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PAIRING VIBRATIONS

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Abstract: We study the properties of the collective states (pairing vibrations) which are associated with fields changing the numbers of particles. In particular, we discuss which processes may be enhanced by the coherence in the pairing-vibration state.

1. Introduction

A large number of excited 0^+ states are known in the low-energy nuclear spectrum. Several mechanisms may produce collective states of this spin and parity. The best studied ones correspond to oscillations in the shape of or in the size of the nucleus (quadrupole and monopole vibrations). These modes are associated with a change in the binding field of each particle (i.e. with a field which conserves the number of particles).

In addition to the previous modes, Bohr and Mottelson [1]) have suggested the existence of vibrational modes based on fields which create or annihilate two particles. One of such fields is produced by the pairing interaction. The corresponding collective mode is the pairing vibration.

With a pairing-type interaction, and assuming the existence of at least two different groups of levels, it is possible to imagine a coherent excitation of pairs of particles from one set to the other. A simple case of this collective motion has been studied by Hogaasen [2]), using a model of Ω-particles moving in two Ω-degenerate shells and coupled by pairing forces. It has also been implicitly included in studies on single-closed shell nuclei [3]) and β-vibrations [4, 5]).

Although in the actual nuclear spectrum the pairing vibration is usually mixed with other collective effects [3-5]), the aim of this work is to study the pairing mechanism by itself. Thus, we assume that the pairing fluctuations are uncoupled with other degrees of freedom. We study the enhancements characteristic of this collective mode and the possibility of detecting them in actual nuclei, within the framework of a pairing interaction with constant matrix elements.

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2. Pairing Vibrations

Our Hamiltonian describes the motion of independent particles coupled by pairing forces. The single-particle energies, measured from the Fermi surface, are denoted by e_ν. As usual, the single-particle creation operator $c_{\bar{\nu}}^\dagger$ is obtained from $c_\nu^\dagger$ by the time-reversal operation

$$
\begin{aligned}
H &= H_{\text{s.p.}} + H_{\text{p}}, \\
H_{\text{s.p.}} &= \sum_\nu e_\nu (c_\nu^\dagger c_\nu + c_{\bar{\nu}}^\dagger c_{\bar{\nu}}), \\
H_{\text{p}} &= -GP^\dagger P, \qquad P^\dagger = \sum_\nu P_\nu^\dagger, \qquad P_\nu^\dagger = c_\nu^\dagger c_{\bar{\nu}}^\dagger .
\end{aligned}
\tag{1}
$$

As in the case of quadrupole forces, the separability properties of the pairing interaction suggest the existence of simple collective degrees of freedom.

The analogy with the quadrupole vibrations can be further extended; using the pairing interaction, one can distinguish two cases according to whether the ground state has a superconducting character or not. In the case of the quadrupole force, this corresponds to the development of a stable equilibrium deformation and to the spherical shape, respectively (see appendix).

2.1. SUPERCONDUCTING NUCLEI

2.1.1. *Qualitative considerations.* The quasi-boson creation operators are the linear combinations †

$$
\Gamma_n^\dagger = \sum_\nu a_{n\nu} \Gamma_\nu^\dagger + \sum_\nu b_{n\nu} \Gamma_\nu , \tag{2}
$$

where

$$
\begin{aligned}
&\Gamma_\nu^\dagger \equiv \alpha_\nu^\dagger \alpha_{\bar{\nu}}^\dagger, \qquad \alpha_\nu^\dagger \equiv U_\nu c_\nu^\dagger - V_\nu c_{\bar{\nu}}, \qquad \alpha_{\bar{\nu}}^\dagger \equiv U_\nu c_{\bar{\nu}}^\dagger + V_\nu c_\nu , \\
&[\Gamma_\nu , \Gamma_\omega^\dagger] \approx \delta_{\nu,\omega}, \qquad [\Gamma_n , \Gamma_m^\dagger] = \sum_\nu (a_{n\nu} a_{m\nu} - b_{n\nu} b_{m\nu}) \approx \delta_{n,m} .
\end{aligned}
\tag{3}
$$

The unperturbed Hamiltonian and residual pairing interaction can also be approximately expressed in terms of $\Gamma_\nu^\dagger$, Γ_ν

$$
\begin{aligned}
H^0 &\equiv 2\sum_\nu E_\nu \Gamma_\nu^\dagger \Gamma_\nu , \qquad E_\nu \equiv (e_\nu^2 + \Delta^2)^{\frac{1}{2}}, \qquad \Delta = G \sum_\nu U_\nu V_\nu , \\
H_{\text{p}}' &\equiv -\tfrac{1}{4} G \Big[\sum_\nu (U_\nu^2 - V_\nu^2)(\Gamma_\nu^\dagger + \Gamma_\nu) \Big]^2 , \\
H_{\text{p}}'' &\equiv \tfrac{1}{4} G \Big[\sum_\nu (\Gamma_\nu^\dagger - \Gamma_\nu) \Big]^2 .
\end{aligned}
\tag{4}
$$

Using (2) and (4), the linear relations

$$
[H, \Gamma_n^\dagger] = W_n \Gamma_n^\dagger \tag{5}
$$

† In the following, Greek subindices will denote unperturbed two-quasi-particle states, while the phonons are denoted by Latin subindices.

determine the excitation energies W_n. The lowest (collective) root associated with the Hamiltonian $H = H^0 + H_p''$ has zero energy and corresponds to the spurious state[2,4]). The pairing vibration is the lowest root of the Hamiltonian $H = H^0 + H_p'$.

We shall neglect for the moment H_p'' and later examine the validity of this approximation.

The condition (5) yields the dispersion relation

$$1/G = \sum_\nu (U_\nu^2 - V_\nu^2)^2 2E_\nu/(4E_\nu^2 - W_n^2), \tag{6}$$

which can be cast, with the usual superconductivity equations, into the form

$$(W_n^2 - 4\Delta^2)[\sum_\nu 1/E_\nu(4E_\nu^2 - W_n^2)] = 0. \tag{7}$$

The coefficients $a_{n\nu}$, $b_{n\nu}$ are given by

$$\begin{aligned} a_{n\nu} &= \Lambda_n(U_\nu^2 - V_\nu^2)/(2E_\nu - W_n), \\ b_{n\nu} &= -\Lambda_n(U_\nu^2 - V_\nu^2)/(2E_\nu + W_n), \\ \Lambda_n &\equiv \tfrac{1}{2}[\sum_\nu (U_\nu^2 - V_\nu^2)2E_\nu W_n/(4E_\nu^2 - W_n^2)^2]^{-\frac{1}{2}}. \end{aligned} \tag{8}$$

The part of an arbitrary operator R, which is effective in connecting the ground state with a one-phonon state, is given by

$$R = \sum_\nu r_\nu(\Gamma_\nu^\dagger + \Gamma_\nu) = \sum_{\nu, n} r_\nu(a_{n\nu} - b_{n\nu})(\Gamma_n^\dagger + \Gamma_n), \tag{9}$$

and thus, the corresponding matrix element is

$$\begin{aligned} R_n &\equiv \langle 0|R|n\rangle = \sum_\nu r_\nu(a_{n\nu} - b_{n\nu}) = \Lambda_n \Sigma_n, \\ \Sigma_n &\equiv 2\sum_\nu r_\nu(U_\nu^2 - V_\nu^2)2E_\nu/(4E_\nu^2 - W_n^2). \end{aligned} \tag{10}$$

In particular, the energy and the matrix element corresponding to the lowest root (pairing vibration) are

$$W_1 = 2\Delta, \tag{7'}$$

$$\Lambda_1 = (\sum_\nu \Delta/E_\nu e_\nu^2)^{-\frac{1}{2}}, \tag{8'}$$

$$\Sigma_1 = \sum_\nu n_\nu/e_\nu. \tag{10'}$$

In order to establish the existence of a collective degree of freedom, we must obtain the "specific" operator F. Such an operator has to be related to the pairing vibrations in the same way as the mass quadrupole operator is connected with quadrupole vibrations. In this last case, the specific operator is also given by the terms of the quadrupole interaction which create (and destroy) two-quasi-particles. Similarly, the $(H_p)_{20}$ component of the pairing force can be written

$$(H_p)_{20} = -\Delta \sum_\nu (U_\nu^2 - V_\nu^2)(\Gamma_\nu^\dagger + \Gamma_\nu),$$

and therefore we conclude that the specific operator has the form

$$F = \sum_\nu f_\nu (\Gamma_\nu^\dagger + \Gamma_\nu), \qquad f_\nu \equiv U_\nu^2 - V_\nu^2. \tag{11}$$

Eq. (11) is also an immediate consequence of the form of H'_p (4) or of the dispersion relation (6). We note in addition that the operator F can be written as a combination of two transfer operators:

$$F = \sum_\nu (c_\nu^\dagger c_{\bar\nu}^\dagger + c_{\bar\nu} c_\nu), \tag{12}$$

where we have neglected, as usual, terms conserving the number of quasi-particles. Thus, the pairing vibration can be excited from the ground state of a neighbouring even nucleus. This is a natural result for a collective state based on a field which changes the number of particles.

However, because in superconducting nuclei the number of particles is not conserved, the pairing vibration can also be excited through operators which conserve the number of particles; suppose we produce displacements in the single-particle energy levels having different signs according to their position with respect to the Fermi surface: when the energy of the levels above is decreased and the energy from those below increased, the Fermi surface tends to be smoothed. Consequently, there is a coherent transfer of particles from states below to states above the Fermi surface. Thus, we may also produce pairing vibrations by adding to the single-particle Hamiltonian a term of the form

$$H'_{\text{s.p.}} = -\mu' \sum_\nu e_\nu (c_\nu^\dagger c_\nu + c_{\bar\nu}^\dagger c_{\bar\nu}) \approx -\mu' \Delta \sum_\nu (U_\nu^2 - V_\nu^2)(\Gamma_\nu^\dagger + \Gamma_\nu) = -\mu F. \tag{12'}$$

In particular, the enhancement in the matrix element F measures the degree of "collectiveness" which the state has acquired. In order to evaluate F, we notice that for any root W_n the value Σ_n (10) is given by

$$\Sigma_n = 2/G, \qquad F_n = 2\Lambda_n/G. \tag{13}$$

Moreover, the factor Λ_1 in eq. (8′) is almost independent of G. We may approximate

$$\sum_\nu \Delta / E_\nu e_\nu^2 = \sum_\nu 1/e_\nu^2,$$

and thus the matrix element F_1 vanishes as one of the single-particle levels approaches the Fermi surface. In the opposite case, in which all the levels are pushed away from the Fermi energy, both the increase in e_ν^2 and the decrease in Δ tend to enhance the matrix element.

If we use the simplified model of uniformly spaced levels ($e_\nu = \pm \nu e$; $\nu = 1, 3, 5, \ldots$),

$$\sum_\nu 1/e_\nu^2 = \pi^2/4e^2, \qquad F_1 \approx 4e/\pi G. \tag{14}$$

In actual deformed nuclei the distance between single-particle levels is about 0.3 MeV, while the value of G is about 0.1 MeV.

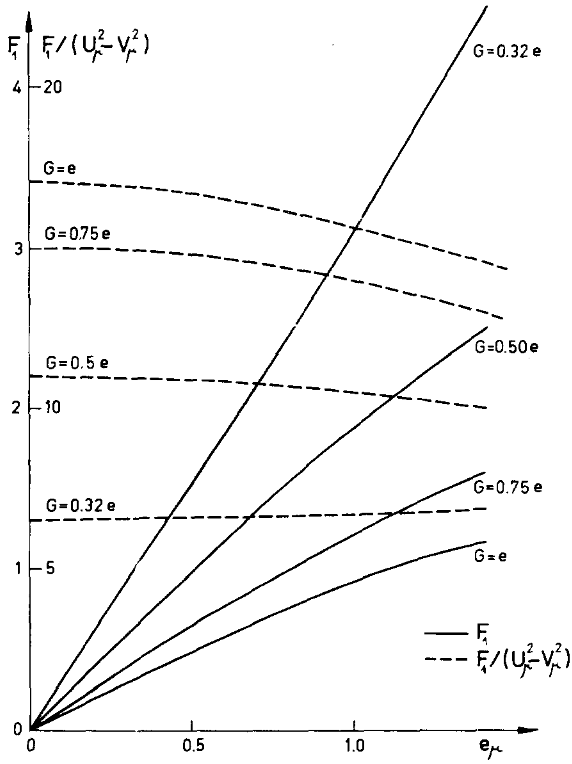

Fig. 1. The matrix elements F_1 and their relation with the unperturbed values $F_1/(U_\mu{}^2 - V_\mu{}^2)$ are represented for various values of the smallest single-particle distance e_μ. All other levels correspond to a symmetrical, uniformly spaced system of 20 levels.

Thus F_1 is expected to be about 2. The ratio between this value and the single-particle estimate $U_\mu^2 - V_\mu^2$ corresponding to the lowest excited state with single-particle energy e_μ

$$F_1/(U_\mu^2 - V_\mu^2) \approx 4\Delta/\pi G, \tag{15}$$

is about 7 for nuclei in the deformed region.

In order to verify these estimates, we have evaluated (14) and (15) for a system of 20 particles moving in 20 uniformly spaced levels (fig. 1). The two lowest single-particle levels (both with the same $|e_\nu| = e_\mu$) are allowed to be displaced. The situation approaching the realistic case arises for $G = \frac{1}{2}e$. The values of F_1 (unlike those of $F_1/(U_\mu^2-V_\mu^2)$) depend strongly on the single-particle spectrum. In the interval $\frac{1}{2}e < e_\mu < \frac{3}{2}e$ the changes in F_1 around the mean value are of the order of $\frac{1}{2}F_1$. As similar fluctuations in the distance of single-particle levels from the Fermi surface do occur in neighbouring deformed nuclei, we may expect corresponding changes in the value of F_1.

Instead of using uniformly spaced levels with essentially only one parameter e/G, we can consider a two-parameter model. Let us assume that we have 2Ω particles moving in two equal sets of levels. Each set admits 2Ω particles, and it is uniformly spread over an energy interval a, while b is the distance between the centres of gravity of the two sets. Let us use as parameters

$$\begin{aligned} &x = 2\Delta/b, && y \equiv a/b, && z \equiv a/G\Omega, \\ &0 \leqq x \leqq \infty, && 0 \leqq y \leqq 1, && 0 \leqq z \leqq 2y. \end{aligned} \tag{16}$$

The three parameters are not independent, the superconductivity equations giving us a relation between them. Replacing sums by integrals, this relation is

$$x^2 \sinh z = (1-y)[x^2+(1-y)^2]^{\frac{1}{2}}-(1-y)[x^2+(1+y)^2]^{\frac{1}{2}}. \tag{17}$$

We can also obtain F_1 and the total sum rule $\mathscr{F} = \sum_\nu (U_\nu^2-V_\nu^2)^2$ corrresponding to the operator F in the absence of the ground-state correlations. Their ratio is given by

$$F_1^2/\mathscr{F} = z^2(1+y^2)/x \sinh z[2y-x \operatorname{tg}^{-1}(2yx/(1+x^2-y^2))]. \tag{18}$$

(i) The limit $y \to 0$ represents the two degenerate-shell case (Hogaasen model). In this case

$$\begin{aligned} z = \sinh z &= 2y/x, \\ F_1^2/\mathscr{F} &= 1+1/x^2 = (b^2+4\Delta^2)/4\Delta^2, \end{aligned}$$

which means that our collective state carries not only the whole of the imperturbed sum rule but, in addition, there is an enhancement by a factor which depends on the ratio of the unperturbed energy $(b^2+4\Delta^2)^{\frac{1}{2}}$ and the energy of the collective state 2Δ.

(ii) The limit $y \to 1$ corresponds to the uniformly spaced model (deformed nuclei).

$$\begin{aligned} \sinh z &= 2/x, \\ F_1^2/\mathscr{F} &= z^2(1-y^2)/(8-2\pi x). \end{aligned}$$

Thus, $F_1^2/\mathscr{F}$ tends to zero as $y \to 1$. This is the reason why we found relative weak enhancements in the previous pages.

(iii) The limit $x \to \infty$ corresponds to merging the two shells into one. Then y measures the spread of the levels, and the parameter z is closely related to Belyaev's[6]) η for a similar model

$$\begin{aligned} z = \sinh z &= 2y/x, \\ F_1^2/\mathscr{F} &= 3(1-y^2)/(3+y^2) \leqq 1, \end{aligned}$$

which shows that the enhancement discussed in (i) tends to disappear when we only have one shell.

(iv) The limit of $x \to 0$ corresponds to the collapse of the superconductivity solution

$$\begin{aligned} \sinh z &= 2y/(1-y^2), \\ F_1^2/\mathscr{F} &= z^2(1-y^2)^2/4y^2x. \end{aligned}$$

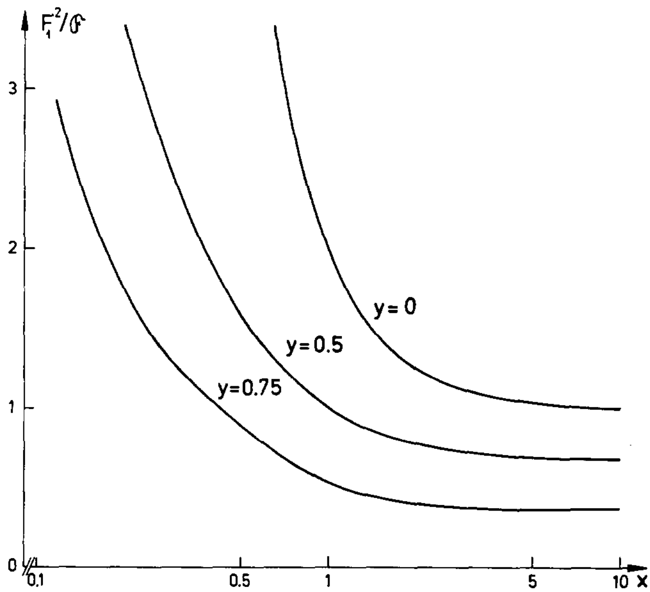

Fig. 2. The ratio between the matrix element F_1 and the total unperturbed sum rule of the specific operator F is plotted as a function of the parameters x and y.

The divergence of $F_1^2/\mathscr{F}$ as $x \to 0$ reflects the increase in coherence which follows the decrease in energy. However, it is also an expression of the failure of the quasi-boson approximation to give reliable results in cases where the collective state is too much displaced from the unperturbed energies.

Eq. (18) is represented in fig. 2. We may obtain from it the maximum value of y, which is allowed if we require that the collective state should have a certain fraction of the total strength.

We turn our attention now to the higher excited states. They correspond to the roots of the equation

$$\sum_{\nu} 1/E_\nu(4E_\nu^2 - W_n^2) = 0, \tag{19}$$

which shows that there is one root between two neighbouring quasi-particle energies. The matrix elements are given in eqs. (8) and (13).

Again, using the model of 20 particles moving in 20 levels, the values of F_n are given in fig. 3. From this result we conclude (i) the shifts in the lowest single-particle level only affect the matrix element to the two lowest states; (ii) in spite of the fact that there is a relatively small fraction of the total strength in the lowest state, a significant fraction of the total intensity has been displaced from the higher to the lower energies.

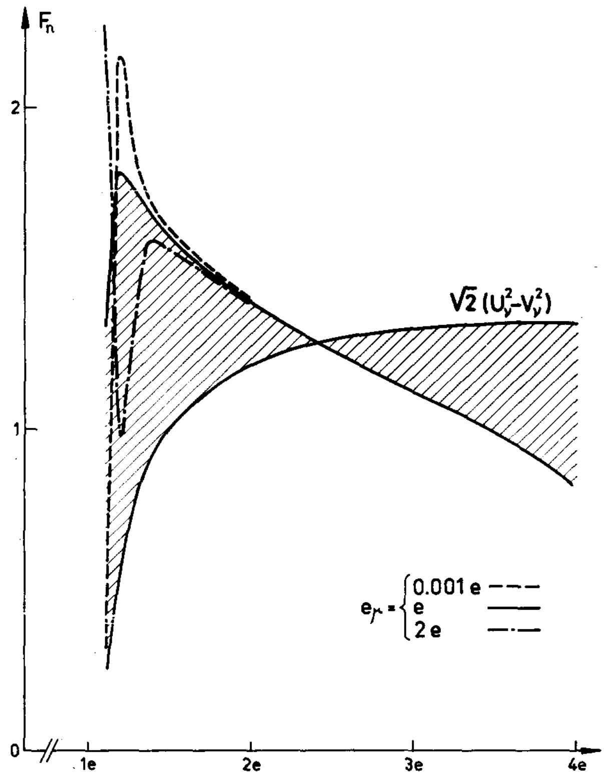

Fig. 3. The matrix element F_n is plotted against the energy of excited states assuming a symmetrical, uniformly spaced model of 20 levels. The values of the unperturbed quantity $\sqrt{2}(U_\nu{}^2 - V_\nu{}^2)$ are also given for comparison. The shadowed areas represent the intensities displaced from higher to lower excitation energies.

This will be the essential effect of the residual pairing interaction that can be expected in most deformed nuclei, namely the disappearance of the reduction factor $U_\nu^2 - V_\nu^2$ for the lowest states.

2.1.2. *Coupling to other degrees of freedom.* The problem of uncoupling the pairing fluctuation and the spurious state is solved by adding to the Hamiltonian $H = H^0 + H_p'$ the term H_p'' in eq. (4) and carrying out the linearization procedure. The excitation energies are given as roots of the determinant equation [2])

$$\begin{vmatrix} \sum_\nu 2E_\nu/(4E_\nu^2-W_n^2)-1/G & \sum_\nu (U_\nu^2-V_\nu^2)/(4E_\nu^2-W_n^2) \\ W_n^2\sum_\nu (U_\nu^2-V_\nu^2)/(4E_\nu^2-W_n^2) & \sum_\nu 2E_\nu(U_\nu^2-V_\nu^2)^2/(4E_\nu^2-W_n^2)-1/G \end{vmatrix}$$
$$= W_n^2[(W_n^2-4\Delta^2)(\sum_\nu 1/2E_\nu(4E_\nu^2-W_n^2))^2-(\sum_\nu (U_\nu^2-V_\nu^2)/(4E_\nu^2-W_n^2))^2] = 0. \quad (21)$$

Aside from the root $W = 0$, the lowest frequency is given by

$$W_1 \geqq 2\Delta,$$

the equality sign being valid when the non-diagonal terms in the determinant vanish. If $W_1 = 2\Delta$, this condition is equivalent to

$$\sum_\nu 1/E_\nu e_\nu = 0. \quad (22)$$

In this case the pairing fluctuation is orthogonal to the spurious state.

The more restrictive requirement that the non-diagonal terms in (21) should vanish for any frequency, implies a symmetric distribution of levels around the Fermi surface. In this case, the Hamiltonian $H = H^0+H'_\mathrm{p}+H''_\mathrm{p}$ (eq. (4)), is invariant under a transformation changing the sign of the single-particle energies e_ν. Accordingly, we may use a representation labelled by the eigenvalues ± 1 of the operator corresponding to the transformation. From (8) and (11) we conclude that the pairing vibration has the eigenvalue -1 and the specific operator F is odd under this transformation. In the same way, it is easy to show that the spurious state is the collective state associated with the eigenvalue $+1$. All other roots of (21) are given by (19) and they are two-fold degenerate.

The condition of a symmetric distribution of levels is fulfilled by the schematic single-particle models previously used. It is approximately satisfied by nuclei in the deformed region, for half-filled or completely filled shells, etc.

If condition (22) is not satisfied, there is a coupling between the spurious root and the pair fluctuation, and the first excited state may be higher than the first two-quasi-particle energy, $2E_\mu$. In order to predict whether this happens, we evaluate (21) in the vicinity of the quasi-particle energy $2E_\mu - W \to +0$

$$\frac{\Omega_\mu^2}{(W^2-4E_\mu^2)4E_\mu^2}\left[1+8\sum_{\nu\neq\mu}\frac{E_\mu}{\Omega_\mu E_\nu}\frac{1}{1+e_\nu/e_\mu}\right], \quad (23)$$

where $2\Omega_\mu$ is the number of single-particle states at energy e_μ. As the left-hand side of (21) is always negative for $W = 0$, there is a root $W_1 < 2E_\mu$, if the expression within brackets in (23) is negative. This occurs if there is a second single-particle energy having different sign and such as to make the factor in parentheses negative and sufficiently great (for instance, this happens in the symmetric model).

Unlike the ordinary quasi-boson dispersion relation, eq. (21) may have zero, one or two roots within an interval limited by two neighbouring quasi-particle energies

$2E_\rho$, $2E_\sigma$ ($E_\sigma > E_\rho$). Let us evaluate (21) at the beginning and at the end of such an interval.

$$W^2-4E_\rho^2 \to +0) \frac{\Omega_\rho E_\rho}{8(W^2-4E_\rho^2)(e_\sigma^2-e_\rho^2)}\left[\frac{2\Omega_\sigma e_\rho}{E_\sigma(e_\rho+e_\sigma)}+\frac{\Omega_\rho}{E_\rho}+4\sum_{\nu\neq\rho,\sigma}\frac{(e_\rho e_\nu-e_\sigma^2)}{E_\nu(4E_\nu^2-W^2)}\right],$$
$$W^2-4E_\sigma^2 \to -0) \frac{\Omega_\sigma E_\sigma}{8(W^2-4E_\sigma^2)(e_\sigma^2-e_\rho^2)}\left[\frac{2\Omega_\rho e_\rho}{E_\rho(e_\rho+e_\sigma)}+\frac{\Omega_\sigma}{E_\sigma}+4\sum_{\nu\neq\rho,\sigma}\frac{(e_\sigma e_\nu-e_\rho^2)}{E_\nu(4E_\nu^2-W^2)}\right]. \tag{24}$$

We see that, if sign e_σ = sign e_ρ, all terms within brackets have the same sign. Thus, the function (21) has a different sign at the beginning and at the end of the interval, and there is an odd number of roots present. On the contrary, if sign e_ρ = $-$sign e_σ, the first term within brackets changes sign. If the two absolute single-particle energies are sufficiently close, this term predominates over the others, the function in (21) has the same sign in both extremes, and there is an even number of roots within the interval.

The coefficients $a_{n\nu}$, $b_{n\nu}$ of the transformation (2) are given now by

$$\begin{aligned}
a_{n\nu} &= [\Lambda_{1n}(U_\nu^2-V_\nu^2)+\Lambda_{2n}]/(2E_\nu-W_n),\\
b_{n\nu} &= [-\Lambda_{1n}(U_\nu^2-V_\nu^2)+\Lambda_{2n}]/(2E_\nu+W_n),\\
(\Lambda_{2n}/\Lambda_{1n}) &= -\frac{\sum_\nu (U_\nu^2-V_\nu^2)/(4E_\nu^2-W_n^2)}{W_n\sum_\nu 1/2E_\nu(4E_\nu^2-W_n^2)},
\end{aligned} \tag{25}$$
$$\Lambda_{1n} = \tfrac{1}{2}\left[\sum_\nu (U_\nu^2-V_\nu^2)^2 2E_\nu W_n/(4E_\nu^2-W_n^2)^2 + (\Lambda_{2n}/\Lambda_{1n})\sum_\nu\frac{(U_\nu^2-V_\nu^2)(4E_\nu^2+W_n^2)}{(4E_\nu^2-W_n^2)^2}+(\Lambda_{2n}/\Lambda_{1n})^2\sum_\nu 2E_\nu W_n/(4E_\nu^2-W_n^2)^2\right]^{-1},$$
$$F_n = 2\Lambda_{1n}/G.$$

Any interaction which can be treated using the linearization approximation may be incorporated in the expression determining the frequencies W_n. In particular, this was done in ref. [7]) for the quadrupole force leading to a 3×3 determinant:

$$H_Q = -\tfrac{1}{2}\chi Q^2,$$
$$Q = \sum_i q_i(\Gamma_i^\dagger+\Gamma_i) = \sum_\nu q_{\nu\nu}(\Gamma_\nu^\dagger+\Gamma_\nu)\Delta/E_\nu + \text{terms containing non-diagonal single-particle quadrupole matrix-elements},$$
$$0 = \begin{vmatrix}
\sum_\nu \frac{2E_\nu}{4E_\nu^2-W_n^2}-\frac{1}{G} & \sum_\nu\frac{U_\nu^2-V_\nu^2}{4E_\nu^2-W_n^2} & \sum_\nu\frac{q_{\nu\nu}2U_\nu V_\nu}{4E_\nu^2-W_n^2}\\
W_n^2\sum_\nu\frac{U_\nu^2-V_\nu^2}{4E_\nu^2-W_n^2} & \sum_\nu\frac{(U_\nu^2-V_\nu^2)^2 2E_\nu}{4E_\nu^2-W_n^2}-\frac{1}{G} & \sum_\nu\frac{q_{\nu\nu}2U_\nu V_\nu(U_\nu^2-V_\nu^2)}{4E_\nu^2-W_n^2}\\
W_n^2\sum_\nu\frac{q_{\nu\nu}2U_\nu V_\nu}{4E_\nu^2-W_n^2} & \sum_\nu\frac{(U_\nu^2-V_\nu^2)q_{\nu\nu}2U_\nu V_\nu}{4E_\nu^2-W_n^2} & \sum_i\frac{q_i^2\varepsilon_i}{\varepsilon_i^2-W_n^2}-\frac{1}{2\chi}
\end{vmatrix}, \tag{26}$$

where the variable i denotes a two-quasi-particle excited state connected with the ground state by the matrix element q_i of the quadrupole operator.

2.1.3. *Physical operators related to pairing fluctuations.* A collective state implies the existence of enhancements in the matrix elements corresponding to physical observables. For instance, because the electric quadrupole operator is very similar to the mass quadrupole operator, it can take full advantage of the coherence of the quadrupole collective state.

The essential characteristic of the specific pairing operator F is that it is odd under the transformation which changes sign of the single-particle energies. In addition, the absolute value of the matrix element f_ν (11) increases with $|e_\nu|$.

The relationship between pairing vibrations and fields changing the number of particles, and the interpretation of the specific operator as a sum of transfer operators suggest that the two-body transfer of identical particles is enhanced by the coherence of our collective state.

The operator corresponding to the absorption of two identical particles with zero angular momentum is given by

$$S = \sum_\nu s_\nu c_\nu^\dagger c_{\bar{\nu}}^\dagger \approx \sum_\nu s_\nu U_\nu V_\nu + \sum_\nu s_\nu U_\nu^2 \Gamma_\nu^\dagger - \sum_\nu s_\nu V_\nu^2 \Gamma_\nu, \tag{27}$$

where s_ν are weighting factors depending on radial integrals. We shall assume them to be all equal ($s_\nu = s$). The matrix element of S between the ground and a one-phonon state is given by †

$$S_n = \Lambda_n s \sum_\nu \frac{(U_\nu^2 - V_\nu^2)^2 2E_\nu}{4E_\nu^2 - W_n^2} = \Lambda_n s/G = \tfrac{1}{2} s F_n. \tag{28}$$

Thus the operator S ($r_\nu \approx U_\nu^2 s$) partly fulfills the previous conditions for enhancement, as it contains a significant odd contribution. (This is obtained by replacing U_ν^2 by $\frac{1}{2}+\frac{1}{2}(U_\nu^2 - V_\nu^2)$ and taking the second term).

It has been emphasized that the two-nucleon transfer is specially enhanced for processes connecting the ground state of two superconducting systems ((t, p) reactions, α-decay [8]), etc.). The corresponding matrix elements are given by the ground-state expectation value of (27)

$$\langle 0|S|0\rangle = s \sum_\nu U_\nu V_\nu = s\Delta/G. \tag{29}$$

Thus, the ratio between the cross sections of the reactions populating the pairing vibration and the ground state is roughtly given by

$$|S_1|^2/\langle 0|S|0\rangle^2 \approx \Lambda_1^2/\Delta^2 \approx (2e/\pi\Delta)^2, \tag{30}$$

which is about 0.02 for realistic parameters in deformed nuclei. The existence of two parallel transitions has the obvious advantage of allowing us to use relative cross

† Eq. (28) holds exactly in the symmetric case. Otherwise, $S_n = \frac{1}{2} s F_n (1 + \Lambda_{2n}/\Lambda_{1n})$ and the corresponding matrix element for the pick-up reaction is $(S^\dagger)_n = \frac{1}{2} s F_n (1 - \Lambda_{2n}/\Lambda_{1n})$ (see eq. (25)).

sections (and thus eliminating s and G from (30)). However, this advantage is partially cancelled by the great enhancement in the transition to the ground state.

Again, this is very similar to the case of β-vibrations in deformed, axially symmetric nuclei. Most of the strength of the $r^2 Y_{20}(\theta, \varphi)$ operator lies in the expectation value corresponding to the lowest intrinsic state, and the ratio between the reduced electromagnetic transition probabilities to the 2^+ member of the β-band and to the 2^+ member of the ground state band †, has the same order of magnitude.

Let us consider now single-particle operators. The monopole operator has a component which is odd under the previous transformation, because the states above the Fermi surface will have, in general, greater radii than those below. However, we do not expect any important enhancement, because the fluctuations in the radii are small, and because the second condition is not fulfilled (due to the damping factor $U_\nu V_\nu$ in r_ν).

In deformed nuclei the quadrupole operator can also be used because states with positive quadrupole moment lie lower than those with negative moment. However, (i) there exists also here the damping factor $U_\nu V_\nu$. (ii) According to our previous estimations, only few levels (which are close enough to the Fermi surface) may participate in the collective state in actual deformed nuclei; these usually have similar intrinsic quadrupole moments. Thus, in most deformed nuclei, the quadrupole transitions will not feel the coherence of our state. However, we may have a more favourable case where low-energy β-vibrations do occur. If the diagonal single-particle matrix elements of the quadrupole operator $q_{\nu\nu}$ are odd, the determinant (26) yields the dispersion relation corresponding to even states, and the expression

$$4\Delta^2 \left[\left(\sum_\nu \frac{(U_\nu^2 - V_\nu^2)^2 E_\nu}{4E_\nu^2 - W_n^2} \right) \left(\sum_\nu \frac{q_{\nu\nu}^2}{E_\nu(4E_\nu^2 - W_n^2)} \right) - \left(\sum_\nu \frac{q_{\nu\nu}(U_\nu^2 - V_\nu^2)}{4E_\nu^2 - W_n^2} \right)^2 \right]$$
$$= \frac{2\Delta^2}{G} \sum_\nu \frac{q_{\nu\nu}^2}{E_\nu(4E_\nu^2 - W_n^2)} + \frac{1}{\chi} \sum_\nu \frac{(U_\nu^2 - V_\nu^2)^2 E_\nu}{4E_\nu^2 - W_n^2} - \frac{1}{2\chi G}. \quad (31)$$

The left-hand side is a small quantity. For instance, it is exactly zero for a degenerate half-filled shell plus a quadrupole field ($e_\nu = \mu q_{\nu\nu}$). In this case, eq. (31) reduces to

$$\sum_\nu (q_{\nu\nu} 2U_\nu V_\nu)^2 2E_\nu / (4E_\nu^2 - W_n^2) = 1/(2\chi + \mu^2 G/\Delta^2). \quad (31')$$

By comparison with the usual dispersion relation we see that the strength of the quadrupole interaction is effectively increased. Moreover, the self-consistency condition for the equilibrium deformation can be written

$$\mu = -\chi Q = -\chi \sum_\nu q_{\nu\nu} 2V_\nu^2 = -\chi\mu \sum_\nu q_{\nu\nu}^2 / E_\nu,$$

and therefore

$$1/\chi = \sum_\nu q_{\nu\nu}^2 / E_\nu .$$

† This ratio has the value 0.02 in ^{152}Sm (ref. [16]) and 0.01 in ^{232}Th and ^{238}U (ref. [17]).

Now, the right-hand side of (31′) can be evaluated using the last relation

$$\frac{1}{2\chi r\mu^2 G/\Delta^2} = \left[\frac{2}{\sum_\nu q_{\nu\nu}^2/E_\nu} + \frac{(G/\Delta^2)(\sum_\nu e_\nu^2/E_\nu)}{\sum_\nu q_{\nu\nu}^2/E_\nu}\right]^{-1} = \frac{(G/\Delta^2)\sum_\nu E_\nu}{\sum_\nu q_{\nu\nu}^2/E_\nu},$$

and thus (31′) is equivalent to

$$\sum_\nu \frac{q_{\nu\nu}^2}{E_\nu(4E_\nu^2 - W_n^2)} = \frac{(\sum_\nu q_{\nu\nu}^2/E_\nu)(\sum_\nu 1/E_\nu)}{4\sum_\nu E_\nu}.$$

In the limit in which the quadrupole splitting is small ($E_\nu = \Delta$) or in the limit in which there are only two single-particle matrix elements $q_{\nu\nu}$ (with the same absolute value), all the quasi-particle energies are equal and the frequency of the lowest state is zero. The coupling between the β- and the pairing vibrations produces an unstable collective state. This unstability corresponds to the fact that if we increase the deformation, the change in binding energy produced by an increase in the quadrupole interaction $-\frac{1}{2}\chi Q^2$ is cancelled by a decrease in the pairing interaction Δ^2/G. Consequently, in this over-simplified situation no equilibrium position occurs (in second order).

The opposite situation arises when all $q_{\nu\nu}$ are similar to each other. In this case, the quadrupole moment is an even operator, and the pairing vibration is uncoupled from both the β-vibration and the spurious state. It is also easy to show that here only the spurious state is expected to be within the gap. Physically, this corresponds to the fact that a variation in energy does not change the value of Q if all single-particle states have similar $q_{\nu\nu}$; and consequently, the potential energy surface represented as a function of Q presents a sharp minimum.

However, in addition to the coupling with pairing vibrations, it is also possible to obtain low-energy β-vibrations through non-diagonal single-particle matrix elements or through diagonal $q_{\nu\nu}$ oscillating in sign at both sides of the Fermi surface. Only a calculation with realistic single-particle energies can inform us which of the cases occurs in actual nuclei and whether the predicted energy of β-vibrations [4,5]) has been lowered by renormalizations similar to (31′).

Another operator characteristic of deformed nuclei corresponds to the coupling between vibrational and rotational degrees of freedom. In second order perturbation theory, the Coriolis force H_C couples the ground state with two-quasi-particle states $\Gamma_\nu^\dagger|0\rangle$

$$\langle \nu, I|H_C|0, I\rangle = \left(\frac{\hbar^2}{2\mathscr{I}}\right)^2 z_\nu I(I+1), \qquad z_\nu = 2\sum_\nu \langle 0|J_-|\sigma\rangle\langle\sigma|J_+\Gamma_\nu^\dagger|0\rangle/\varepsilon_\sigma, \quad (32)$$

where the intermediate states $|\sigma\rangle$ have $K = 1$ and energy ε_σ. Here $\mathscr{I}$ is the moment of inertia, I the total angular momentum and $J_\pm$ the components of the angular momentum operator corresponding to the intrinsic motion. Evaluating (32) one obtains

$$z_\nu = \Delta \sum_\omega |\langle\omega|J_\pm|\nu\rangle|^2(e_\nu - e_\omega)/E_\nu E_\omega(E_\nu + E_\omega). \tag{33}$$

Obviously the value of z_ν depends on the details of the single-particle structure. For instance, the uniformly-spaced model can represent the central region of a large, deformed j-shell. In this case the matrix elements of $J_\pm$ are

$$|\langle jm|J_\pm|jm'\rangle|^2 = \tfrac{3}{4}j^2\delta_{m',m\mp 1},$$

and thus

$$z_\nu = 2\Delta j^2 e[1/E_{\nu-1}(E_\nu + E_{\nu-1}) - 1/E_{\nu+1}(E_\nu + E_{\nu+1})]/E_\nu \approx j^2 3\Delta e(E_{\nu+1} - E_{\nu-1})/E_\nu^4, \tag{34}$$

which changes sign when crossing the Fermi surface. Moreover, for states lying away from the Fermi surface ($|e_\nu| \gg e$),

$$E_{\nu+1} - E_{\nu-1} \approx 4(U_\nu^2 - V_\nu^2),$$

and thus z_ν approaches the specific matrix element f_ν. The total mixing matrix element z is obtained by replacing in (10) r_ν by z_ν. In the case of 20 particles moving in 20 equally-spaced levels, z/z_μ is about one third of the value (15) corresponding to the specific operator. (z_μ is the value which would be obtained in the absence of the pairing vibrations). This reduction is due to the existence of the damping factor $1/E_\nu^4$.

So far we have been looking for processes depending linearly on the amplitudes. Because, using realistic parameters, the coefficients $a_{n\nu}$ in eq. (2) decrease rapidly with the single-particle energy, the coherence of our state cannot be exhibited in physical processes depending quadratically on the amplitudes.

2.1.4. *Quantitative calculations.* The single-particle parameters and interaction constants corresponding to deformed nuclei are the same as those used in ref. [4]).

The relevant results concerning the neutron states in rare-earth elements are listed in tables 1 and 2 and in fig. 4. In table 1, the values of F_n are listed for the three lowest roots while in table 2 (δ = 0.25 and δ = 0.20) only the values corresponding to the two lowest roots are given. The mean value for the first root is 2.2 with a mean square deviation of 1.3, in agreement with the previous estimates (14). The oscillations can be understood on the basis of Nilsson's energies; the enhancements for 90, 96 and 102 neutrons correspond to small gaps in the single-particle spectrum; for 98 and 100 neutrons the two (521$\frac{1}{2}$) and (633$\frac{7}{2}$) states are extremely close to each other and to the Fermi surface, and the coherence in the first excited state disappears. The predicted gap between the (624$\frac{9}{2}$) level and the (510$\frac{1}{2}$) level again increases the matrix element for 108 neutrons (table 2). The average matrix elements for the (t, p) reaction is 1.0 with a root mean square deviation of 0.3, thus predicting the cross section to the collective excitation to be on the average about 40 times smaller than the cross section to the ground state, again in agreement with previous qualitative estimates. However in the case of 108 neutrons it is predicted to be only 15 times smaller.

In deformed nuclei, a decrease in the single-particle level density has been experimentally found for 104 (ref. [9]) and 152 (ref. [10]) neutrons. It should be very interesting to measure whether corresponding enhancements occur in these cases.

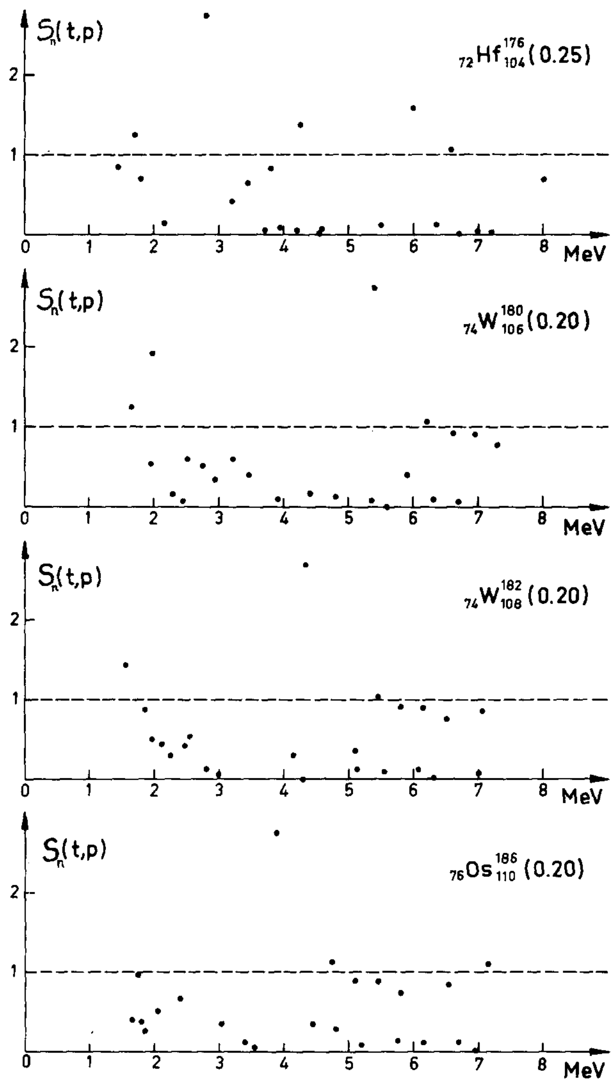

Fig. 4. The matrix element S_n for the (t, p) transfer is given as a function of the excitation energy for several neutron numbers in the rare-earth region, assuming the Nilsson single-particle energies.

The third column of table 1, and especially fig. 4, suggest the possibility of finding significant enhancements in the population of some excited states. In fig. 4 we see a state with high cross section, which is lowered in energy as the number of neutrons increases. It is due to a gap appearing in the theoretical single-particle spectrum for 118 neutrons.

Table 2 also presents the values of the matrix elements corresponding to the operators F and S when the part of the pairing force H_p', which is responsible for the pairing vibration, is neglected (H_p'' is kept in order to eliminate the spurious state).

TABLE 1

Distorted nuclei with $\delta = 0.30$

N	Δ/G	W_1	F_1	S_1	W_2	F_2	S_2	W_3	F_3	S_3
90	6.5	1.77	3.6	1.8	2.06	1.2	0.8	2.13	1.6	0.6
92	6.5	1.75	1.2	0.7	1.94	1.7	1.2	2.10	3.2	1.0
94	6.4	1.69	1.9	0.9	1.83	1.2	0.4	1.99	2.9	2.0
96	6.4	1.64	3.0	1.5	1.89	1.8	0.6	1.91	0.4	0.2
98	6.3	1.63	0.4	0.2	1.66	0.4	0.2	1.82	2.6	0.9
100	6.0	1.54	0.4	0.1	1.59	0.4	0.1	1.92	3.5	2.2
102	5.4	1.33	4.6	1.7	1.81	2.0	1.3	1.84	0.4	0.1
104	5.6	1.37	2.1	1.0	1.64	1.7	1.0	1.95	3.8	0.9
106	5.9	1.42	1.8	0.8	1.57	1.8	0.6	1.67	1.9	1.2

The ratio Δ/G and the energy, matrix element F_n and matrix element S_n for the (t, p) reaction are given for the three lowest roots of (21).

TABLE 2

Distorted nuclei with $\delta = 0.25$ and $\delta = 0.20$

N	δ	Δ/G	W_1	F_1	S_1	W_2	F_2	S_2	W_1'	F_1'	S_1'
104	0.25	6.0	1.45	2.1	0.8	1.70	2.2	1.3	1.50	0.4	0.3
106	0.25	5.7	1.36	2.5	1.0	1.66	2.0	0.5	1.43	0.4	0.4
106	0.20	6.9	1.65	3.0	1.2	1.95	1.8	0.5	1.74	0.5	0.3
108	0.25	5.7	1.35	4.3	1.8	1.72	2.4	0.6	1.57	0.7	1.1
108	0.20	6.8	1.59	3.2	1.4	1.88	3.0	0.8	1.70	0.5	0.2
110	0.20	7.0	1.67	0.8	0.4	1.75	2.7	1.0	1.67	0.09	0.2
112	0.20	7.0	1.63	0.8	0.3	1.63	0.7	0.3	1.63	0.003	0.02

The ratio Δ/G and the energy, matrix element F_1, and matrix element S_n for the (t, p) reaction are given for the two lowest roots of (21). The last three columns give the corresponding values that are obtained for the first root if one omts in the residual pairing Hamiltonian the second term in (4).

The resulting mean values for F_1 and S_1 are 0.37 and 0.36, respectively, thus showing a decrease of a factor of 6 and 3, respectively, with respect to the values obtained including the pairing-vibration correlations.

The coupling between the β- and pairing vibrations can be evaluated by either considering (26) or, in a simpler way, by solving (21) and by calculating the B(E2) value corresponding to the transition between the ground state and the pairing vibration.

The predicted B(E2) values do not show significative enhancements, and moreover, the larger B(E2) values are not correlated at all with corresponding enlargements in the matrix elements F_n. This is also true for inelastic scattering processes of the type (p, p′), monopole matrix elements and the parameter measuring the coupling of rotational and vibrational motion.

TABLE 3

Spherical single-closed shell nuclei

Protons	Δ/G	W_1	S_1	W_2	S_2	W_3	S_3	W_4	S_4	W_1'	S_1'
56	4.5	1.7	1.1	3.8	2.0	6.0	0.7	6.6	0.4	1.9	0.7
58	4.7	1.6	0.9	3.3	2.0	5.5	0.8	6.1	0.4	1.8	0.5
60	4.7	1.7	0.7	2.7	2.1	4.9	0.8	5.5	0.4	1.9	0.01
62	4.6	2.0	2.1	2.1	0.5	4.3	0.8	4.8	0.4	2.1	0.6
					82 – neutrons						
Neutrons											
64	6.5	2.7	0.5	2.8	1.3	3.4	0.4	3.9	0.3	2.7	0.02
66	6.5	2.6	1.1	2.8	0.5	3.5	0.3	3.9	0.3	2.8	0.2
68	6.5	2.7	0.9	3.0	0.4	3.1	0.2	4.3	0.3	2.9	0.4
70	6.5	2.7	0.2	2.8	0.6	3.3	0.3	4.8	0.2	2.7	0.1
72	6.4	2.5	0.2	3.1	0.5	3.7	0.3	5.2	0.2	2.5	0.1
					Sn isotopes						

The ratio Δ/G and the energy and matrix element S_n for the (t, p) reaction, are given for all the roots of (21). The last two columns list the values corresponding to the first root if one omits in the residual pairing Hamiltonian the second term in (4).

Typical spherical superconducting nuclei are the single-closed shell nuclei in which the shell starting at 50 particles is being filled. In the Sn isotopes the $d_{\frac{5}{2}}$, $g_{\frac{7}{2}}$ and $s_{\frac{1}{2}}$ states are below the Fermi surface, while the $h_{\frac{11}{2}}$ and $d_{\frac{3}{2}}$ levels lie above. Using the same energies as Kisslinger and Sorensen [11]), our parameters x and y are ≈ 1.3 and 0.4, respectively. According to fig. 2, these numbers would correspond to an enhancement of the collective properties associated with the pairing-fluctuation state. However, this estimation is handicapped (i) because of the presence of the $s_{\frac{1}{2}}$ level relatively close to the Fermi surface, the lowest excited estate can be approximated as a mixture of the configuration $(s_{\frac{1}{2}})^2$ and the pair fluctuation; and (ii) the single-particle levels do not have the same degeneracies; in consequence, the matrix element for the (t, p) reaction strongly favours the population of the $(h_{\frac{11}{2}})^2$ configuration. Thus, the reaction is somewhat intermediate between a first and a second order process. (The latter ones depending on the admixture of a particular state, which is a small number unless the state is very close to the Fermi surface.) Due to both disadvantages, there appears to be a relative enhancement of the reaction populating the first state † only

† These calculations have been performed in collaboration with C. Veje.

if we compare the predicted matrix elements with those obtained by simply projecting the spurious state out of the same level (table 3).

2.2. NON-SUPERCONDUCTING NUCLEI

2.2.1. *Qualitative considerations.* Let us consider now those nuclei in which no superconducting solution exists. The subscripts ν will denote the single-particle states which are filled in the unperturbed ground state $|0\rangle$; the label ω will indicate empty states. The quasi-phonon creation operators are now

$$\Gamma_b^\dagger = \sum_\omega b_\omega \Gamma_\omega^\dagger + \sum_\nu b_\nu \Gamma_\nu, \tag{35a}$$

$$\Gamma_a^\dagger = \sum_\nu a_\nu \Gamma_\nu^\dagger + \sum_\omega a_\omega \Gamma_\omega, \tag{35b}$$

where

$$\Gamma_\omega^\dagger \equiv c_\omega^\dagger c_{\bar\omega}^\dagger, \qquad \Gamma_\nu^\dagger \equiv c_{\bar\nu} c_\nu .$$

The linearization equations (5) are now equivalent to

$$1/G = \sum_\omega 1/(2e_\omega - W_b) + \sum_\nu 1/(2e_\nu + W_b), \tag{36a}$$

$$1/G = \sum_\omega 1/(2e_\omega + W_a) + \sum_\nu 1/(2e_\nu - W_a), \tag{36b}$$

where all energies e_ν, e_ω, W_a, W_b, in the denominators are positive and measured from the common minimum of the curves representing the right-hand side of (36a) and (36b) as a function of W (see fig. 5). This minimum coincides with the value of the Fermi surface in the limit in which the superconducting solution disappears.

The coefficients in the transformation (35) are given by

$$b_\omega = \Lambda_b/(2e_\omega - W_b), \quad b_\nu = \Lambda_b/(2e_\nu + W_b), \quad a_\nu = \Lambda_a/(2e_\nu - W_a), \quad a_\omega = \Lambda_a/(2e_\omega + W_a), \tag{37}$$

with the normalization condition

$$1 = \Lambda_b^2 [\sum_\omega 1/(2e_\omega - W_b)^2 - \sum_\nu 1/(2e_\nu + W_b)^2] \tag{38a}$$

$$= \Lambda_a^2 [\sum_\nu 1/(2e_\nu - W_a)^2 - \sum_\omega 1/(2e_\omega + W_a)^2]. \tag{38b}$$

Because $\Lambda_b^2(\Lambda_a^2)$ is the inverse of the derivative of the right member of (36) with respect to $W_b(W_a)$, both Λ_b and Λ_a go to infinity when W_b tends to zero and, moreover, from (38)

$$\lim_{W_b \to 0} \Lambda_a/\Lambda_b = 1. \tag{39}$$

As $\Gamma_a^\dagger|\tilde{0}\rangle(\Gamma_b^\dagger|\tilde{0}\rangle)$ represents a state with two more (less) particles, the excitations of our system are represented by the two-phonon states

$$|ab\rangle = \Gamma_a^\dagger \Gamma_b^\dagger |\tilde{0}\rangle,$$
$$H|ab\rangle = (E_0 + W_a + W_b)|ab\rangle,$$

where E_0 is the energy of the correlated ground state $|\tilde{0}\rangle$. Therefore, the excitation energy of the lowest state 0^+ has been lowered due to the pairing interaction from the smallest of the distances $2(e_\nu+e_\omega)$ to the smallest sum (W_a+W_b).

Our description yields also the wave functions and energies of the states belonging to neighbouring nuclei. In particular, the increase in binding energy due to the pairing interaction of the nucleus with two particles moving in ω-levels, with respect to the binding of the closed-shell nucleus, is given in this model by the difference

$$2e_\omega - W_b,$$

and, therefore, the decrease in the energy of the excited state is determined by the increase in binding of nuclei with $n+2$ and $n-2$ particles (fig. 5).

The "specific" operators correspond to the transfer reaction of two identical particles F

$$F = \sum_\omega \Gamma_\omega^\dagger + \sum_\nu \Gamma_\nu = \sum_b \left(\sum_\omega b_\omega - \sum_\nu b_\nu\right)\Gamma_b^\dagger + \sum_a \left(\sum_\nu a_\nu - \sum_\omega a_\omega\right)\Gamma_a \tag{40}$$

and to the inverse process $F^\dagger$. In particular, the matrix elements corresponding to the transfer of two nucleons to the ground state and to an excited state are

$$\langle\tilde{0}|\Gamma_a F|\tilde{0}\rangle = \Lambda_a/G, \tag{41a}$$

$$\langle\tilde{0}|\Gamma_a F|a'b\rangle = \Lambda_b\delta_{aa'}/G. \tag{41b}$$

Because of the delta factor in (41b), we can populate only those excited states having the same hole configuration as the ground state of the $n-2$ system $(\Gamma_a^\dagger|\tilde{0}\rangle)$. There are as many such excited states as solutions of (36b). If the lowest root W_b is sufficiently displaced from the unperturbed energy, the stripping of two identical particles is characterized by two strong peaks corresponding to the ground and first excited state, respectively. The ratio between these two cross sections is given by

$$\sigma(^10^+)/\sigma(^20^+) = (\Lambda_a/\Lambda_b)^2, \tag{42}$$

which depends on the ratio between the degeneracies of the lower and upper shells. As the ratio Λ_a/Λ_b tends to 1, when $W_b \to 0$ (eq. (39)), the correlations tend to equalize the population of both levels.

The two-shell model introduced in (16) can also be applied now. The energy of the lowest state is given by

$$2W_1 = b(1+y^2-2y\coth z)^{\frac{1}{2}}, \tag{43}$$

while the matrix element F_1 is

$$F_1 = (6\Omega/2W_1)^{\frac{1}{2}}(z/\sinh z)$$

In the limit $y = 0$ (no spread within the shells)

$$y = a = z = 0,$$
$$zW_1 = (b^2-2G\Omega b)^{\frac{1}{2}},$$
$$F_1^2/\Omega = b/(b^2-2G\Omega b)^{\frac{1}{2}},$$

which is greater than 1, showing the same enhancement as in the superconducting case. As the spread increases, (keeping the energy W_1 constant) the cross section decreases due to the increase in z.

2.2.2. *Quantitative calculations for* ^{208}Pb. It is known from Hogaasen's calculations [2]) on the two-shell model that the predicted decrease in excitation energy (as G approaches the critical value) is a real effect only in the limit of high degeneracies; because of this, and because the distance between two closed shells is smaller for heavy nuclei, the most proper case for testing the model is ^{208}Pb. The single-particle levels are experimentally well established [12, 13]) but for the highest proton shell; for these levels we have identified the three known energies with the $h_{\frac{9}{2}}$, $f_{\frac{7}{2}}$ and $i_{\frac{13}{2}}$ levels, respectively. and assume that the other three levels have the same relative energies as in the corresponding neutron shell. In any case, their exact position is completely unimportant, because of the very low degeneracy associated with them. As the sum of the pairing binding energies in ^{206}Pb and ^{210}Pb is 2 MeV (ref. [12])), we predict a excited 0^+ neutron state at 4.9 MeV. The ratio between the matrix element populating the first excited state and the ground state should be 1.3; from the data [12]) on ^{210}Pb, we find that a similar proton state should lie higher, at about 6.7 MeV. The values of G, which correspond to these energies, are $G_n = 18$ MeV/A and $G_p = 26$ MeV/A, in fair agreement with those used by Kisslinger and Sorensen [11]) for the neighbouring nuclei ($G = 24$ MeV/A), taking into account that this last value includes renormalization effects due to inter-shell transitions which we explicitly consider here.

The experimental situation partially supports this picture. Both from α-decay [14]) and the (t, p) process [15]) there is experimental evidence indicating the existence of a state at 3.20 MeV, which is populated in the (t, p) reaction with a total cross section which is 0.6 of the cross section populating the ground state. This strength is consistent with a 0^+ assignment, which is furthermore supported by the fact that pairs have been observed in connection with the α-decay.

We can lower the excited neutron state in order to fit the experimental energy by means of a relatively small increase in the pairing interaction; $G_n = 22$ MeV/A. The corresponding ratio between the matrix elements is decreased to 1.1, a remaining difficulty being the fit of the experimental binding energies in ^{206}Pb and ^{210}Pb (fig. 5). It should be stressed that this discrepancy will arise whenever we impose the linearization equation (5) and, in particular, it is independent of any assumed two-body interaction.

The previous strengths of the pairing interaction are close to the critical value, i.e., to the value for which appears a superconducting solution $(G_C)_n = 25$ MeV/A. For $G_n = 30$ MeV/A, both 2Δ and the pairing vibration lie at 3.20 MeV, and the ratio between the matrix elements is further decreased to 0.35. For this value of G_n, a corresponding difficulty appears in the description of ^{206}Pb: the value of the gap 2Δ increases to 3.32 MeV, in contradiction with the energy of the first excited 0^+ state at 1.15 MeV.

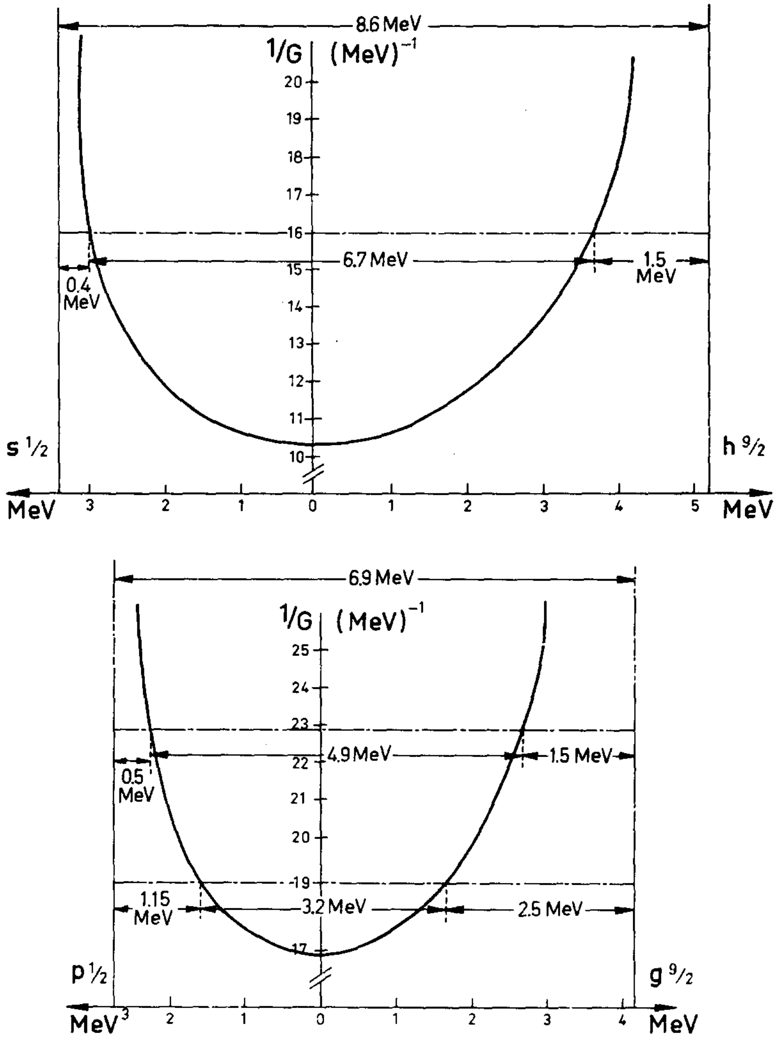

Fig. 5. The left-hand side of eq. (31) is plotted for both neutron and proton levels in the case of ^{208}Pb, in the region between the two neighbouring shells. For each G there is a straight line, which is divided by our curve in three sections. The first one from the left corresponds to the pairing energy of the nucleus with $n-2$ particles; the segment at the centre represents the excitation energy of a closed shell nucleus; and the third is the pairing energy of the $n+2$ system. The figure above corresponds to the neutron case; the one below, to the protons.

3. Conclusions

The pairing vibration appears as a low-energy collective mode in the case of a residual two-body interaction such that the nucleus should be sufficiently close to the transition point between the single-particle and a superconducting system, and if there exists at least two well-defined groups of single-particle levels in such a way that the spread on energy within each group should be significantly smaller than the distance between the two groups.

The operator which specifically feels the coherence of the collective state corresponds to the two-body transfer processes. A difficulty existing in superconducting nuclei is the existence of large competing matrix elements corresponding to the population of the ground and spurious states.

Deformed even nuclei are not very far from satisfying the first condition for the existence of pairing vibrations. The second one is only partly fulfilled whenever a gap appears in the single-particle spectrum. These gaps are predicted to produce enhancements in the reactions populating either the first of higher excited states.

In spherical nuclei, the most promising cases are the closed-shell nuclei. The main characteristic of the resulting spectrum is the existence of a low-energy 0^+ state which is populated with the same intensity as the ground state. Such spectrum appears to exist in ^{208}Pb although residual effects must have at least quantitative importance. It should be interesting to perform also the (t, p) reactions on nuclei with 80 neutrons.

It should be stressed that the pairing fluctuations previously considered occur (or not) within the framework of a simplified model consisting of particles interacting via a pairing force with constant matrix element. This model has proved its validity mainly in connection with the properties of the ground state. However, one could also produce two sets of levels and the corresponding pairing fluctuations through the introduction of some additional selection rule in the pairing matrix elements. It is probable that these refinements of the pairing model are needed in order to explain further details in the excitation spectrum.

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Appendix

In an attempt to clarify the physical meaning of the pairing vibrations, we try in this appendix to emphasize further the analogy between quadrupole and pairing collective states.

Let us start considering spherical, non-superconducting nuclei. We construct the states by means of the independent-particle operators $c^\dagger_{jm}$ and c_{jm}. These operato create and destroy the Mayer-Jensen "particles". An essential characteristic of these degrees of freedom is that they carry a definite angular momentum and that they create (or destroy) one nucleon.

However, the Hartree-Fock equations may yield deformed states. This happens, for instance, if the residual force has a strong quadrupole component

$$H_Q = -\tfrac{1}{2}\chi \sum_{\mu} Q_{\mu}^{\dagger} Q_{\mu}. \tag{A.1}$$

Let us assume that these solutions maintain the axial and reflection symmetries. We note:

(i) The nuclear distortion is measured by the magnitude of the static quadrupole moment Q. The ground state is determined by minimizing the expectation value of the Hamiltonian with respect to Q.

(ii) The Mayer-Jensen "particles" are replaced by Nilsson "particles", which move independently in a deformed field. In consequence, they may be expressed as linear combinations of the form

$$\gamma_m^{\dagger} = \sum_j a_j c_{jm}^{\dagger}. \tag{A.2}$$

(iii) A fundamental invariance principle has been violated, namely the conservation of total angular momentum. One insures, however, that the average value of the total angular momentum has a prescribed value I, by using the technique of the Lagrange multipliers.

In analogy with eq. (1.A), we may expand the total wave function in eigenstates of the total angular momentum.

$$|0\rangle = \sum_I D_{MK}^I |0, I\rangle. \tag{A.3}$$

The set of states $|0, I\rangle$ associated with the intrinsic state $|0\rangle$, constitute a rotational band. These bands are characterized by an enormous enhancement of the quadrupole matrix elements between states belonging to the same band.

(iv) Finally, we must study also the transition region, i.e., the domain in which neither the spherical nor the deformed description is predominant. Most of the work performed so far in this respect assumes a definite equilibrium position and small departures from it (quadrupole vibrations in spherical nuclei, β-vibrations in deformed nuclei.)

Instead of treating a distortion in the shape of the nucleus, we can consider a superconducting distortion, if our Hamiltonian includes a pairing interaction term

$$H_p = -GP^{\dagger}P \tag{A.1'}$$

In this case we may remark:

(i) The distortion is measured by the gap parameter Δ/G which is the expectation value of the pairing operator P. Furthermore, the value of the gap is used as a minimization parameter in the determination of the wave function.

(ii) The Mayer-Jensen "particles" are replaced by Bogolyubov-Valatin "particles". These latter ones are given by the linear combinations

$$\begin{aligned} \alpha_{jm}^{\dagger} &= U_j c_{jm}^{\dagger} - V_j c_{j\bar{m}}, \\ \alpha_{jm}^{\dagger} &= V_j c_{jm} + U_j c_{j\bar{m}}^{\dagger}, \end{aligned} \tag{A2'}$$

(iii) These "particles" move independently in a distorted field which does not conserve the numbers of particles. One requires that the system has a fixed average number of particles N by using a Lagrange multiplier. In fact, the non-conservation of the number of particles is not an unfortunate feature of the superconducting solution, but rather a fundamental feature. It is at least as essential as the lack of conservation of angular momentum in a deformed field.

The wave functions can be expanded in terms of states corresponding to a definite number of particles

$$|0\rangle = \sum c_N |0, N\rangle. \tag{A.3'}$$

The set of states $|0, N\rangle$ associated with the intrinsic state $|0\rangle$ represents, for instance, the set of ground states of neighbouring even nuclei. They form a superconducting band. These bands are characterized by the great enhancement of the two-body transfer operator P between states belonging to the same band.

(iv) As in the quadrupole case, one may attack the problem of the transition region by studying the small fluctuations about the equilibrium position. This has been the aim of the present work. The oscillations around $\Delta = 0$ correspond to quadrupole vibrations in spherical nuclei. In this last case, the angular momentum is a good quantum number and the low-energy 0^+ states have two phonons. Correspondingly, the conservation of the number of particles implies the use of two pairing phonons in order to obtain an excited estate of a non-superconducting nucleus. Because the previous symmetries are lost, the collective oscillations correspond to only one phonon, both for β-vibrations and for pairing-vibrations in superconducting nuclei.

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