A New Technique for Analog Integration and Differentiation*

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Summary—**A technique is described which enables an approach to ideal analog integration or differentiation by means of passive elements only. A series of RC circuits in a cascade arrangement, uncoupled to each other, provides the first, second, third, etc., integrals or derivatives (according to the connection of the RC circuits) of the input function. The theory establishes that if the outputs of each are fed to an analog summing amplifier, its output becomes arbitrarily close to the ideal integral or derivative of the input function as the number of RC stages is raised indefinitely. A device has been developed according to this idea to perform one of these operations (integration) and to check the results obtained in theory. Mathematical proofs of the theory are given in the paper.**

Introduction

ANALOG differentiation or integration is based upon basic RC circuits connected in each case according to the corresponding use. They can give allowable results only for relatively short computing intervals of time. In order to extend their performance, active elements have been introduced in many ways (*e.g.*, amplifiers). This investigation presents the possibility of approaching ideal integration or differentiation without voltage amplifiers. Only RC circuits and ideal devices with special characteristics as defined below are used. The purpose of these elements is to decouple the RC circuits connected in cascade. In the practical example, impedance decoupling elements such as cathode followers have been used as practical approaches to the elements required in the theory.

General Description

Fig. 1 is a schematic diagram of an ideal circuit, developed according to the mathematical pattern of the theory. The blocks named S have the following characteristics: input impedance $=\infty$, output impedance $=0$, voltage transfer $=1$. A series of RC circuits with S elements placed in between are connected in a cascade arrangement. In this particular case, the RC connection corresponds to analog integration.

The signals obtained from the capacitor nodes are fed to the summing device Σ. Its output $e_0(t)$ is an approximate integral function of the input signal $e_i(t)$. The output voltage $e_0(t)$ differs from the ideal integral function of $e_i(t)$ for two reasons: 1) the summing device is not a perfect one, though it can be improved without limit; 2) the number of RC-S stages is not infinite;

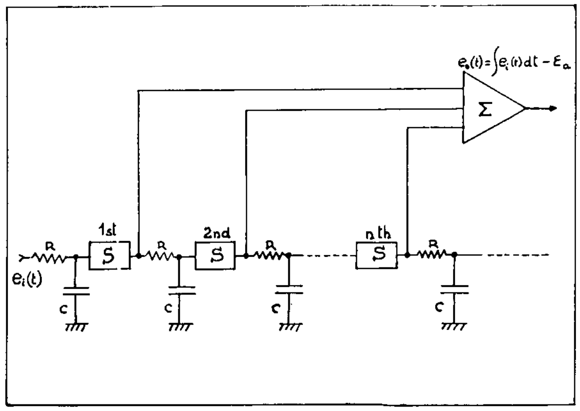

Fig. 1—Schematic diagram of a cascade-integrating circuit with RC stages and decoupling elements s connected in between. The outputs of the RC circuits are fed to an analog summing device Σ.

nevertheless, it can be increased without limit, at least in theory.

Before giving a rigorous demonstration, we can visualize the phenomena through an intuitive description according to the following line of thought. The first RC-S stage gives a voltage that approximates $\int e_i(t)dt$ to a degree dependent on t/τ, where t is the computation-time interval, and τ is the time constant of the RC circuit. This approximation obtains with an absolute negative error for any value of t. If in any instant we add to that voltage the necessary amount to make that error vanish, we obtain just the true integral function. That is exactly what happens in the summing circuit when the number of stages (RC-S) is raised indefinitely.

In other words, the absolute negative error of voltage obtained for an analog integral function in a cascaded RC-S circuit of indefinitely large extent is equal to the sum of the voltages obtained from all the following RC stages. The same considerations can be applied to a derivative-computing device developed according to the same pattern.

Mathematical Proofs

The transfer function in the Laplace domain corresponding to an RC-S stage is

$$\frac{1}{1+s\tau}.$$

The signals obtained in each of the node capacitators corresponding to an input of Laplace transform $e_i(s)$ are:

* Received by the PGEC, March 7, 1960; revised manuscript received, May 31, 1960.
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first stage $\dfrac{e_i(s)}{1+s\tau}$,

second stage $\dfrac{e_i(s)}{(1+s\tau)^2}$,

nth stage $\dfrac{e_i(s)}{(1+s\tau)^n}$.

The sum of this voltage transformed for n stages is

$$e_0(s) = e_i(s)\left[\frac{1}{1+s\tau} + \frac{1}{(1+s\tau)^2} + \cdots + \frac{1}{(1+s\tau)^n}\right].$$

Evaluation of the sum yields

$$e_0(s) = e_i(s)\left[\frac{1}{1+s\tau}\cdot\frac{\dfrac{1}{(1+s\tau)^n}-1}{\dfrac{1}{1+s\tau}-1}\right].$$

It is convenient to write it as follows:

$$e_0(s) = \frac{e_i(s)}{s\tau} - \frac{e_i(s)}{s\tau(1+s\tau)^n}.$$

$e_0(s)$ is then formed by two terms: 1) the exact integral function of the input; 2) the negative error inherent to the method. It follows immediately that the error term becomes arbitrarily close to zero as n is raised arbitrarily, so that $e_0(s)$ becomes the perfect integral of the input.

It is possible to differentiate functions following the same general pattern used for integration, substituting RC differentiators for the RC integrators, in the diagram of Fig. 1. The output voltage is then:

$$e_0(s) = e_i(s)s\tau - e_i(s)s\tau\left(\frac{s\tau}{1+s\tau}\right)^n.$$

As in the integration case, $e_0(s)$ is formed by two terms: 1) an ideal derivative of the input function; 2) a negative error inherent to the method. When n is raised arbitrarily, the last term becomes arbitrarily close to zero. It follows that in the limit, $e_0(s)$ becomes the derivative of the input function.

In the laboratory the method was checked with a three-stage integrator and an input-step function $e_i(t) = E$. The expected errors were previously calculated for 1, 2, 3, and 4 integrating stages. The absolute error in voltage ϵ_a was calculated from the inverse Laplace transform of

$$-\frac{e_i(s)}{s\tau(1+s\tau)^n},$$

with $e_i(s)$ replaced by $1/s$ (thus setting $E=1$). Letting $t/\tau=k$, the results are:

$$n = 1:\ \epsilon_a = -e^{-k} + 1 - k,$$

$$n = 2:\ \epsilon_a = -e^{-k}(2+k) + 2 - k,$$

$$n = 3:\ \epsilon_a = -e^{-k}\left(3 + 2k + \frac{k^2}{2!}\right) + 3 - k,$$

$$n = 4:\ \epsilon_a = -e^{-k}\left(4 + 3k + 2\frac{k^2}{2!} + \frac{k^3}{3!}\right) + 4 - k.$$

From the equations we can induce a general formula for the absolute error ϵ_a that will result from integrating a unit step of voltage in a cascade of n RC-S circuits,

$$\epsilon_a = -e^{-k}\left[n + (n-1)k + (n-2)\frac{k^2}{2!} + \cdots + \frac{k^{n-1}}{(n-1)!}\right] + n - k.$$

Giving values to k we obtain Table I, in which the percentage error has been calculated from $\epsilon_a/k \times 100$, because $k=t/\tau$ is the exact integral of a unit-step input.

TABLE I

	n	$K=t/\tau$							
		1	2	3	4	5	6	7	8
Absolute Errors	1	−0.37	−1.14	−2.05	−3	−4	−5	−6	−7
	2	−0.10	−0.54	−1.25	−2.12	−3.03	−4	−5	−6
	3	−0.04	−0.22	−0.68	−1.35	−2.17	−3.08	−4	−5
	4	−0.02	−0.07	−0.33	−0.85	−1.43	−2.23	−3.12	−4.04
Percentage Errors	1	37	57	68	75	80	83	86	87
	2	10	27	42	53	61	67	71	75
	3	4	11	23	34	43	51	57	62
	4	2	3.5	16	21	29	37	44	50

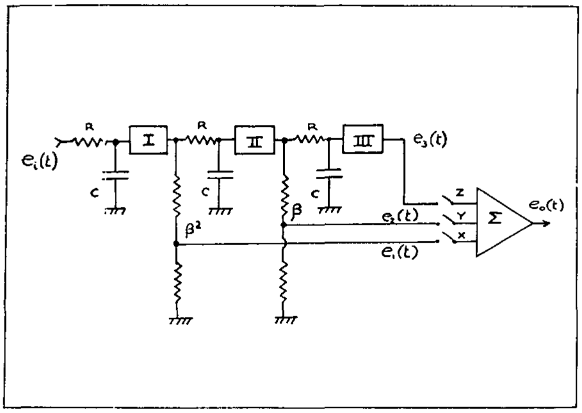

Fig. 2—Three RC integrators circuits with three cathode followers I, II, and III, and the attenuators β^2 and β used to compensate for the corresponding slight voltage attenuations of II·III and III respectively.

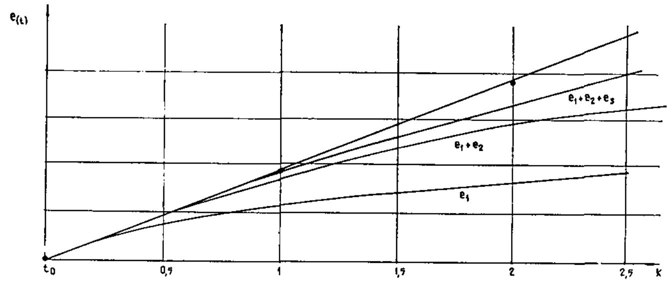

Fig. 3—Output voltages e_1, e_1+e_2, and $e_1+e_2+e_3$ corresponding to the circuit of Fig. 2. The straight line corresponds to the ideal integration that should be obtained with $N=\infty$.

Circuit

The experimental circuit shown in Fig. 2 includes three RC integrators and uses three cathode followers as buffer elements. A cathode follower was used, since it is a circuit that approximates the requirements of the theory.

The signals fed to the summing device from Stages I and II have been attenuated by β^2 and β to compensate the corresponding attenuations of II·III and III respectively. By means of switches x, y, and z, we can select and record the output signals e_1, e_1+e_2, and $e_1+e_2+e_3$, corresponding to $n=1$, 2, 3. The ideal integral of the input-step function was superimposed on the same graph of Fig. 3 by drawing a straight line of slope E/τ through the origin. A check of the error calculations was made by plotting in Fig. 3 the points corresponding to $k=1$ and $k=2$ for the exponential $e_1+e_2+e_3(n=3)$, using the experimental data plus the magnitude of the error term as given in Table I.

The use of electron tube cathode followers is not considered to be an essential part of the method. Any suitable decoupling device may be used.

Acknowledgment

The author wishes to thank the Comisión Nacional de la Energía Atómica, Buenos Aires, Argentina, for their support and for permission to publish this paper. Thanks are also due to Gernot Oehley for the wiring of the circuit and the measurements.

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Reprinted from IRE TRANSACTIONS
ON *ELECTRONIC COMPUTERS*
Volume EC-9, Number 4, December, 1960

PRINTED IN THE U.S.A.