of the incident photon), acted upon by the electric field of the incident electromagnetic wave. Then each electron oscillates with a frequency equal to that of the incident radiation and in a direction parallel to that of the electric vector. This oscillating charge radiates as an oscillating dipole and its radiation field is the scattered radiation.

If a wave whose electric intensity is E, traverses an electron of charge e and mass m, the acceleration on this is

$$\frac{\vec{E}\, e}{m} .$$

Substituting this acceleration in the expression for the radiation field we have:

$$E_\theta = \frac{e \sin\theta}{r\, c^2} \cdot \frac{E\, e}{m} = \frac{E\, e^2 \sin\theta}{r\, m\, c^2}$$

$\theta \equiv$ angle between the electron's acceleration and the ray we are considering.

Since the intensities of both the primary and the secondary rays are proportional to the square of the electrics vectors, we have

$$\frac{I_\theta}{I} = \frac{E_0^2}{E^2} = \frac{e^4 \sin^2\theta}{r^2\, m^2\, c^4} \qquad (1)$$

If the primary ray is unpolarized, the acceleration of the scattered electron will be at a random direction on a plane perpendicular to the primary beam.

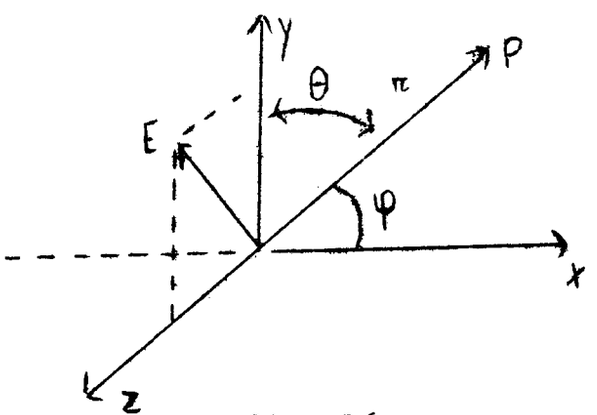


fig. 16

Let us take two rectangular axes in this plane, O_y and O_z, such, that O_y is in the plane POx in which lies the scattered ray we are studying. The electric vector of the primary ray may be resolved into two components, E_y and E_z, such that:

$$E^2 = E_y^2 + E_z^2$$

Since the direction of E in the yOz plane is at random, E_y is on the average equal to E_z, whence on the average

$$E_y^2 = E_z^2 = \frac{1}{2} E^2$$

thus:

$$I_y = I_z = \frac{1}{2} I$$

where E_y and E_z represent the intensities of the y and z components of the primary beam.

The intensity of the scattered beam at P due to E_y is, from (1)

$$I_{\theta\, y} = I_y \frac{e^4 \sin^2\theta_y}{r^2 m^2 c^4} - \frac{1}{2} \frac{e^4}{r^2 m^2 c^4} \cos^2\phi \quad (2)$$

where ϕ is the scattering angle i.e. angle between the primary and the scattered rays.- Similarly that for the z compo-

nent is:

$$I_{\theta z} = \frac{1}{2} I \frac{e^4}{r^2 m^2 c^4} \quad (\text{since } \theta_z = \pi/2) \qquad (3)$$

Thus we have for the intensity of the scattered beam from a single electron:

$$I)_e = I_{\theta y} + I_{\theta z} = I \frac{e^4}{2r^2 m^2 c^4} (1 + \cos^2 \phi)$$

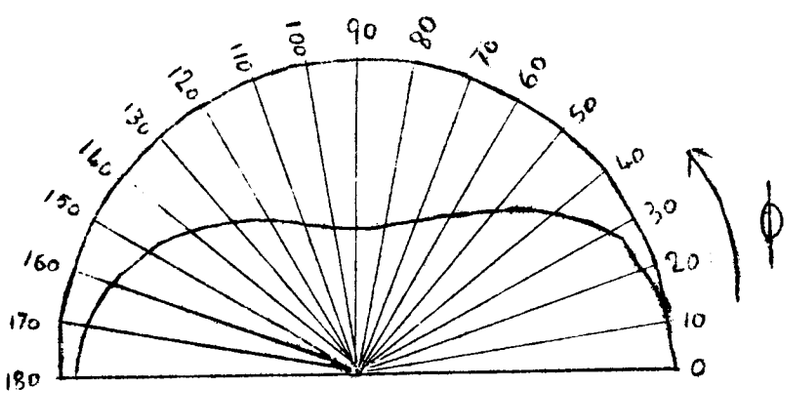


fig. 17

This angular distribution is the low energy limit for the Compton scattering $h\nu \gtrsim 10$ kev. as described by the Klein-Nishina formula taht we will see below.

Polarization of scattered X-Rays: A significant confirmation of the principles underlying Thompson's theory of X-Rays scattering comes from a study of the polarization of scattered X-Rays. According to equation (2) the intensity of the $I_{\theta y}$ component is zero at right angles with the primary beam ($\phi = \pi/2$) whereas the oppositly polarized component, equation (3), keeps its normal intensity. Thus in this direction the scattered beam should be completely linearly polarized.

Klein-Nishina Formula:(*) An accurate description of the Compton interaction can't be deduced from classical electrodynamics alone, after Dirac devellopped his relativistic theory of the electron; Klein and Nishina applied Dirac's theory to

(*) For details on the theoretical derivation of the Klein-Nishina Formulas, see the article by J. Kallen in vol. V, part 1 of Encyclopedia of Physics)

this problem and obtain the expression:

$$d(\sigma_e) = \frac{r_o^2}{4} d\Omega \left(\frac{\nu'}{\nu_o}\right)^2 \left[\frac{\nu_o}{\nu'} + \frac{\nu'}{\nu_o} - 2 + 4\cos^2\Theta\right] cm^2/\text{electron}$$

for the differential collis. cross section in the case of linearly polarized incident radiation, interacting with stationary and unbound electrons whose magnetic moments are randomly oriented.

Here Θ is the angle between the electric vector of the incident radiation $\vec{E}_o$ and that of the scattered radiation $\vec{E}'$; $r_o = \frac{e^2}{m c^2}$ is the classical electron radius and $d\Omega$ is the element of solid angle through which the scattered photon emerges after the collision.-

BREHMSTRAHLUNG

According to classical electromagnetic theory an accelerated electron radiates energy. In order to calculate the radiation in any particular case we have to set up a condition for the negative acceleration and investigate the energy radiated in this way. The earliest thoeretical attempt was the pulse theory developped by Stokes and J.J. Thompson.-

Stokes considered the irregular negative accelerations of the electrons being stopped in a target and the resulting irregular pulses of electromagnetic radiation.

Thompson simplified the theory by considering the electrons as suffering a uniform negative acceleration in a direction opposite to the motion.-

On the basis of this oversimplified assumption we have:

1) The X-Rays should be completely polarized with the electric vector in the same plane as the negative acceleration.
2) The spatial distribution of the intensity should be given by (as we have seen)

$$I = \frac{a^2 e^2}{4\pi r^2 c^3} \sin^2\Theta$$

Where

a is the acceleration of the electron at the previous time

r is the distance to the point of observation

θ is the angle between the direction of the electron beam and the direction of observation.

I would then be zero at 0 and 180º and a maximun at 90º rather than at samller angles as is observed experimentally.

Relativistic considerations: If one considers the bombarding electrons to be subject to a constant negative acceleration along their direction of motion and besides that the velocity of these electrons is appreciable with respect to the velocity of light, relativistic corrections become important. These corrections were made by Sommerfeld (4) with the result that the intensity at distance r and angle θ when the electron acceleration is a at the previous time -. r/c is as we have shown before

$$I = \frac{a^2 e^2 \sin^2 \theta}{4\pi r^2 c^3 (1-\beta \cos)^6}$$

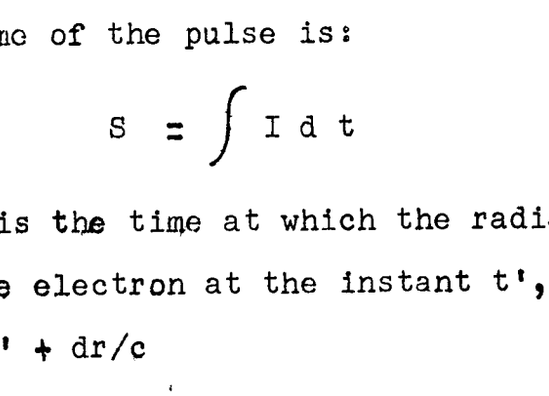


fig. 18

The total intensity at the point (θ,r) integrated over the entire time of the pulse is:

$$S = \int I \, d \, t \qquad (1)$$

If t is the time at which the radiation reaches P which left the electron at the instant t', then $t = t' + r/c$

Thus $dt = dt' + dr/c$

But by the figure 18 we see that

$$dr = v\,dt'\cos\theta = -\beta c\cos\theta\,dt'$$

Hence

$$dt = dt' - \beta\cos\theta\,dt' = dt'(1-\beta\cos\theta)dt'$$

but

$$a = c\,d\beta/dt' \quad \therefore\ dt' = \frac{c}{a}\,d\beta$$

and

$$dt = \frac{c}{a}(1-\beta\cos\theta)\,d\beta$$

Substituting this expression in (1), we have

$$S = \int_{\beta}^{0} \frac{a^2 e^2 \sin^2\theta}{4\pi r^2 c^3(1-\beta\cos\theta)^6}\ \frac{c}{a}(1-\beta\cos\theta)\,d\beta$$

(The upper limit of the integration corresponds t o the radiation emitted by an electron that stop in terget).

$$S = \frac{a e^2}{4\pi r^2 c^2}\sin^2\theta\int_{\beta}^{0}\frac{d\beta}{(1-\beta\cos\theta)^5} = \frac{a e^2}{16\pi^2 c^2}\ \frac{\sin^2\theta}{\cos\theta}\left[\frac{1}{(1-\beta\cos\theta)^4}-1\right]$$

This expression is plotted in fig. for different values of β.

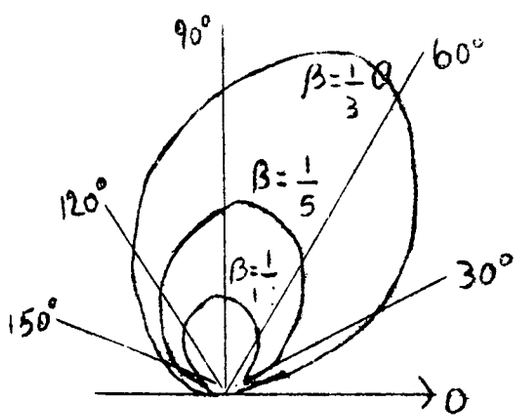


fig. 19

Two significant differencies in spatial distribution appear between the classical electromagnetic relativistic theory and experiment: compare fig.19 with fig. 20

1) The angle of maximun intensity is greater according to this theory than that which is actually observed.- This may be accounted for by the fact that Sommerfeld assumed that electrons are brought completely to rest in the target (thik target). In experiments with thin targets the electrons were not

greatly retarded in passing through the foil.

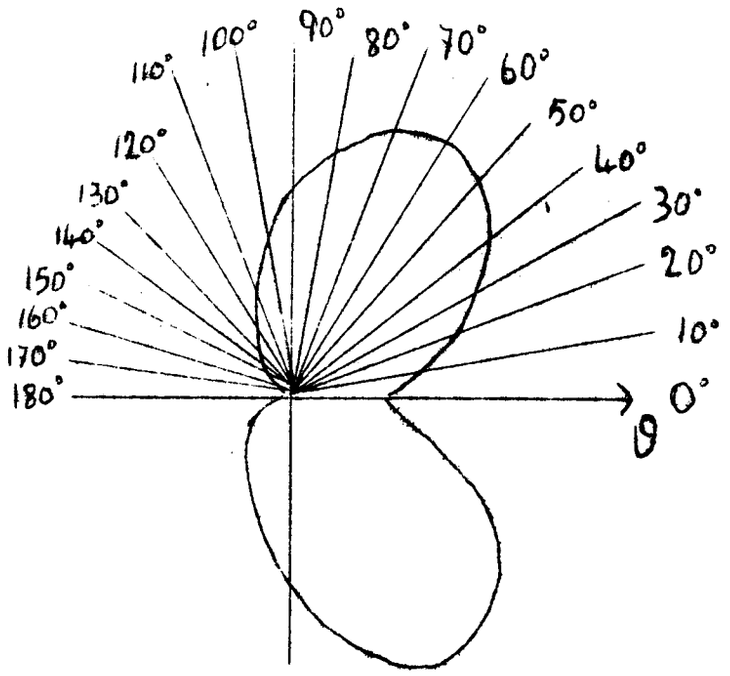


fig. 20

2) The intensity at 0º is predicted by the theory to be zero whereas the experiments yield substantial intensity at 0º. Perhaps the difference is caused by the fact that the electrons are not all slowed down in the direction of motion; rather some transverse motion is certain to take place under experimental conditions.

The dependence of the cross section for brehmstrahlung on the atomic number Z for relativistic electrons has been calculated by Bethe and Heitler both for unshielded nuclei and for nuclei screened by the orbital electrons in the atoms; this is a quantum mechanical calculation.-

The energy and angular distribution of radiation from fast electrons in very thin targets has been considered on the basis of Bethe-Heitler theory of brehmstrahlung by Schiff[5] and by Lawson[6].

Screening is taken into account; the cross section that gives the energy distribution is then

$$\sigma(k,x)dk\,dx = \frac{4Z^2}{137}\left(\frac{e^2}{m\,c^2}\right)^2 \frac{d\,k}{k}\,x\,dx\left\{\frac{16\,x^2E}{(x^2+1)^4E_o} - \frac{(E_o+E)^2}{(x^2+1)^2\,E_o^2} + \right.$$

$$\left. + \left[\frac{E_o^2+E^2}{(x^2+1)^2E_o^2} - \frac{4\,x^2\,E}{(x^2+1)^4E_o}\right] \ln\, M\,(x)\right\}$$

where

$$x = \frac{E_o \theta_o}{\mu} \; ; \; \frac{1}{M(x)} = \left(\frac{\mu k}{2 E_o E}\right)^2 + \left(\frac{Z^{1/2}}{e(x^2+1)}\right)^2$$

E_o = energy of the incident electron

E = energy of the scattered electron

$\mu = m_o c^2$ = rest energy of an electron

$k = E_o - E$ energy of the radiated quantum

θ_o = angle between the quantum and the incident electron.

x is the reduced angle

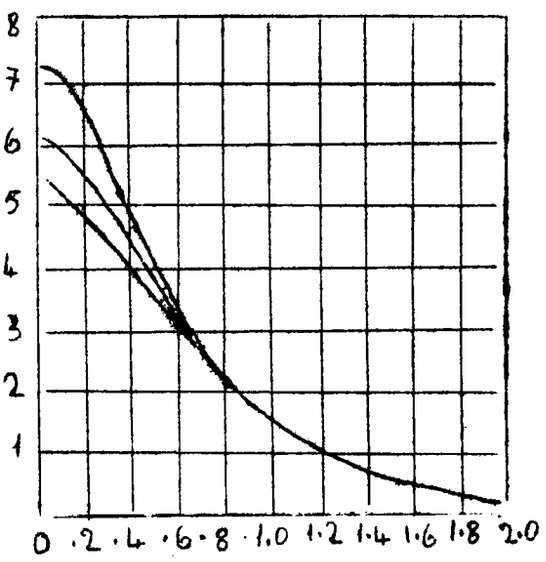


fig. 21

The effect of multiple scattering on angular distribution using thick targets has received satisfactory experimental verification, as have also the effects of multiple scattering and radiation loss on the energy spectrum.-

Suppose a beam of electrons traversing a thick target fig 22 . The distribution of the electrons after they have suffered multiple scattering according to Williams is

$$f(\theta, t) = \frac{1}{2\pi \bar{\theta}_t^{\,2}} \; e^{-\theta^2 / 2 \bar{\theta}_t^{\,2}}$$

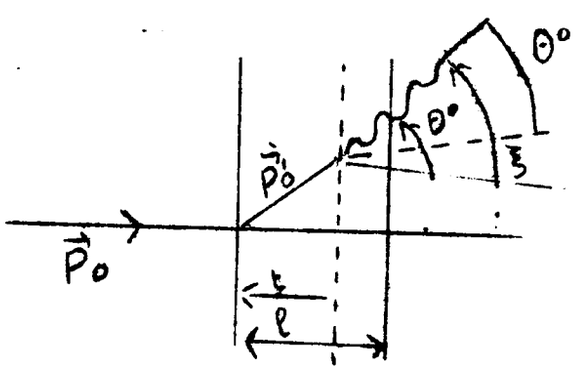


fig. 22

where

$$\bar{\theta}_t = \frac{m_o c^2}{E} \times 4{,}5 \times 10^{-2} \frac{Z}{\sqrt{A}} \sqrt{t}$$

Z = atomic number

t = thickness in mg/cm^2.

A = atomic mass

Before irradiating the electron may travel in any direction.

Consider an incident electron $\vec{p}_o$ that has a probability $p(\theta, \varphi)$ of being scattered in the direction $[\theta = (\vec{p}_o', \vec{p}_o), \varphi]$. Then it radiates and emits a photon in the direction $\theta_o = (\vec{p}_o', \vec{k})$ with a probability $d\sigma(\vartheta_o)$.-

The probability of observing a photon in an angle ξ with the initial direction $\vec{p}_o$ is the product of those two probabilities:

$$p(\theta,\varphi)\, d\sigma(\vartheta_o)$$

such that the combination of (θ,φ) and θ_o gives ξ.
To a given value (θ,φ) corresponds one value θ_o for fixed ξ.

The probability $p(\theta,\varphi)$ is proportional to the total number of electrons which make an angle (θ,φ) with the initial direction, then

$$p(\theta,\varphi)\, d\Omega = \left[\int_{t=0}^{\ell} f(\theta,t)\, dt \right] d\Omega$$

Then we get for the angular distribution of the photon, taking into account the multiple scattering inside the target:

$$d\sigma(\xi) = \int p(\theta,\varphi)\, d\sigma(\theta_o)\, d\Omega$$

i.e. this gives the number of photons emitted in Θ_o direction by the electrons scattered in the solid angle $d\Omega$ arround the Θ direction.-

Referencies:

1) L.I. Schiff, Phys. Rev. 83, 252, 1951

2) P.T. Mac Cornick et al. Phys. Rev. 103, 29, 1956

3) Rapport C.E.A. nº 655, 1957

PRODUCTION OF CIRCULARLY POLARIZED RADIATION

In order to understand the production of circularly polarized radiation we will discuss first the Zeeman effect.

Larmor's thoerem states that, to a first approximation the change in the motion of an electron around a nucleus, produced by the introduction of a magnetic field, is a precession of the orbit about the field direction with uniform frequency

$$\omega_L = \frac{H e}{2 m c}$$

This can be demonstrated using classical electrodynamics. Suppose an electron is moving in a fixed orbit of radius r, and velocity v.-

Then it makes $\frac{v}{2\pi r}$ revolutions por second, which is equivalent a current

$$i = \frac{e v}{2 \pi r}$$

But this current produces a magnetic moment $\mu = \frac{i A}{c}$ where A in our case is πr^2; hence

$$\mu = \frac{e v a}{2 c} \quad (a = r)$$

But the electron will have an angular momentum about the center given by : p = mva, so that we have:

$$\frac{\mu}{p} = \frac{e}{2 m c} :$$

giromagnetic ratio.- Suppose now we introduce a magnetic field $\mathcal{H}$, which will alter the motion of the electron. Let the vector magnetic moment $\vec{\mu}$, and angular momentum $\vec{p}$ make an angle θ with the magnetic field $\mathcal{H}$ as seen in the fig. 23

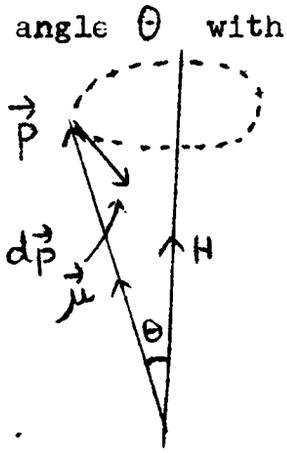


fig. 23

Then the torque produced by $\vec{H}$ on $\vec{p}$ will add an increment $dp = \vec{\mu} \vec{H} \sin\theta \, dt$ Thus it can be sees that the end of vector $\vec{p}$ will precess around the axis $\vec{H}$, describing a circle of ra-

dius $r = p \sin\theta$.- During a time dt, it will revolve trough an angle

$$d\theta = \frac{d\,p}{p \sin\theta} = \frac{\mu H \sin\theta\, dt}{p \sin\theta} \; ; \quad \frac{d\theta'}{dt} = \frac{\mu H}{p}$$

Thus it will precess with an angular velocity:

$$\omega_L = \frac{\mu H}{p} = \frac{e H}{2 m c}$$

A simple illustration of this theorem is a top with a constant angular momentum. When it is actied on by a torque, which tends to change the direction of its axis, it is observed to acquire a rotation, or to precess about a third axis with an angular velocity ω . It is interesting to know that the frequency of precession is independent of the orientation, angle θ . Hence we can conclude from Larmor's theorem that the introduction of a magnetic field modifies the classical motion of an electron in a field of a nucleus chiefly as if an uniform rotation at angular velocity ω_L, were superposed upon the original motion.

ZEEMAN EFFECT.- CLASSICAL TREATMENT.-

In 1896, Zeeman observed that when a sodium source was put in a strong magnetic field, then the spectral lines of the first principal doublet were considerably broadened. Later he was able to analyze this effect using a nicol prism and quarter wave plate, and verify Lorentz theory, namely, that if the spectrum is viewed in a direction parallel to the magnetic field, the light should be circularly polarized, and if the direction is perpendicular to the field, then the polarization is linear. Let us consider an assembly of electrons moving in orbits oriented at random in space. Let us take one electron and resolve its motion in three components x, y , and z. Then along each of this axis the motion will be a simple armonic one. If we do this

for all the electrons, the average amplitude along each of this axis will be thesame. Then the radiation emited, as viewed in the direction z, will be unpolarized because it cames from electrons randomly oriented in space. Let us now introduce a magnetic field parallel to the z axis. The effect of the field will be such that, as was seen be Larmor's theorem, the angular velocity ω_o will be modified by an amount

$$\omega_L = \frac{e\,H}{2mc}$$

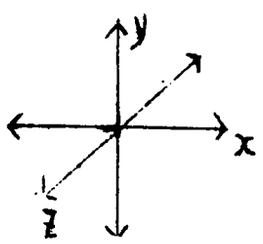
If the field is normal to and up from the page, then the electrons moving in the counterclockwise direction in the plane of the page, are speed up by an amount ω_L moving with a frequency:

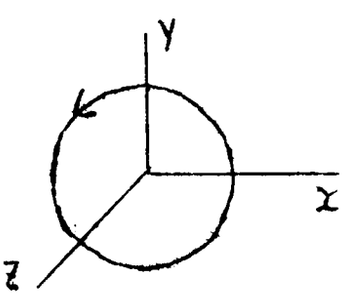
$$\nu_o + \frac{e\,H}{4\pi\, m\, c} = \nu_o + \nu_L$$

while the electrons in the clockwise direction are slowed down by the same amount and have a final frequency:

$$\nu_o - \frac{e\,H}{4\,\pi\, mc} = \nu_o - \nu_L$$

Since the magnetic field is in the z direction it will have no effect on the z component of the electrons motion, but will modify the x and y components, in such a way that the x^+ and y^+(counterclockwise) components will speed up, while the x^- and y^-(clockwise) components will slow down, by an amount ν_L. If we combine now the new x^+, y^+ and x^-, y^- components we will obtain resultant circular motions of frequency $\nu_o + \nu_L$ and $\nu_o - \nu_L$

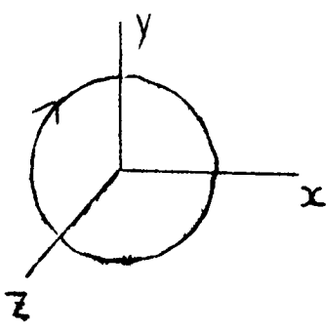




resultant of the new motion x^+ y^+

Hence the effect of magnetic field on the motion of the electrons is

as if one third of the electrons were moving with unchanged frequency ν_o along the z axis, one third moving with a counter-clockwise circular motion normal to be z axis with frequency



$\nu_o + \nu_L$, and the other third moving wirh clockwise circular motion normal to the z axis of frequency $\nu_o - \nu_L$, fig. . In this way, when viewed in the direction of the field, the Z motion does not emit light in the z direction, on the x and y motions, so that we shall observe circularly polarized light of frequency $\nu_o + \nu_L$ and $\nu_o - \nu_L$ which cones from those two circular motions. But when viewed fron an x or y directions, then the z motion will emit linearly polarized light, and with electric vector parallel to the direction of the field. The circular motion on the other hand, viewed fron the x or y direction, that is edge on, are observed as linearly polarized with electric vector perpendicular to the field. Thus we see that the Zeeman effect consist of splitting a spectral line of unpolarized light of frequency ν_o by introducing a magnetic field, into three polarized lines of frequencies $\nu_o - \nu_L$, ν_o and $\nu_o + \nu_L$.-

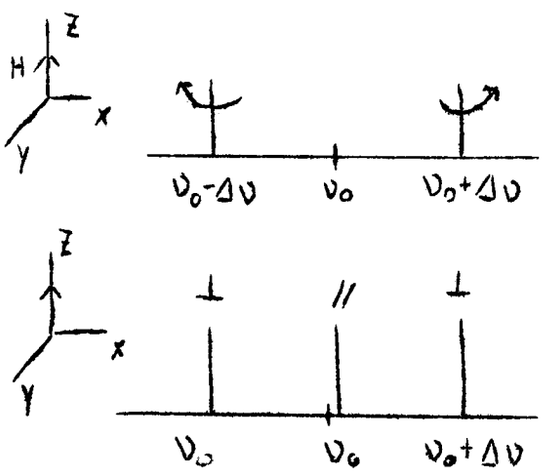


fig. 24

ZEEMEN EFFECT. QUANTUM MECHANICAL TREATMENT.-

Let us consider the simple case of an atom having a single electron. Then the eigenfunctions in polar coordinates are:

$$U_{n\ell m} = R_{n\ell}(r)\, P_{m}(\theta)\, e^{im\varphi}\, \frac{1}{\sqrt{2\pi}}$$

The transitions from the state with quantum numbers n, ℓ m, to the state n', ℓ', m' are given by the matrix elements of the coordinates z, x and y. Let us then examine first the matrix element $Z^{n'\ell'\,m'}_{n\,\ell\,m}$ and since $z = r\cos\theta$ we have

$$Z^{n'\ell'\,m'}_{n\,\ell\,m} \int U^{*}_{n'\ell'\,m'}\, Z\, U_{n\ell\,m}\, d\tau =$$

$$= \int_0^{\infty} r^2\, dr\, R_{n'\ell'}(r)\, R_{n\ell}(r) \int_0^{\pi} P_{\ell'm'}(\theta) P_{\ell m}(\theta) \cos\theta \sin\theta\, d\theta \int_0^{2\pi} \frac{1}{2\pi} e^{i(m-m')\varphi}\, d\varphi$$

The integral over φ will vanish unless $m = m'$, so we get a selection rule for the magnetic quantum number $\boxed{\Delta m = 0}$ and this will correspond to radiation emited with linear polarization parallel to z.

For the purpose of evaluating the integral over θ, we use the recursion formula for Legendre polinomials

$$P_{\ell\,m}(\cos\theta)\cos\theta = C_1 P_{\ell+1,m} + C_2 P_{\ell-1,m}$$

Substituting in the integral θ in (2) we get

$$\int_0^{\pi} C_1 P_{\ell'm'}(\theta) P_{\ell+1,m}(\theta) \sin\theta\, d\theta + \int_0^{\pi} C_2 P_{\ell'm'}(\theta) P_{\ell-1,m}(\theta) \sin\theta\, d\theta$$

And by ortogonality property of the Legendre functions we have

$$\int_0^{\pi} P_{\ell'm'}(\theta)\, P_{\ell m}(\theta) \sin\theta\, d\theta = \delta_{\ell\ell'}$$

We conclude that unless $\ell' = \ell \pm 1$ the integral over θ will vanish, so we have

$$\Delta\ell = \pm 1$$

and the matrix element reduces to

$$z \begin{cases} Z^{n'\ell'+1,\,m}_{n\ell\,m} = C_1\, R^{n'\ell'+1}_{n\ell} \\ Z^{n'\ell'-1,m}_{n\ell\,m} = C_2\, R^{n'\ell'-1}_{n\ell} \\ Z^{n'\ell'\,m}_{n\ell\,m} = 0 \qquad \text{for all other} \end{cases}$$

The constant C_1 and C_2 depend only on (l) an m

In order to evaluate the matrix elements of the x and y components we consider the linear combination of both of them:

$$x + i\,y = r \sin\theta\, e^{i\varphi} \qquad x - i\,y = r \sin\theta\, e^{-i\varphi}$$

So the matrix element for a transition $n, \ell\, m$ to n', ℓ', m', will be given by .

$$(x \pm iy)^{n'\ell' m'}_{n\ell m} = R^{n'\ell'}_{n\ell} \int_0^{\pi} P_{\ell' m'}\, P_{\ell m} \sin\theta \sin\theta\, d\theta \int_0^{2\pi} \frac{e}{2\pi}^{i(m \pm 1 - m')\varphi} d\varphi$$

Where:

$$R^{n'\ell'}_{n\ell} = \int R_{m'\ell'}(r)\; R_{n\ell}(r)\; r^3\, dr$$

The integral over φ vanish unless $m' = m \pm 1$ so we get the selection rule $\Delta m = \pm 1$. This will correspond to polarization parallel to x and y axis, but not the z axis. So that a linear combination will give a circularly polarized light with $m' = m - 1$ for right handed and $m' = m + 1$ for left handed polarization. Evaluating the integral over θ in a similar manner as before we get the selection rule $\Delta \ell = \pm 1$ and the matrix elements become

$$(x + iy)^{n', \ell' + 1, m+1}_{n\ell\, m} = C_3\, R^{n'\ell'+1}_{n\ell} \qquad (x+iy)^{n'\ell - 1, m+1}_{n\ell\, m} = - C_5\, R^{n'\ell - 1}_{n\ell}$$

$$(x - iy)^{n', \ell + 1, m-1}_{n, \ell, m} = -C_4\, R^{n'\ell+1}_{n\ell} \qquad (x-iy)^{n'\ell - 1\; m-1}_{n\ell\, m} = C_5\, R^{n'\ell - 1}_{n\ell}$$

Let us introduce a magnetic field on the atom, and see how it is afected in energy. We can write the Hamiltonian of the system in polar coordinates as follows (Bohm-Chap.15)

$$H = \frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{\ell(\ell+1)}{\hbar^2 r^2}\right) + \frac{e\mathcal{H}}{2\mu c}\frac{\hbar}{i}\frac{\partial}{\partial \varphi} + \frac{e^2\mathcal{H}^2}{8\mu^2 c^2 \ell^2} + V(r)$$

where $\mathcal{H}$ is the magnetic field. Considering only a weak magnetic fiel, we can neglect the term in $\mathcal{H}^2$, and assuming that

$$L^2 = (\ell + 1)\,\ell\,\hbar^2$$

and $L_Z = mh$, we obtain

$$H = -\frac{\hbar^2}{2\mu}\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2}\right] + \frac{e\mathcal{H}}{2\mu c}\hbar\, m + V(r)$$

The only effect of the magnetic field is then to add a constant to the energy proportional to the azimutal number m. This means that some of the degeneracy is removed because levels of different m now have different energues. Let us illustrate this in a simple example of a transition from level $\ell = 2$ to a level $\ell = 1$

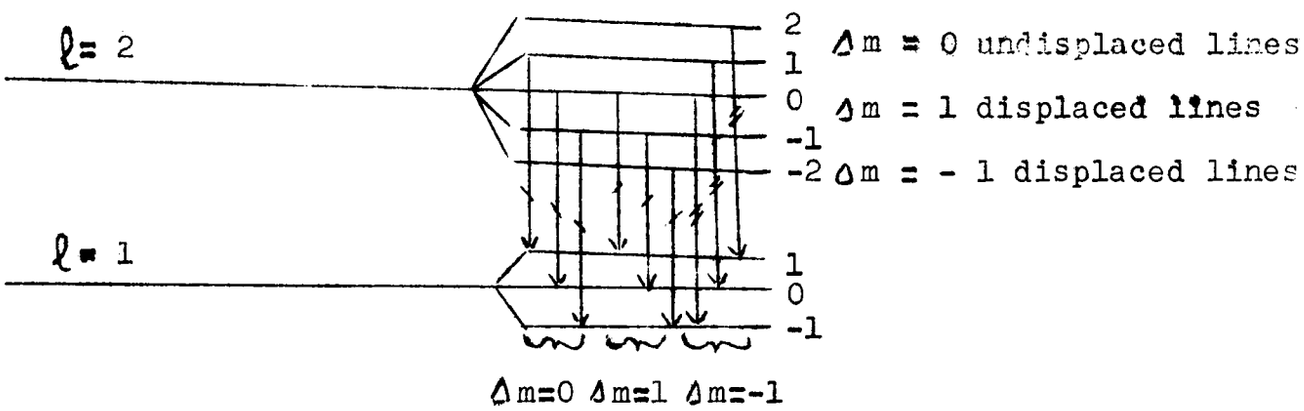


fig. 25

This is a dipole transition $\Delta\ell = 1$; $\Delta m = 0, \pm 1$.- In a transition in which $\Delta m = 0$, the initial level is displaced as much as the final level, when we introduce the magnetic field, which removes the degeneracy in m. Hence the radiated frequency will not be changed. If $\Delta m = -1$ the final level is raised more than the initial level, and the radiated angular frequency is reduced by

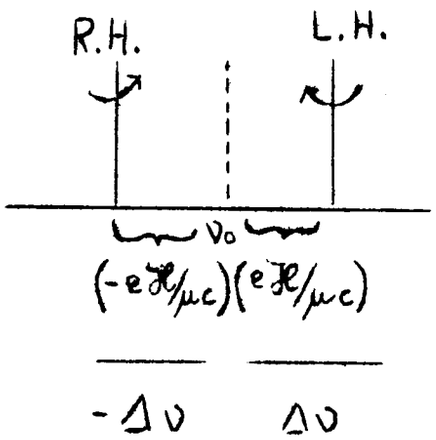
$$\Delta\nu = \frac{\Delta E}{h} = \frac{-e\mathcal{H}}{4\pi\mu c}\ ; \quad \mu = \text{reduce mass}$$

Conversely, when $\underline{\Delta m = -1}$ the radiated angular frequency is increased by:

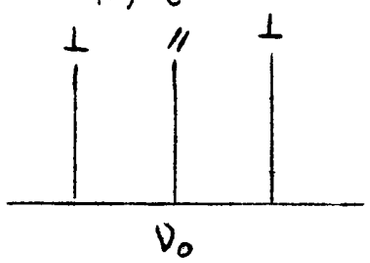
$$\Delta\nu = \frac{e\mathcal{H}}{\mu c}$$

Thus the spectral lines, will in general split into three components, exactly as predicted by the classical

theory. If the radiation is viewed along the direction of the magnetic field, only x and y components appear on the matrix elements, so that the transition $\Delta m = 0$ does not contribute to the line. So that what we see are only the two circularly polarized components, with $\Delta m = +1$ for right handed polarizatio and $\Delta m = -1$ left handed, and with angular frequency difference between them of $\frac{e\mathcal{H}}{2\pi\mu c}$



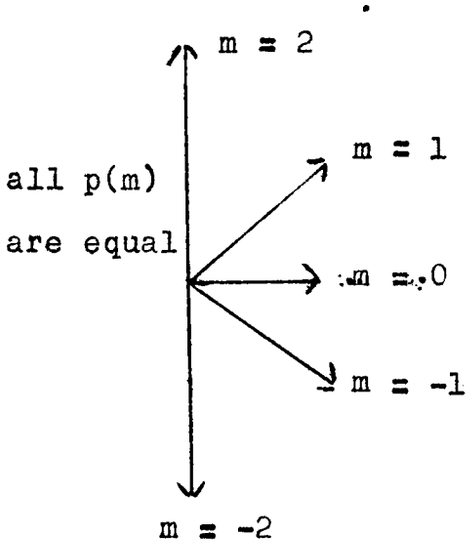
On the other hand if the light is viewed normal to the magnetic field, say in the x direction, then it can be polarized, either in the z or y direction. In a transition with $\Delta m = 0$ we have seen that we have polarization in the z direction, so the net result is that the line is split into three parts, first an undeviated part polarized in the z direction, and second, two parts displaced respectively by $\pm \frac{e\mathcal{H}}{4\pi\mu c}$ and polarized in a direction normal to z. Hence by using optical methods we can obtain polarized light since in the Zeeman effect different polarizations have different frequencies.



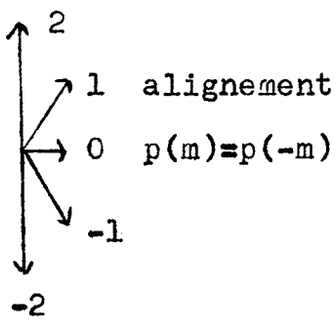
ALIGNEMENT OF NUCLEI.

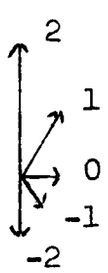
Since the splitting of energy levels due to Larmor's precession is much smaller in nuclear case than in the atomic case by a factor of 1840, we need much greater magnetic fields in order to produce a Zeeman effect in a nucleus. In fact, the necessary magnetic field is so big (at room temperature) that it can not be produced in the laboratory, and in this way other methods have to be used in order to polarize the radiation.

Let p (m) be the population of a substate m. In general, since all the nucleus are oriented at random, the populations of all the substates will be the same. Consider a particular case of $\ell = 2$; then we have $m = \pm 2, \pm 1, 0$ and all p (m) are equal. So we have no polarization, and no alignement. If now by some means we manage to populate some states more than other (for example using very low temperatures or angular correlation) but in such a way that $p(m) = p(-m)$ then we have alignement. If in the other hand we manage to populate only one state, with a definite m, more than all the other states then we have polarization. In particular we have here $p(m) \neq p(-m)$. Let us examine some experimental methods for doing this.



a) Low temperature, high magnetic field: Direct interaction of nuclear magnetic dipole moment with a large external magnetic field, causes the energy levels to split by an amount $\Delta E = \frac{\mu \mathcal{H}}{I}$ where I = total angular momentum; μ nuclear magnetic dipole moment. The population of the individual states is given by the Boltzmann distribution:





$$p(m) = \frac{e^{m\beta}}{\sum_{m'} e^{m'\beta}} \qquad \beta = \frac{\mu \mathcal{H}}{k\,T\,I}$$

Since nuclear states of + m and - m have different energies, then they are differently populated and we can obtain nuclear polarization. In order to achieve high polarization, β must be large, at least of the order of unity, and this leads to

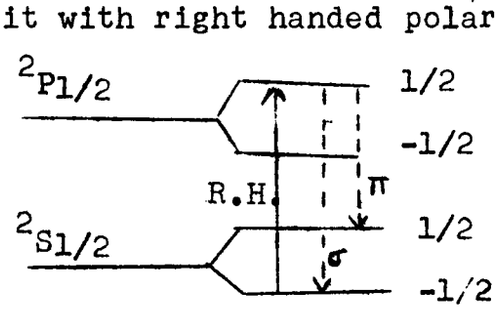
very high magnetic fields and very low temperatures. For example, for I = 2; μ 1 magneton, $\mathcal{H}/T = 5.10^7$ gauss/ºK; which means a magnetic field $\mathcal{H}$ of 50.000 gaus and temperature of 0,001ºK . This conditions are difficult to achieve in a laboratory, but at least one experiment of this kind has been done.

b) Another method was suggested, which avoids such strong magnetic fields by using the fact that the unfilled electron shells produces a magnetic field at the nucleus of a paramagnetic ion, of the order of 100.000 gauss. Thus at very low temperatures (0,01 ºK) and with the help of weak magnetic fields (a few hundred gauss) we can obtain nuclear polarization; since the nucleus is oriented with respect to electron orbit this small field is necessary to orient the orbits.

c) Bleaney and Daniels used a method based on the fact that the energy difference is due to interaction between the inhomogeneus cristallyne electric field and the electronic magnetic moments through the intermediary of the electronic orbits. Then the strong electric field of the neighboring ions or molecules in the chrystal defines an axis of quantization, and if it has an axial simmetry, we get an alignement of electronic magnetic moments relative to this axis. If we decrease the temperature to 0,01 ºK we get nuclear alignements, but no polarization.

This method does not use magnetic field.

d) There is another completely different method which uses the observation of polarized resonance radiation for orienting nuclei, and with subsequent emission of radiation. Consider for example sodium with a $^2S_{\frac{1}{2}}$ ground state. If we illuminate it with right handed polarized resonance radiation, then since



$\Delta m = +1$, the induced transition can be only from $-\frac{1}{2}$ of $^2S_{\frac{1}{2}}$ to $\frac{1}{2}$ of the $^2P_{\frac{1}{2}}$ state. The P state can decay either by emitting right

handed circular polarized radiation to the ground state $m = -\frac{1}{2}$ or by plane polarized to $m = \frac{1}{2}$ (since $\Delta m = 0$).

Thus if there is no interaction between the atoms (rarified gas) then we can obtain a greater population for the ground state $m = \frac{1}{2}$ than any other state, and in this way we can achieve nuclear polarization.

In all this methods we can separate the radiation emited by one definite substate, which is therefore polarized.

References:

1) White - "Introduction to atomic spectra"
2) Richtmayer, Kennard and Lauritsen - "Intr. to Modern Physics."
3) Bohm - "Quantum Mechanics"
4) Handbuch der Physik, vol. XXXV chap. 2.-
5) Id. vol. XLII chap. 6.-

DETECTION METHOD OF LINEARLY POLARIZED RADIATION

As we have seen, one can get linearly polarized radiation by Compton effect.

Let us consider the Klein-Nishina collision differential cross section for linearly polarized radiation abready explained:

$$d\sigma = \frac{r_o^2}{4} \left(\frac{\nu'}{\nu}\right)^2 \left(\frac{\nu}{\nu'} + \frac{\nu'}{\nu} - 2 + 4\cos^2\Theta\right) d\Omega \qquad (1)$$

Our first step is to sum this expression over all possible directions of polarization of the scattered photon. The angles and directions involved are defined in Fig. 26 which shows the polarization details in the plane ODAC taken from Fig. 27

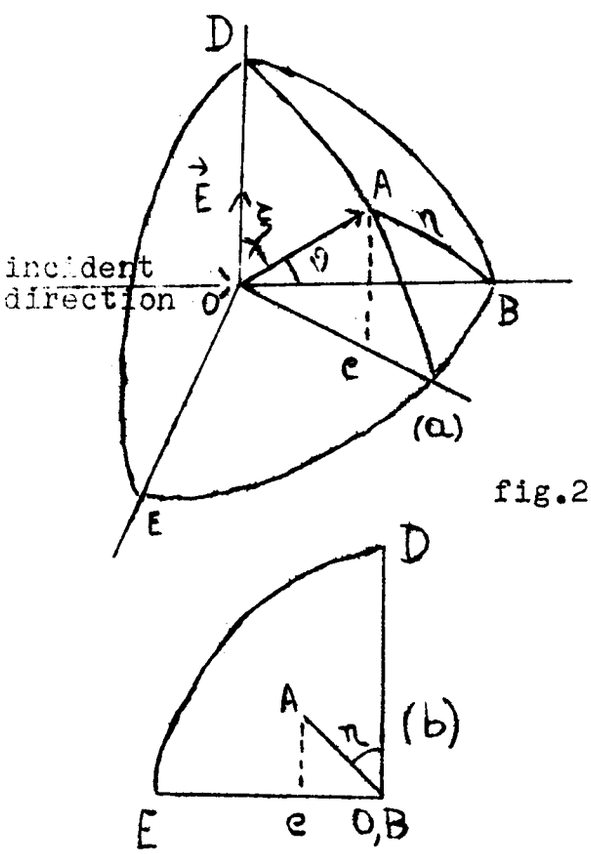


fig.26

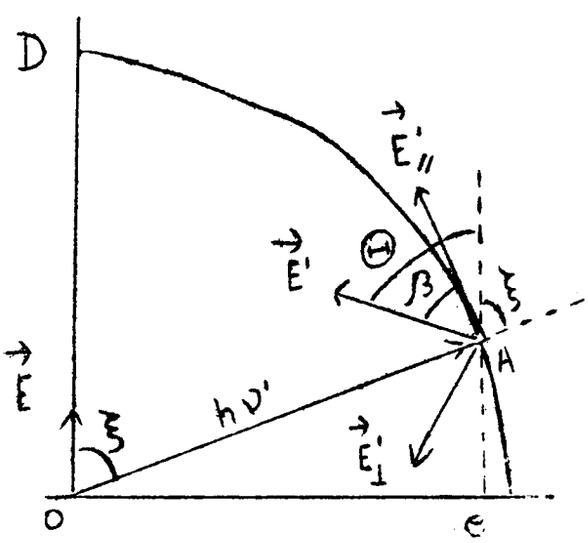


fig. 27

$\vec{E}$ electric vector of the incident linearly polarized radiation.

$\vec{E}'$ electric vector of the scattered radiation projects out of the plane of the page at an angle β so that:

$\vec{E}'_{//}$ component in the plane

$\vec{E}'_{\perp}$ component perpendicular to the plane

$\vec{E}$ and $\vec{E}'$ are transversed vectors

η angle between $\vec{E}$ and the scettering plane.

OAB scattering plane

Θ angle between $\vec{E}$ and $\vec{E}'$

ϑ scattering angle i.e. angle between the scattered photon $h\nu'$ and the direction of the incident photon $h\nu$.

We can see by Fig. that

$$\cos\Theta = \cos\beta \sin\xi$$

Substitutin in (1) we have:

$$d\sigma = \frac{r_o^2}{4}\left(\frac{\nu'}{\nu}\right)^2\left(\frac{\nu}{\nu'} + \frac{\nu'}{\nu} - 2 + 4\cos^2\beta \sin^2\xi\right) d\Omega \quad (2)$$

Since $E'_{//} = E' \cos\beta$ and $E'_{\perp} = E' \sin\beta$ we have

$E'^2 = E'^2 + E'^2_{//}$, the total scattered intensity in the direction ξ is proportional to E'^2 and hence to the sum of the orthogonal components. Then the collision differential cross section for an interaction in which an incident linearly polarized photon is scattered at a particular ϑ and ξ , when summed over all direction of polarization of the scattered photon is:

$$d\sigma = (d\sigma)_{//} + (d\sigma)_{\perp} \quad (3)$$

If we put $\beta = 0$ and $\beta = \frac{\pi}{2}$ in equation (2) we obtain $(d\sigma)_{//}$ and $(d\sigma)_{\perp}$ respectively; then (3) becomes:

$$d\sigma = \frac{r_o^2}{4}\left(\frac{\nu'}{\nu}\right)^2\left(\frac{\nu}{\nu'} + \frac{\nu'}{\nu}\right)-2 +4\sin^2\xi)+ \frac{r_o^2}{4}\left(\frac{\nu'}{\nu}\right)^2\left(\frac{\nu}{\nu'} + \frac{\nu'}{\nu}\right) d\Omega$$

$$d\sigma = \frac{r_o^2}{2}\left(\frac{\nu'}{\nu}\right)^2\left(\frac{\nu}{\nu'} + \frac{\nu'}{\nu} - 2\cos^2\xi\right) d\Omega \quad (4)$$

From Fig. the angle ξ can be written

$$\cos\xi = \sin\vartheta \cos\eta$$

Substituting this expression in (4) one gets:

$$d\sigma = \frac{r_o^2}{2} \left(\frac{\nu'}{\nu}\right)\left(\frac{\nu}{\nu'} + \frac{\nu'}{\nu} - 2 \sin^2\vartheta \cos^2\eta\right)$$

For a fixed value of ϑ the scattering probability assumes its extremes values for $\eta = 0$ and $\eta = \frac{\pi}{2}$. The two extreme angular positions for a linearly polarized radiation is a measure of the sensitivity of the polarimeter.
This ratio we shall call asymmetry ratio R.

For the ideal geometry dealt as this one, R is simply the ratio of the two probability:

$$R = \frac{(d\sigma)_{\eta - \frac{\pi}{2}}}{(d\sigma)_{\eta = 0}}$$

$$R = \frac{\frac{\nu}{\nu'} + \frac{\nu'}{\nu}}{\frac{\nu}{\nu'} + \frac{\nu'}{\nu} - 2 \sin^2\vartheta} \qquad (5)$$

wing to the ϑ dependence of energy of the scattered photon

$$\frac{\nu'}{\nu} = \frac{1}{1 + \alpha\,(1 - \cos\vartheta)}$$

where $\alpha = \frac{h\nu}{m_o c^2}$ the asymmetry ratio R which is unity for $\vartheta = 0$ and $\vartheta = \pi$, reaches its maximun value at an angle ϑ somewhat smaller than $\frac{\pi}{2}$ as we expect by formula (5).
In Fig. 28 the dependence of the angle $\vartheta_{máx}$ for maximum asymmetry is given as a function of the energy of the primary photon.

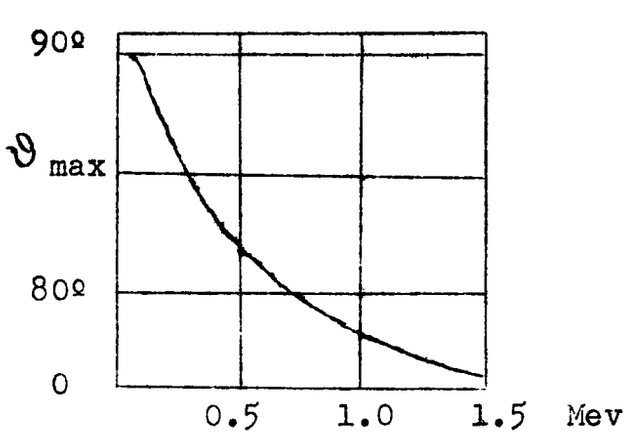


fig. 28

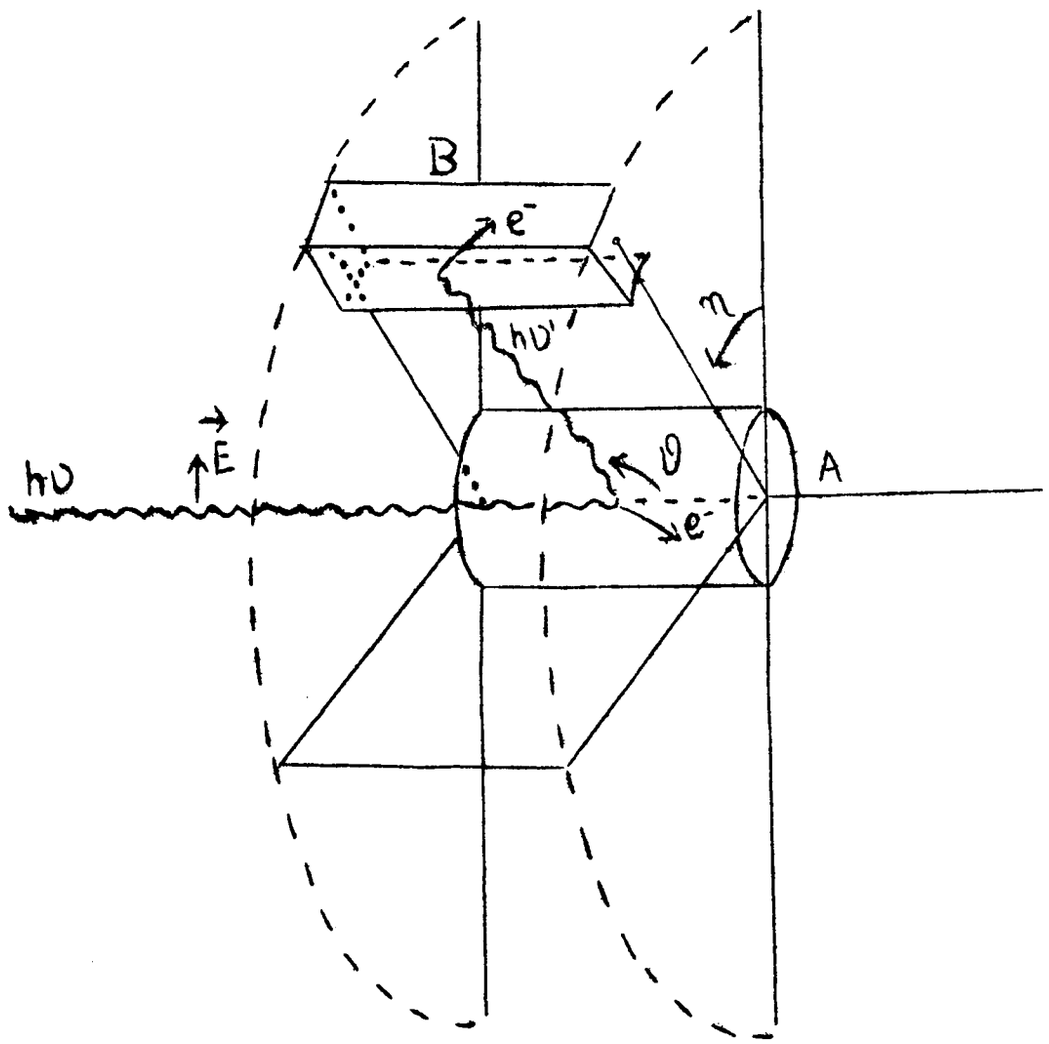


fig. 29

Fig. 29 shows schematically the basic elements for a γ-ray polarimeter. Two scintillation counters are used in a coincidence circuit. A coincidence is registred when the incident photon $h\nu$ produces a Compton recoil electron in the counter A, while the scattered photon at ϑ, η is detected in a second counter B. Two measurements are made i.e. in the positions $\eta = 0$ and $\eta = \frac{\pi}{2}$

ANGULAR CORRELATION

The probability of emission of a particle or quantum by a radioactive nucleus depends in general on the angle between the nuclear spin axis and the direction of emission.

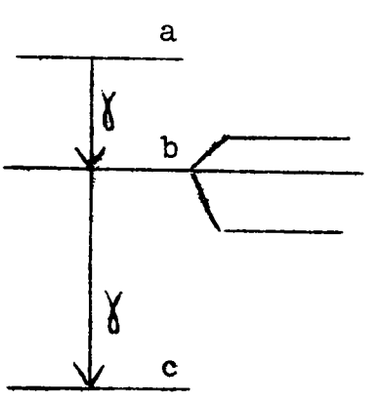
Under ordinary circunstances the total radiation from a radioactive sample is isotropic because the nuclei are randomly oriented in space.

An anisotropic radiation pattern can be observed only from an ensemble of nuclei which are not randomly oriented. One method of arriving at such an ensemble consist in placing the radioactive sample at a very low temperature in a strong magnetic field or electric field gradient, thereby orienting or aligning the nuclei. Another method, discussed bellow, consist in picking out only those nuclei whose spins happen to lie in a preferred direction. This can be realized if the nuclei decay through sucessive emission of two radiations R_1 and R_2.

A complete descripción of the emission requires the quantum theory of radiation, hewever considerable information can be obtained from a study of the "classical radiation field" emitted by a distribution of charges and currents which vary with time.

We illustrate the existence of angular correlation with a simple example.

Let us assume the following spins of the three states



a,b,c,$(E_a > E_b > E_c)$ $I_a = I_c = 0$ and $I_b = 1$ see Fig.

Then two succesive dipole transition take place.

In the transition a $\longrightarrow$ b the three possibilities

$$\Delta m = +1,\ 0,\ -1$$

are equally probable.

The directions distribution in each of these transition is given by: (see appendix)

$$(1)\quad Z_{\ell,m}(\theta,\phi) = \frac{1}{2}\left[1 - \frac{m\,(m+1)}{\ell(\ell+1)}\right]\left|Y_{\ell,m+1}\right|^2 +$$

$$+ \frac{1}{2}\left[1 - \frac{m\,(m-1)}{\ell(\ell+1)}\right]\left|Y_{\ell,m-1}\right|^2 + \frac{m^2}{\ell(\ell+1)}\left|Y_{\ell,m}\right|^2$$

Then for the transition of a ED (electric dipole $\ell = 1$) we have three types of angular distribution:

$$Z_{1,0}(\theta) = c\,(1 - \cos^2\theta) \qquad \Delta m = 0$$

$$(2)\quad Z_{1,1}(\theta) = \frac{c}{2}\,(1 + \cos^2\theta) \qquad \Delta m = +1$$

$$Z_{1,-1}(\theta) = \frac{c}{2}\,(1 + \cos^2\theta) \qquad \Delta m = -1$$

Here c is a constant independent of the angles.

It follows from (2) that the first quantum is equally distributed over all directions since the sum over the three possibilities gives a constant.

The choice of our z direction is quite arbitrary, it determines a polar coordinate system and it also defines the substates $m = \pm 1,\ 0$ of state b. Without any loss of generality we may choose the z direction parallel to the direction of the first light quantum. Then the first transition cannot lead to state b with magnetic quantum number $m = 0$ since a dipole emission with $\Delta m = 0$ does not give rise to any radiation in the z direction. The first transition a $\rightarrow$ b is therefore of type $\Delta m = +1$ or $\Delta m = -1$. The second transition b $\rightarrow$ c must then be of type $\Delta m = -1$ or $\Delta m = +1$ respectively.

Either of these transition gives rise to the second directional distribution (2). Hence we find that the direction of the second quantum is not arbitrary but is distributed statistically according to the law:

$$W(\theta_{1,2})\, d\Omega = \frac{3}{16\pi} (1 + \cos^2 \theta_{1,2})\, d\Omega$$

The factor $\frac{3}{16\pi}$ serves to normalize the total emission probability of the second quantum.

Here $W(\theta_{1,2})\, d\Omega$ is the probability that the direction of the second quantum lies within the solid angle element $d\Omega$ at an angle $\theta_{1,2}$ with respect to the first quantum (our z axis).

The case treated here is a special example of two sucessive gamma-ray transitions. In general, the angular correlation depends on the multipole order of the two emitted quanta and also on the spin values I of the three states.

1) Theoretical Nuclear Physics (Blatt-Weisshopf)

2) Robley D. Evans (Enciclopedia of Physics vol. XXXIV)

3) Beta and Gamma ray spectroscopy (Siegbahn)

METHOD OF MEASSURING LINEAR POLARIZATION

Photoelectric effect: Photoelectric effect has not classical analogy; it is a purely quantum mechanical phenomen.

When a beam of radiation heats a target, a photoelectron may be ejected from an atom. This photoelectron can be emitted either from one of the outer shells, where the binding energy is small, or from the inner shells. (K-shell predominates in general).-

The energy of the K-photoelectrons is given by $E_e = h\nu - E_K$ where E_K is the binding energy of an electron in the K-shell. It would be expected that, like in the case of a dipole emission, the electrons would be emitted in the direction of the electric vector. This is true only for very small energies, but for higher ones we have two other effects which modofies the direction of emission, namely, relativistic effect and the effect of the magnetic vector.

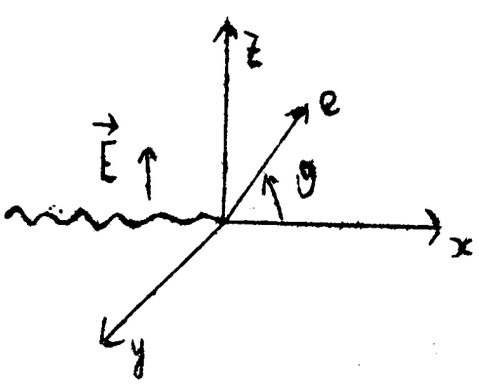


fig. 30

Consider the electron is emitted at an angle ϑ with respect to the direction of propagation ——, and with momentum $p = mv$. Neglecting the effect of nuclear recoil —— which is very small, we have

$$\cos\vartheta = \frac{h\nu/c}{mv} \quad \text{but} \quad h\nu = \tfrac{1}{2}\, mv$$

hence: $\cos\vartheta = \tfrac{1}{2}v/c = \tfrac{1}{2}\beta$

Thus we see that for $v \ll c$, $\cos\vartheta \simeq 0$, and the emission is in the direction of the electric field. But as $v \to c$, we have $\cos \to 1$, and the emission in the forward direction.

It has been shown theoretically that for higher energies the magnetic vector becomes predominant, so that most of the

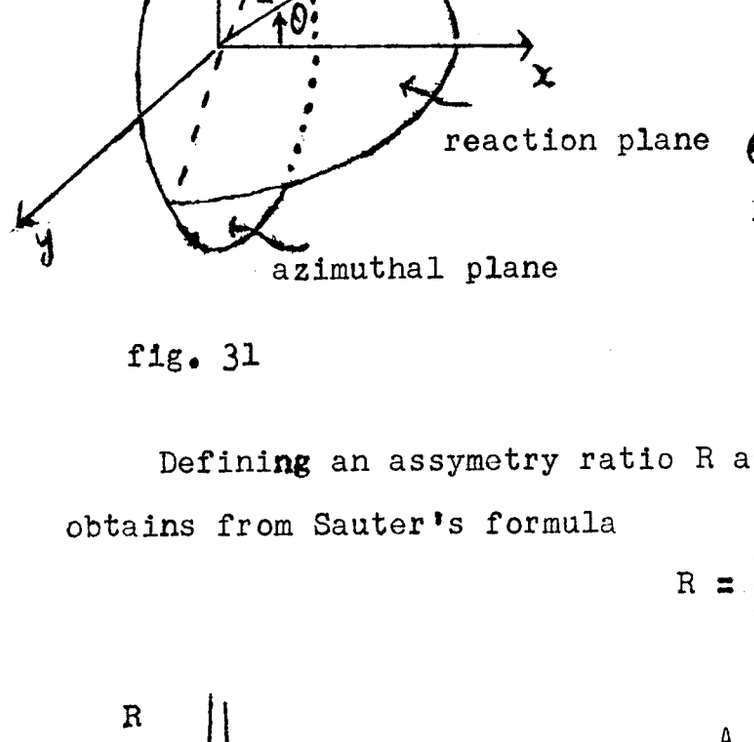
photoelectrons are emitted in the direction normal to the electric vector.

In 1931 Sauter calculated the differential cross section for the photoelectric effect, and obtained the following formula:

$$d\sigma = C\,(A + B\cos^2\phi)\,d\Omega$$

where

$$A = \frac{\epsilon^2}{4}\,(1 - \beta\cos\vartheta)$$



$$B = (1-\beta^2)^{\frac{1}{2}} - \frac{\epsilon}{2}\,(1 - \beta\cos\vartheta)$$

$$C = \left(\frac{1}{\epsilon}\right)^4 \frac{(1-\beta)^{\frac{1}{2}}\beta^2\,\sin^2\vartheta}{(1-\beta\cos\vartheta)^4}$$

ϵ = kinetic energy of the photoelectron in units m_0c^2.-

fig. 31

Defining an assymetry ratio R as $d\sigma_{\vartheta=0} / d\sigma_{\vartheta = 90^\circ}$ one obtains from Sauter's formula

$$R = \frac{d\sigma_{0^\circ}}{d\sigma_{90^\circ}} = \frac{A}{A+B} = F(\epsilon, \vartheta)$$

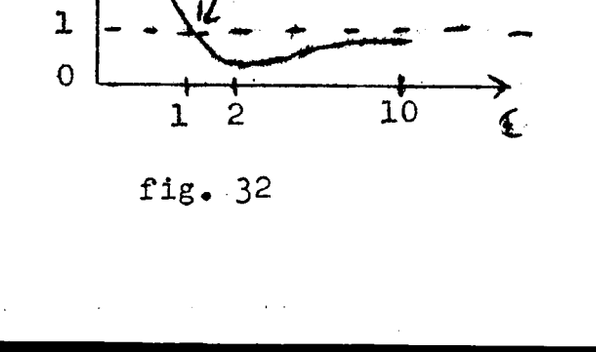


fig. 32

A graph of R for $\vartheta = \pi/2$ is given here. Hence we see that for low energies (below 500 kev) most of the photoelectons are emittod in direction of electric vector, and R is very large..

So in this region the photo-

electric effect is a very good polarimeter. For energies above 500 kev, the magnetic field is predominant, but the effect is very small.-

If we plott R for different θ , we verify that R (ϵ) becomes smaller, and the cross-over point tends toward higher energies, while the influence of the magnetic field becomes smaller too. In this way, by putting the counter in a forward angle we can have a polarimeter for slightly higher energies.

The photoelectric effect as polarimeter has a great disadvantage over the Compton effect, namely, it is much more difficult to count low energy electrons than gammas, due to scattering effects.

The extent of the influence of the magnetic field on the direction of emission of photoelectrons, is not quite clear yet. About 1950 Archibald proposed a theory, by which allthough R decreasses with ϵ , there is no cross-over point, so that the electric field is allways predominant. Two experiments have been made, one by Hereferd and Mc Master in 1954, which corroborates Sauter's thoery, and another by Brini et al. in 1957, agreeing with Archibald. A third experiment is at the moment being carried out by a group in Rio de Janeiro (E. W. Cybulska and L. Moura).

Photodesintegration of the Deuteron.- We have seen that the photoelectric and Compton effects are good polarimeters for low energies, below 2 or 3 Mev. In the region of higher energies from 4 to 12 Mev the photodesintegration of the deuteron is an excelent polarimeter. Since the threshold is: 2,225 Mev, then this method can not be used for low energies.

Bethe and Longnire calculated the cross section for the desintegration of the deuteron assuming a central force

potencial, and valid for energies below 10 mev. The differential cross section for the electric dipole desintegration is given by

$$d\sigma_e = 2 \frac{e^2}{\hbar c} \left[\frac{1}{1-\gamma r_o t} \right] \left(\frac{1}{1-\gamma r_o t} \right) \cos^2 \alpha \, d\Omega$$

where the γ and k are given by

$$E = h\nu - W_1 = \frac{h^2 k^2}{M} \quad ; \quad W_1 = \frac{h^2 \gamma^2}{M}$$

M is the mass of the nucleon

W_1 is the binding energy of the deuteron.

α is the angle between electric vector and the motion of the proton or neutron as defined in figure 31 . Since this is a two body problem, the energy is divided equally between proton and neutron so that the angular distribution will the same for both of them.

The term

$$\left(\frac{1}{1-\gamma r_o t} \right)$$

is the correction to the zero-range approximation for the cross section.

Writing the cross section in terms of the angle ϑ and ϕ , since

$$\cos \alpha = \sin \vartheta \cos \phi$$

we have

$$d\sigma_\phi^e = \frac{2 e^2}{hc} F(\gamma k) \sin^2 \vartheta \cos^2 \phi \, d\Omega$$

then by putting $\vartheta = \quad /2$ as before we have

$$R = \frac{d\sigma_{\phi=0}^e}{d\sigma_{\phi = \quad /2}^e} = \infty$$

Thus this method has a perfect response to polarization, and could be a very good polarimeter.

However, this reaction does not proceed entirely by electric dipole transitions, but also has a small amount of magnetic dipole interaction, which yields an isotropic distribution of photoprotons. The cross section for this process is given by

$$\sigma^m = \frac{2}{3} \frac{e^2}{\hbar c} \left(\frac{\hbar}{M c}\right)^2 (\mu_n - \mu_p) \frac{k\gamma(\bar{\gamma} - 1/a_s)^2}{(k^2 + \gamma^2)(k^2 + 1/a_s^2)} R(E)$$

where μ_n and μ_p are the magnetic moments of neutrons and protons, a_s is the singlet scattering lenght and R (ϵ) is a correction for the non-zero range of the neutron-proton force. Thus the magnetic dipole interaction diminishes the polarization sensitivity. If we compare the total cross section for the magnetic and electric dipole interaction we obtain the figure.-

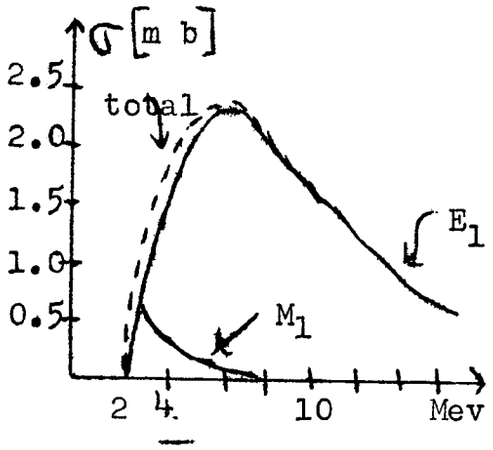


fig. 33

This was confirmed very well by experimental results. Hence we see that about 4 Mev the influence of magnetic dipole is very small, and we can have a very good polarimeter.

At energies higher than 10 Mev there are other effects which diminish the polarization sensibility. The nature of this effects is not quite clear yet, but in general is is believed to be due to the tensorial character of the nuclear force. Also, since all those calculations were made in the center of mass system, when we pass to the laboratory system then there is a marked tendency for a more for ward emission of photoprotons and neutrons, and the angle θ decrease with higher energies.

Another disadvantage of using photodesintegration of deuterons as a polarimeter is its small cross section, which

is of the order of milibarns, while for the Compton effect, we ahve of the order of 10 barns.

There are 2 ways of meassuring the polarization by the use of the photodesintegration of the deuteron.

(1) Using nuclear emulsions, which are soaked in heavy water. The γ -ray beam hits the emulsion which is placed perpendicular to the direction of propagation of the γ's. Since the angular distribution of the photoprotons is proportional to $\sin^2 \vartheta$, then the trakcs of the protons will lie on the plane of the emulsion. By counting the number of tracks for $\phi = \pi/2$ and $\phi = 0º$ we can measure the degree of polarization of the incident beam.

(2) Electronic method: this method is entirely similar to that used in Compton polarimeter. Here we detect electromically the photoprotons emited, and the neutrons, at angles $\vartheta = \pi/2$; $\phi = \pi/2$ and $\vartheta = \pi/2$; $\phi = 0$. The disadvantage of this method is that although proton detectors have very high efficiency, the protons themselves have a very short range, so, high vacuum is necessary, and short distances must be used. The neutrons on the other hand has a very long range, but the detectors have an efficiency of about 1%.-

METHOD OF MEASURING CIRCULAR POLARIZATION

In recent years there has been much interest in the measure of circular polarization due to investigations of non conservation of parity in weak interactions. This can be done by using Compton scattering of circularly polarized photons from polarized electrons. The total cross section consist of one term σ_0, which is the Klein Nishina formula for unpolarized radiation, and another term σ_1 which depends on circular polarization $\sigma = \sigma_0 \pm \sigma_1$. Thus in order to measure the

circular polarization we nust first polarize the electrons. This is done by magnetizing the iron. But the great disadvantage is that only 2 electrons out of 26 from an iron atom are polarized So that in the most ideal situation the net effect which can be obtained is 8% of right hand, and 8% of left hand polarization.

So far, in our study of polarized γ radiation, we did not consider the orientation of the electrons, in other words, in the Compton thoery an average is made over all the spin directions of the electrons. This can not be done any longer, since the electron is oriented in such a way that its spin makes a polar angle ψ with the z-axes in figure
Then the differential cross sectio becomes

$$\frac{d\sigma}{d\omega} = \frac{1}{2}\gamma_o^2 \frac{k^2}{k_o^2}\left\{\left[\frac{k_o}{k} + \frac{k}{k_o} - \sin^2\vartheta\right] \mp \left[\left(\frac{k_o}{k} - \frac{k}{k_o}\right)\cos\vartheta\cos\psi + \right.\right.$$

$$\left.\left. + \left(1 - \frac{k}{k_o}\right)\sin\vartheta \sin\phi \sin\psi\right]\right\}$$

where $$d\omega = d\phi \cdot d(\cos\vartheta)$$

If we integrate over the sphere, then we obtain the total cross section $\sigma = \sigma_o \pm \sigma_1$.- But in reality σ_1 is weighted by ν = Nº of oriented electrons per atom which in the best case, iron, is 8%. It happens that σ_o and σ_1 are practically equal in magnitude so that we can write

$$\sigma_R = \sigma_o + \nu\sigma_1 = \sigma_o(1+\nu)$$

$$\sigma_L = \sigma_o - \nu\sigma_1 \simeq \sigma_o(1-\nu)$$

$$R = \frac{\sigma_R}{\sigma_L} = \frac{1+\nu}{1-\nu} = \frac{1.08}{0.92} = 16\%$$

So the maximum sensitivity is 16%, if the beam of gamma

rays is completely polarized.

Let us now consider the transmission through a ferromagnet of a beam of unpolarized radiation (which can be thought as made up of 2 circularly polarized beam in opposite direction). The ferromagnet obsorber has N atoms per unit volume, uniform magnetization along the x-axis, and ν polarized electrons per average atoms. Circularly polarized photons of a certain sense, propagating along the x-axis can undergo Compton scattering with the

$$\frac{1}{2}\,(2-\nu)\;\;N\,d\,x$$

electrons, which have spins oriented along one direction and with the

$$\frac{1}{2}\,(2-\nu)\;\;N\,d\,x$$

electrons, whose spins are in the opposite direction.

The probability that the photons having no circular polarization will survive (in good geometry) a lenght L of the ferromagnet is

$$T(\nu)=\frac{I}{I_o}=\frac{1}{2}\exp\left\{-NL\left[(\sigma_o+\sigma_1)\,\frac{1}{2}(2+\nu)+(\sigma_o-\sigma_1)\,\frac{1}{2}\,(2-\nu)+\sigma_2\right]\right\}+$$

$$+\frac{1}{2}\exp\left\{-NL\left[(\sigma_o-\sigma_1)\,\frac{1}{2}(2+\nu)+(\sigma_o-\sigma_1)\,\frac{1}{2}\,(2-\nu)+\sigma_2\right]\right\}$$

$$=\cosh\,(NL\nu\sigma_1)\exp\left\{-NL\,(2\sigma_o+\sigma_2)\right\}$$

where σ_o and σ_1 as defined above, and σ_2 represents all other processes in addition to Compton scattering.

Polarization effects in σ_2 are taken to be nil; for example, photoeffect involving the polarized electrons has a very small cross section compared to σ_1 at energies of interest.

The ratio of magnetized transmission to unmagnetized transmission is

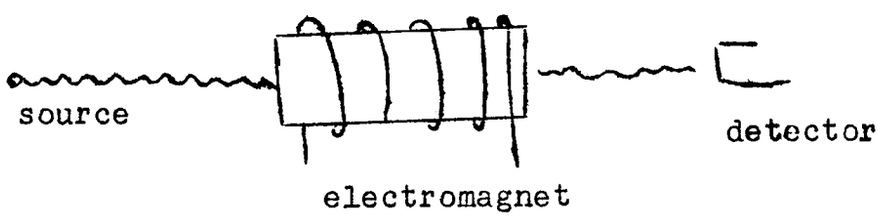
$$T(\nu)/T(0)=\cosh\;(NL\nu\sigma_1).$$

The transmission ratio R of left circular polarization to right circular polarization photons is

$$R = \exp (2 NL \nu \sigma_1)$$

where the magnetization is in the direction of transmission.

The experimental tecnique is to measure the absorption through a cilindrical electromagnet of about 12 " long



The transmission can be used for polarizing a beam of unpolarized gamma rays or for the analysis of the polarization of a partially polarized beam of light.

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Fagg & Hanna - Rev. Mod. Phys. 711, 31 (1959)

Gunst & Page - Phys. Rev. 92, 970 (1953)

A P P E N D I X

The electric multipole field of order ℓ, m at a given point $\vec{r} = r, \vartheta, \phi$ is given by

$$\begin{cases} \vec{\mathcal{E}}_E(\ell, m, \vec{r}) = \dfrac{i}{K} \left[\dfrac{U_\ell^{(+)}(r)}{K r} X_{\ell, m}(\vartheta, \phi) \right] \\ \vec{H}_E(\ell, m, \vec{r}) = \dfrac{U_\ell^{(+)}(r)}{K r} X_{\ell, m}(\vartheta, \phi) \end{cases} \tag{1}$$

where $$X_{\ell, m}(\vartheta, \phi) = \frac{L\, Y_{\ell, m}(\vartheta\phi)}{\sqrt{\ell(\ell+1)}}$$

vector spherical harmonics

$L = -i \vec{r} \times \vec{\nabla}$ differential operator

$Y_{\ell,m}$ scalar spherical harmonic

K wave number of the emitted quanta

The flow of energy in the multipole radiation is determined by the Poynting vector $\vec{S}$:

$$\vec{S} = \frac{C}{4\pi} \vec{\mathcal{E}} \times \vec{H}$$

Far away from the source of the radiation in the "wave zone" $\vec{\mathcal{E}}$ and $\vec{H}$ are perpendicular to each other and to the radial direction $\vec{r}$ and they are equal in magnitude, then we have:

$$|\vec{S}| \cong \frac{C}{4\pi} \mathcal{E}^2 \cong \frac{C}{4\pi} H^2 \tag{2}$$

Let us suppose a large sphere of radius R around the radiating system. At any point of the sphere, $|\vec{S}|$ gives the energy escaping per square centimeter per second. We get the energy emitted into the polid angle element $d\Omega$ around the

direction θ, ϕ by multiplication of $|\vec{S}|$ by the area $R^2 \, d\Omega$ subtended by that solid angle at the surface of our sphere. We call this rate of emission of energy by $R(\Omega) \, d\Omega$

The electric field are rapidly varying function of time but we are interested in their average values i.e. $\langle \mathcal{E}^2(\vec{r}, t) \rangle$

Suppose then we have

$$\vec{\mathcal{E}}(\vec{r}, t) = \vec{\mathcal{E}}(\vec{r}) e^{-i\omega t} + \vec{\mathcal{E}}^*(\vec{r}) e^{+i\omega t}$$

(that definition of the electric field makes the transition to the quantum mechanical case easier).

Then we have

$$\langle \mathcal{E}^2 (\vec{r}, t) \rangle = 2 \vec{\mathcal{E}}^* (\vec{r}) \cdot \vec{\mathcal{E}} (\vec{r}) \qquad (3)$$

Now to have the energy $U(\ell, m, \Omega) \, d\Omega$ in a pure electric multipole radiation ℓ, m with amplitude $a(\ell, m)$ emitted per second into the solid angle element $d\Omega$, owes combine equations (1), (2) and (3).

The results is

$$U(\ell, m, \Omega) = \frac{e}{2\pi K^2} \left| a(\ell, m) \right|^2 Z_{\ell, m}(\theta, \phi)$$

where $Z_{\ell, m}(\theta, \phi)$ gives the directions distribution.