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DEAD-TIME CORRECTIONS IN DETECTION EXPERIMENTS ON NUCLEAR STATISTICS

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Dead-time corrections to be applied in nuclear detection experiments are obtained. A probabilistic analysis of the problem is made, starting from the different possible states of a detection system. The correlations present in the count loss are avoided comparing probabilities of different states if the time intervals are suitably choiced.

The formalism is then applied to the two most important statistics in nuclear detection: Poisson Statistics and Negative Binomial Statistics. Application to some experiments as the α-Feynman of neutron correlations is straightforward.

The accuracy of the expressions was verified experimentally in Poisson distribution experiencces and proved to fit it very well trough the chi-square test. This also proved the present one to be better than Pacilio's formulation. Finally the parameter p of correction was determined experimentally for a big number of conditions and the results showed the goodness of the theoretical hypotheses.

1. Introduction

The most interesting statistics in the field of detection of nuclear processes are the Poisson Statistics (PS), and the Negative Binomial Statistics (NBS). As it is well known, the first describes the distribution of the unit-time number of decays of excited nuclei. The second describes with good accuracy[1]) the fluctuations of the neutron population in a steady-state-operating multiplicative assembly as is the case of a reactor.

We are interested in the dead-time corrections to apply in detection experiences of nuclear particles, carried out in "pulse mode". Starting from the different possible states of a detection system, it will be found the π_k probability of losing one count because of dead time, if the detection system was active (opened to detection) during a time interval τ in which arrived k true counts, and consequently, $k-1$ observed counts.

The knowledge of π_k will lead us to the desired correction expression for both types of experiences: nuclear emission of excited nuclei (PS) and neutron detection in a multiplicative assembly (NBS). In this last case the expressions can be directly applied, in particular, in the α-Feynman experiences[2]) in which the time correlation is measured from the relative variance of the distribution. (This correlation arises from the multiplicative properties of the fissionable material.)

2. Principal features of the π probabilities

We formulate the problem in the way made by Pacilio[3]). A detection system is active during fixed time intervals τ and we perform a number N of records. If N_k^c is the number of records with observed k content, we have:

$$N_k^c = N_k + \sum_{i>k} N_i \pi(i\to k) - \sum_{i<k} N_k \pi(k\to i), \qquad (1)$$

where N_k is the true number of records that would occur if the dead time DT were zero. $\pi(j\to n)$ is the transition probability of a record from a true content equal to j, to an observed content equal to n. Of course, always it is $n<j$.

The total number of records gives the condition:

$$\sum_{k=0}^{\infty} N_k = \sum_{k=0}^{\infty} N_k^c = N. \qquad (2)$$

The actual problem is to find the expressions for $\pi(j\to n)$. For all usual cases we have:

$$p_0 \equiv \mathrm{CR}\cdot\mathrm{DT} \ll 1, \qquad (3)$$

where CR is the true counting rate; and as $\pi(k\to k-n) \propto (p_0)^n$ for n not too big, it will be

$$\pi(k\to k-1) \gg \pi(k\to k-2), \qquad (4)$$

according to eq. (3), except for k too big ($k \gtrsim 100$). This enables us to take only first order transitions in eq. (1):

$$N_k^c = N_k + N_{k+1}\cdot\pi_{k+1} - N_k\cdot\pi_k, \qquad (5)$$

where $\pi_j = \pi(j\to j-1)$.

Pacilio took $\pi_k = k\Delta$, where Δ = relative counting loss, must be a function only of CR and DT. However, we observed experimentally a strong dependence of Δ on

$$\bar{n} = N^{-1} \sum_{k=0}^{\infty} kN_k,$$

the mean value of k, or in other words on the time interval τ of detection. A second objection is the following:

$\pi_1 \equiv \pi(1\to 0)$, is exactly the probability of losing the only one true count arrived during τ. Then it should coincide with the value obtained by physical considerations. Such a value happens to be practically zero, because the only case of loss holds when that count, arrives at the moment the gate is opened and in addition, another count had arrived inside a time interval DT before. This case has a probability of the order of Δ^2 since it requires the time-coincidence of three events during a time DT; so if we take π_k only in first order of Δ we must put $\pi_1 = 0$.

We have found an expression for π_k different from that adopted by Pacilio, studying when the correlation of events can be neglected in the observed counts. Through the chi-square test[4]), experiences 1 and 2 at the end, it is observed that the values so corrected fit better the Poisson distribution than those corrected by Pacilio's expression. From these experiences it is also seen that the changes of sign in the error $\delta N_k^c = N_k^c - N_k^{\text{Poisson}}$ of the corrected experimental data from the theoretical values found from Poisson distribution, are distributed with more randomness in the present formulation.

Finally our expressions are in agreement with those obtained by Babala[5]).

3. Expression of the π_k probability

We divide the time interval τ of detection in M equal intervals $t_0 > \text{DT}$ such that they make possible to neglect the interaction between the events occurred in different intervals, but sufficiently short, so that if in an interval t_0 arrive two true counts, its time separation be less than DT in almost all the cases. It is possible to prove that these limitations hold very well for $t_0 \simeq 2\text{DT}$. For this t_0, the probability of having two observed counts inside t_0 is very little.

In this case it is sufficient to take only 3 possible states of the detection system during an interval t_0:

E_1: no count arrived to the system;
E_2: one count arrived (and of course, it is observed);
E_3: two counts arrived, but only one is counted.

If we call $p = t_0 \cdot \text{CR}$, we have seen that $p \ll 1$, eq. (3), but this is the probability of having one count (true count) during t_0, with very good accuracy. In the same order of approximation p^2 will be the probability of having two true counts in t_0. Now, the probabilities of the different states are:

$$\begin{aligned} p_1 &\equiv p(E_1) = 1 - p - p^2, \\ p_2 &\equiv p(E_2) = p, \\ p_3 &\equiv p(E_3) = p^2. \end{aligned} \tag{6}$$

If we call $P(\alpha_1,\alpha_2,\alpha_3)$ the probability of having a number $\alpha_1,\alpha_2,\alpha_3$ of states E_1,E_2,E_3 respectively in the set of M intervals of the detection time τ, the independence of events in distinct intervals allows us to put:

$$P(\alpha_1,\alpha_2,\alpha_3) = \{M!/(\alpha_1!\alpha_2!\alpha_3!)\}p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdot p_3^{\alpha_3}, \tag{7}$$

where $\alpha_1 + \alpha_2 + \alpha_3 = M = \tau/t_0$. The π_k was defined as:

$$\pi_k = P(M-k+1;\, k-2;\, 1)/B(M-k;\, k), \tag{8}$$

where $B(M-k;k)$ is the binomial distribution of $M-k$ and k; since according to our hypotheses have arrived k true counts in a pure binomial form during τ. Its probabilities must be $1-p$ and p respectively, since p is the true detection probability during a time t_0. So we have:

$$\pi_k = \{[M!/(M-k+1)!(k-2)!\,1!]\cdot(1-p-p^2)^{M-k+1}\cdot \\ \cdot p^{k-2}\cdot p^2\}\cdot\{[M!/(M-k)!\,k!]\cdot(1-p)^{M-k}\cdot p^k\}^{-1}. \tag{9}$$

Since in all the cases is $t_0 \ll \tau$ we will have $M \gg 1$ and $M \gg k$ except for k too big, which are of very little importance for the first moments ($\bar{k}$ and $\bar{k}^2$) of the PS and NBS, since their probabilities in both distributions are very little. Besides, we have seen that $1 \gg p \gg p^2$ so it is:

$$\pi_k = k(k-1)/M, \tag{10}$$

or remembering that

$$1/M = t_0/\tau = t_0\cdot\text{CR}/\bar{n} = p/\bar{n},$$

where

$$\bar{n} = N^{-1}\sum_{k=0}^{\infty} kN_k,$$

is the mean value of the distribution, we have

$$\pi_k = k(k-1)p/\bar{n}, \tag{11}$$

which verifies that $\pi_1 = 0$.

A better expression for π_k taking into account the following term is:

$$\pi_k = k(k-1)(p/\bar{n})[1-\{(\bar{n}-1)(k-1)/\bar{n}\}p], \tag{12}$$

but the calculations with it are too complicated and the final results are not better than those given by eq. (11).

4. Dead-time corrections for the Poisson distribution

Remembering that the probability p_k of having k events has the property of:

$$p_{k+1} = \{\bar{n}/(k+1)\}p_k \tag{13}$$

and the first moments of the PS are related by the following equations:

$$\overline{k^2} = (\bar{k})^2 + \bar{k}, \tag{14}$$

$$\overline{k^3} = (\bar{k})^3 + 3(\bar{k})^2 + \bar{k}, \tag{15}$$

$$\overline{k^4} = (\bar{k})^4 + 6(\bar{k})^3 + 7(\bar{k})^2 + \bar{k}. \tag{16}$$

Eqs. (5), introducing (11) and (13) becomes:

$$N_k^c = N_k[1 + p(1+\bar{n}^{-1})k - (p/\bar{n})k^2], \tag{17}$$

where for the sake of clarity we have used $\bar{n}$ instead of $\bar{k}$. If we multiply eq. (17) by k, make a sum on k from 0 to ∞ and then divide by N according to eq. (2), we have:

$$\bar{n}^c = \bar{n} + p(1+\bar{n}^{-1})\overline{n^2} - (p/\bar{n})\overline{n^3} \tag{18}$$

and introducing eqs. (14) and (15):

$$\bar{n}^c = \bar{n}(1-p), \tag{19}$$

which allows to know the true mean value of the distribution $\bar{n}$, if previously we have determined the value of p corresponding to the present counting rate CR. Now if we multiply eq. (17) by k^2, and another time sum on k divided by N, we obtain:

$$\overline{n^{2c}} = \overline{n^2} + p(1+\bar{n}^{-1})\overline{n^3} - (p/\bar{n})\overline{n^4} \tag{20}$$

and introducing eqs. (14), (15) and (16):

$$\overline{n^{2c}} = (\bar{n})^2(1-2p) + \bar{n}(1-3p). \tag{21}$$

For any distribution the absolute and relative variance $\sigma^2(\bar{n})$ and V are:

$$\sigma^2(\bar{n}) = \overline{n^2} - (\bar{n})^2; \qquad V = \{\overline{n^2} - (\bar{n})^2\}/\bar{n}. \tag{22}$$

For the PS is $V = 1$ according to eq. (14), but the relative experimental variance will be:

$$V^c = \{\overline{n^{2c}} - (\bar{n}^c)^2\}/\bar{n}^c < 1, \tag{23}$$

as can be seen from eqs. (19) and (21) because $p > 0$ i.e. DT$\neq 0$. Physically the dead time introduces a negative correlation in the detection process.

Now putting $\overline{n^{2c}} - (\bar{n}^c)^2$, from eqs. (21) and (19),

TABLE 1
Poisson distribution.

Present formulation

k	N_k^e	N_k^c	N_k^P	δN_k^c	δN_k^2	χ_k^2
0	6733	6733	6700	− 33	1089	0.16
1	21418	20780	20837	+ 57	3249	0.16
2	33925	32567	32402	−165	27225	0.84
3	34696	33591	33590	− 1	1	0.00
4	26248	26133	26116	− 17	289	0.01
5	15435	16144	16244	+100	10000	0.62
6	7424	8359	8420	+ 61	3721	0.44
7	2998	3746	3741	− 5	25	0.01
8	977	1410	1454	+ 44	1936	1.33
9	276	488	503	+ 15	225	0.45
$\geqq$ 10	90	215	212	− 3	9	0.04

$p = 0.03072 \pm 0.00006$ $\quad L = 11-2 = 9$ $\quad \boxed{P = 0.90}$ $\quad \chi^2 = 4.06$

$\bar{n} = 3.11 \pm 0.01$

Pacilio's formulation

k	N_k^e	N_k^c	N_k^P	δN_k^c	δN_k^2	χ_k^2
0	6733	4644	4722	+ 78	6084	1.29
1	21418	16228	16338	+110	12100	0.74
2	33925	28513	28265	−248	61504	2.18
3	34696	32738	32599	−139	19321	0.59
4	26248	28230	28198	− 32	1024	0.03
5	15435	19299	19513	+214	45796	2.35
6	7424	11084	11253	+169	28561	2.54
7	2998	5554	5562	+ 8	64	0.01
8	977	2384	2406	+ 22	484	0.20
$\geqq$ 9	366	1308	1385	+ 77	5929	4.84

$\Delta = 0.130 \pm 0.003$ $\quad L = 10-2 = 8$ $\quad \boxed{P = 0.09}$ $\quad \chi^2 = 14.21$

$\bar{n} = 3.46 \pm 0.08$

TABLE 2
Poisson distribution.

Present formulation.

k	N_k^e	N_k^c	N_k^P	δN_k^c	δN_k^2	χ_k^2
0	6775	6775	6766	− 9	81	0.01
1	21534	20887	20975	+ 88	7744	0.37
2	33894	32528	32511	− 17	289	0.01
3	34867	33753	33595	−158	24964	0.74
4	26111	26007	26036	+ 29	841	0.03
5	15527	16259	16142	−117	13689	0.85
6	7396	8348	8340	− 8	64	0.01
7	2824	3543	3694	+151	22801	6.17
8	961	1397	1431	+ 34	1156	0.81
≧ 9	389	696	703	+ 7	49	0.07

$p = 0.03095 \pm 0.00006$ $\bar{n} = 3.10 \pm 0.01$ $\boxed{P = 0.33}$ $\chi^2 = 9.07$

$L = 10-2 = 8$

Pacilio's formulation

k	N_k^e	N_k^c	N_k^P	δN_k^c	δN_k^2	χ_k^2
0	6775	4676	4749	+ 73	5329	1.12
1	21534	16327	16397	+ 70	4900	0.30
2	33894	28509	28309	−200	40000	1.41
3	34867	32928	32583	−345	119025	3.65
4	26111	28110	28128	+ 18	324	0.01
5	15527	19436	19425	− 11	121	0.01
6	7396	11057	11179	+122	14884	1.33
7	2824	5240	5514	+274	75076	13.62
8	961	2350	2380	+ 30	900	0.38
≧ 9	389	1395	1365	− 30	900	0.66

$A = 0.130 \pm 0.003$ $\bar{n} = 3.45 \pm 0.08$ $\boxed{P < 0.02}$ $\chi^2 = 22.49$

$L = 10-2 = 8$

equal to $V^e\bar{n}^e$ from eq. (23) we obtain an equation for p in function of two experimental (known) values: $\bar{n}^e$ and V^e,

$$F(p) \equiv A_0 + A_1 p + A_2 p^2 = 0, \qquad (24)$$

where the coefficients of eq. (24) are:

$$\begin{aligned} A_0 &= -(1-V^e), \\ A_1 &= 2(2-V^e), \\ A_2 &= V^e + \bar{n}^e - 3, \end{aligned} \qquad (25)$$

so the expression for determining p experimentally is:

$$p = [V^e - 2 \pm \{1 + \bar{n}^e(1-V^e)\}^{\frac{1}{2}}]/(V^e + \bar{n}^e - 3) \quad (26)$$

and we choose the + sign since if DT goes to zero, V^e must go to 1 and $p \rightarrow 0$.

The expression (26) can be reduced taking the first terms of a Taylor expansion in the square root since the factor $1 - V^e$ is always very little:

$$p \cong \tfrac{1}{2}(1-V^e). \qquad (27)$$

According to our formalism the factor p must be determined experimentally from a Poisson distribution through eqs. (26) or (27) for several CR, and adjusting the obtained points of $p = f(\mathrm{CR})$ to a straight line, since we started putting $p = t_0 \cdot \mathrm{CR}$ where t_0 is constant. The resulting graph will be used with any distribution (PS, NBS, etc.) since p is not a function of it, at least in a first approximation. Finally it must be said that we know only CR^e which according to eq. (19) is:

$$\mathrm{CR}^e = \mathrm{CR}(1-p), \qquad (28)$$

so, we can take in a first guess $\mathrm{CR} = \mathrm{CR}^e$ for obtaining p in the graph of p and CR, and with this first value p_0 of p we correct $\mathrm{CR} = \mathrm{CR}^e/(1-p_0)$. With this new CR we will obtain, in a second iteration, a value p_1 for p which is allways sufficient.

5. Dead-time corrections for the Negative Binomial distribution (NBS)

The probability of having k events p_k has the

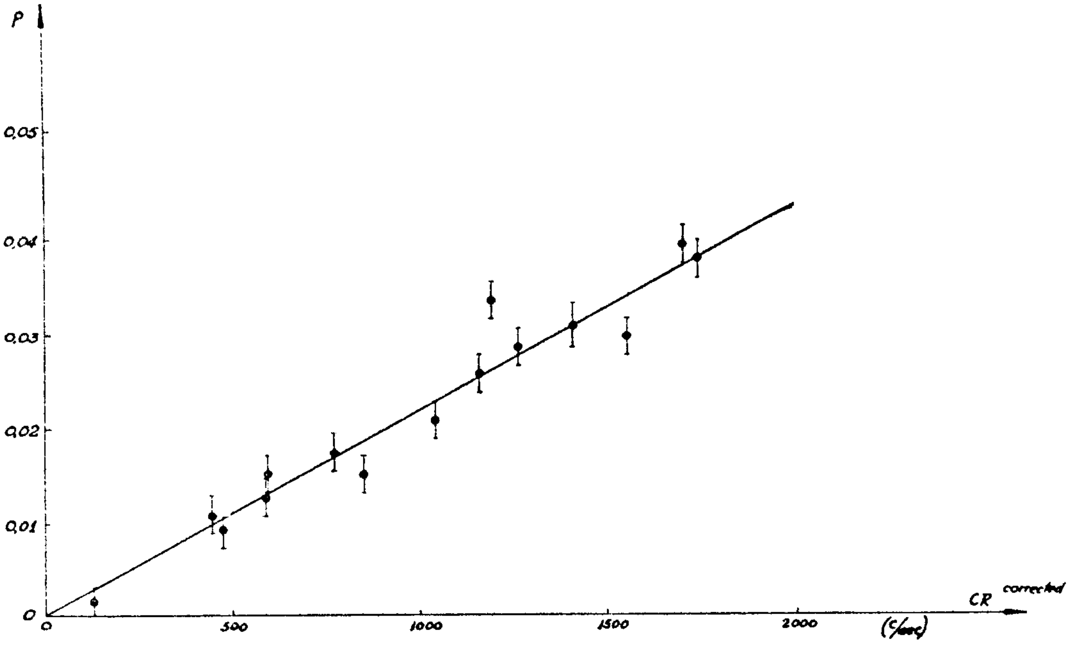

Fig. 1. The p-graph. The experimental points of p vs the corrected CR, adjusted to a straight line according to the first assumption of the theory. The errors shown are only the statistical errors of eq. (40), the other main errors come from instabilities of the electronic apparatus.

following property:

$$p_{k+1} = (1+\psi)^{-1}\{(\bar{k}+k\psi)/(k+1)\}p_k, \qquad (29)$$

and the first moments of the NBS are:

$$\overline{k^2} = (\bar{k})^2+(1+\psi)\cdot\bar{k}, \qquad (30)$$

$$\overline{k^3} = (\bar{k})^3+(1+\psi)\cdot[3(\bar{k})^2+(1+2\psi)\cdot\bar{k}], \qquad (31)$$

$$\overline{k^4} = (\bar{k})^4+(1+\psi)\cdot[6(\bar{k})^3+(7+11\psi)\cdot(\bar{k})^2+ +(1+6\psi+6\psi^2)\cdot\bar{k}]. \qquad (32)$$

The relative variance of the NBS according to eq. (22) can be obtained from the relation (30):

$$V = 1+\psi, \qquad (33)$$

which shows that ψ is a parameter of the NBS that measures the positive correlation of the distribution, since in the PS, where the correlation between events is strictly zero, is $V = 1$. In a multiplicative assembly the fission chains establish a positive correlation in the neutron detection, and in particular the α-Feynman experiences consist in measure $\psi = \psi(\tau)$ and to relate it through a model to the reactivity ρ or to the prompt-neutron decay constant α of the reactor.

Now eq. (5), using eqs. (11) and (29) becomes:

$$N_k^c = N_k[1+pk\{\bar{n}^{-1}+(1+\psi)^{-1}\}-pk^2\{\bar{n}(1+\psi)\}^{-1}], \qquad (34)$$

and in a similar way to that followed in the PS case, we obtain through eqs. (30), (31) and (32):

$$\bar{n}^c = \bar{n}(1-p)-p\psi, \qquad (35)$$

$$\overline{n^2}^c = (\bar{n})^2(1-2p)+\bar{n}(1-3p+\psi-6p\psi)-3p\psi-4p\psi^2. \qquad (36)$$

Finally the true correlation ψ can be obtained from eqs. (35) and (36) eliminating $\bar{n}$ (not yet known), as a function of known values: $\bar{n}^c, \overline{n^{2c}}$ and the corresponding p determined from its graph through the observed counting rate CR^c. The resulting equation for ψ is:

$$F(\psi) = B_2\psi^2+B_1\psi+B_0 = 0, \qquad (37)$$

with

$$\begin{aligned} B_0 &= (\bar{n}^c)^2(1-2p)-\overline{n^2}^c(1-p)^2+\bar{n}^c(1-4p+3p^2), \\ B_1 &= \bar{n}^c(1-5p+2p^2)-2p(1-p), \qquad (38) \\ B_2 &= -p(3-2p). \end{aligned}$$

6. Statistical error of the p parameter

Since p is determined through eqs. (26) and (27) of a Poisson distribution, its error can be found from the absolute variance σ^2 of the relative variance V of the PS:

$$\sigma^2(V) = (2/N)\{1+(2\bar{n})^{-1}\}. \qquad (39)$$

In all usual cases it is possible to use the simplified

TABLE 3
Experimental values of p vs CR^e, corrected CR, Pacilio's Δ and statistical errors.

CR^e (c/sec)	$\bar{n}^e$	V^e	p	CR^c = CR$^e(1-p)^{-1}$ (c/sec)	Δ
130	1.2970	0.9969	0.0016 ± 0.0015	130	0.0460
442	3.5352	0.9779	0.0107 ± 0.0018	447	0.0735
473	3.7869	0.9810	0.0092 ± 0.0017	477	0.0661
584	2.9179	0.9741	0.0126 ± 0.0019	591	0.0859
584	2.9181	0.9684	0.0152 ± 0.0019	593	0.0940
762	3.8095	0.9634	0.0174 ± 0.0018	775	0.0890
840	6.7760	0.9678	0.0151 ± 0.0015	853	0.0645
1024	4.0975	0.9557	0.0208 ± 0.0018	1046	0.0942
1133	2.2674	0.9456	0.0257 ± 0.0017	1162	0.1347
1156	2.3128	0.9277	0.0336 ± 0.0018	1195	0.1504
1230	4.9190	0.9371	0.0285 ± 0.0021	1265	0.1022
1367	2.7332	0.9337	0.0308 ± 0.0019	1410	0.1349
1510	5.9814	0.9337	0.0295 ± 0.0018	1555	0.0952
1637	3.2730	0.9125	0.0394 ± 0.0020	1702	0.1402
1677	3.3546	0.9161	0.0379 ± 0.0015	1741	0.1364

(The statistical errors of Δ are also rather constant, they are between 0.0035 and 0.0045.)

eq. (27) for obtaining the p error:

$$\delta p = \tfrac{1}{2}\delta V^c \cong \tfrac{1}{2}\sigma(V). \qquad (40)$$

7. Experimental tests; Comparison of Pacilio's and present correction

Two experiences as examples of the goodness of the obtained corrections are given in tables 1 and 2. N_k^e, N_k^c and N_k^p are the experimental, corrected (by Pacilio's Δ or by the present p) and the predicted from the Poisson distribution numbers of k-content records. $\delta N_k^c = N_k^p - N_k^c$ is the absolute error of the predicted N_k of each formulation.

Next, table 3 gives a set of experimental values of p for different CR^e. It is seen in fig. 1 a general agreement between the observed results and the first assumption of the theory: $p = \text{const.CR}$. When a graph of the same kind was done for Pacilio's Δ a strong dependence of Δ on $\bar{n}$ was observed, and it was impossible to adjust the experimental values to a straight line as it is required by Δ's definition.

The number N of records of each point of fig. 1 lies between 100000 and 400000, and a number still greater would have been desirable for reducing the statistical error of eq. (40). However, the always present instabilities of the electronic apparatus impose a final limit of accuracy.

8. Experimental equipment

The block diagram shown in fig. 2 is self explanatory. Only the equipment A requires a short description: a scaler receives the detection counts and it is opened and closed by the marker pulses at fixed intervals τ. The accumulation of counts in the scaler during the time τ, is converted through a digital-to-analogue converter to pulses of amplitude proportional to the scaler accumulation. These pulses are then analysed in a multichannel analyser which gives the statistical distri-

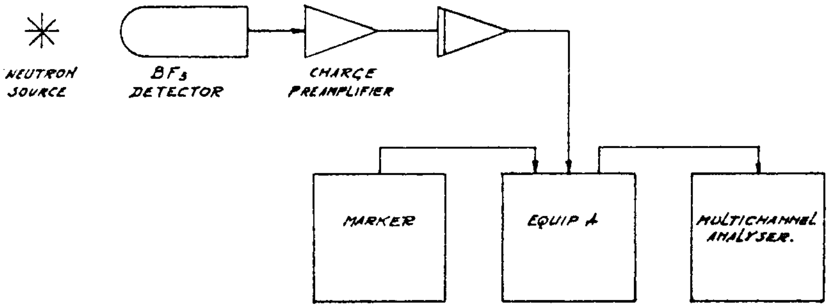

Fig. 2. Block diagram of the electronic apparatus.

bution of counts, putting "number of channel" proportional to "number of counts during τ".

The only source of dead time is, in this case, the preamplifier: a charge amplifier.

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