CAB/1970/3

FINITE ENERGY SUM RULES WITH $\pi\pi \to \pi\pi$ VENEZIANO AMPLITUDES

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Summary.- Veneziano amplitudes for $\pi\pi$ scattering with defined isospin in the crossed channel are tested with regards to finite energy sum rules. It is shown that FESR for the imaginary part of the amplitude with isospin 1 are satisfied up to first daughter but not when second daughter is included, and those for isospin 2 converge to the fixed-poles residues for determined regions of t . FESR for real parts are numerically satisfied for $t = 0$ but discrepancies appear for non-forward scattering. The separation of Veneziano amplitudes into Regge poles and Background is discussed.

Bariloche, May 1970

* Supported in part by Istituto Nazionale di Fisica Nucleare, Sezione di Napoli.

1.- Introduction.

The concept of duality is not very well defined since there are many ways of presenting it. Therefore one has to be careful about the sense in which the Veneziano model (1) is considered to be dual.

One can take the view of "mathematical duality" (2), i.e. that an amplitude can be written alternatively as a sum of pole contributions in the direct channel or in the crossed one, showing in both channels Regge asymptotic behaviour when the corresponding Mandelstam variable is taken out of a finite angle around the real axis. In this sense, because of the properties of beta-functions, the Veneziano model is certainly dual.

However, one can see that a Veneziano amplitude can be split into two terms, one of which contains the s-channel resonances and asymptotically tends to zero out of real s-axis, the second one containing the t-channel resonences and giving rise to Regge behaviour for $s \to \infty$ (3). This splitting looks like an interference model but not in the traditional sense because of the strong correlation between the two parts. Moreover, this separation suggests a division of the amplitude in a series of Regge poles and a background integral contribution (4)(5)(6).

The purpose of this work is to verify the duality of Veneziano amplitude in the original sense of Dolen-Horn-Schmid (7), i.e. to investigate whether this model satisfies finite energy sum rules (FESR). Since, from what is said above, Veneziano model contains a background integral term, it is by no means authomatic that this amplitude should satisfy FESR (8). In fact one expects in general that on the RHS of FESR, apart from the contributions of Regge trajectories with their series of daughters, a term due to background integral could appear even at high values of the energy cut-off.

The original work of Veneziano (1) shows that this model satisfies FESR with integer moments for imaginary part of the amplitude up to the leading Regge pole contribution.

We want now to verify if FESR are satisfied also when daughters, obtained from a J-plane analysis, are considered, studying sum rules both for imaginary and real parts of a scattering amplitude. We use for our analysis the $\pi\pi \rightarrow \pi\pi$ amplitudes which contain in a simple way all the effects we want to study. The general features of FESR for the imaginary part of these amplitudes have been given by Yellin (9). Since we will use a Veneziano model with real trajectory we cannot obtain strict Regge behaviour for $s \rightarrow \infty$ on real axis. But on the other hand we are allowed to work with a well defined mathematical tool without the complications of introducing complex trajectories.

In Section 2 we present for completeness the anlysis of Veneziano amplitudes in Khuri plane, which is for our purposes somewhat simpler than the ordinary J-plane analysis. Then we compute in Section 3 the FESR for amplitudes with defined isospin in the crossed t-channel. We obtain that for isospin $I = 1$ the sum rules for imaginary part are satisfied for the leading order and they are also satisfied for first non-leading contribution only if the energy cut-off is placed half-way between two neighbouring resonances. On the other hand they cannot be satisfied when the second daughter is taken into account. The numerical evaluation of sum rules for the real part shows that for $t = 0$ there is good agreement but for non-forward amplitude severe discrepancies may appear.

Regarding the amplitude with $I = 2$, which has no Regge poles, FESR for its imaginary part tend to zero or to the residues of the fixed poles for right or wrong signature sum rules respectively, but with restrictions on the values of t. For the real part the amount of disagreement of sum rules oscilates and increases in absolute value with increasing $|t|$.

Finally we will discuss in Section 4 the separation of Veneziano amplitude in a sum of Regge poles and a background integral, showing that this separation is not straightforward.

2.- Analysis of Veneziano amplitudes in Khuri plane.

We use the well known Lovelace [10] amplitudes for $\pi\pi$ elastic scattering with defined isospin values in t channel

$$A_2^t = V(s, u) \tag{1a}$$

$$A_1^t = V(s, t) - V(u, t) \tag{1b}$$

$$A_0^t = \frac{3}{2}\left(V(s, t) + V(u, t)\right) - \frac{1}{2} V(s, u) \tag{1c}$$

where

$$V(x,y) = \frac{\Gamma(1-\alpha_\rho(x))\,\Gamma(1-\alpha_\rho(y))}{\Gamma(1-\alpha_\rho(x)-\alpha_\rho(y))} \quad , \quad \alpha_\rho(x) = \alpha_0 + x = \alpha_x . \tag{2}$$

We will be concerned only with A_1^t and A_2^t since the amplitude with $I = 0$ should contain the Pomeranchon contribution which is absent in the present model. Moreover, the mathematical features of A_0^t are a combination of those of the other two amplitudes.

The J-plane analysis in t-channel of this model, performed by several authors (11)(12)(13)(14)(15), shows that the blocks $V(s, t)$ and $V(u, t)$ contain an infinite set of Regge poles at $J = \alpha_t$, $\alpha_t - 1$, ..., without fixed poles, whereas $V(s, u)$ has no Regge poles but wrong-signature fixed poles at $J = -1,-3,....$

For purposes of consistency of our notation we reproduce briefly the analysis in Khuri plane m according to the same procedure used in Ref.(13). We prefer to use Khuri representation because it is somewhat closer related to our analysis of next sections.

The Khuri representation of an amplitude is given by

$$(3) \quad A(\nu,t) = \frac{1}{2}\sum_{m=0}^{\infty}\left\{a_m^+(t)\left[\nu^m + (-\nu)^m\right] + a_m^-(t)\left[\nu^m - (-\nu)^m\right]\right\} ,$$

with $\nu = (s-u)/4\mu$ and where μ is the pion mass. Therefore the analytic continuation is defined by

$$(4) \qquad a_m^{\pm}(t) = \frac{1}{\pi}\int_0^{\infty}\left[\operatorname{Im} A(\nu,t) \mp \operatorname{Im} A(-\nu,t)\right]\frac{1}{\nu^{m+1}}\, d\nu .$$

In our case, since

$$(5) \qquad V(s,t) = -\frac{1}{2\mu}\sum_{n=0}^{\infty}\frac{1}{B(\alpha_t, n+1)}\;\frac{1}{\nu_n - \nu} ,$$

where B is the beta-function and

(6) $$\nu_n = \left(n + 1 + \frac{\alpha_t}{2} - \frac{c}{2}\right)/2\mu \quad , \quad c = 3\alpha_0 + 4\mu^2 ,$$

we obtain

(7) $$\mathrm{Im}\, A_1^t = -\frac{\pi}{2\mu} \sum_{n=0}^{\infty} \frac{1}{B(\alpha_t, n+1)} \left[\delta(\nu-\nu_n) + \delta(\nu+\nu_n)\right],$$

so that (4) gives

(8a) $$a^+_{1_m} = 0 \quad ,$$

(8b) $$a^-_{1_m} = -\frac{1}{\mu} \sum_{n=0}^{\infty} \frac{1}{B(\alpha_t, n+1)} \frac{1}{\nu_n^{m+1}} = \sum_{n=0}^{\infty} a_{m_n} .$$

Extending $a^-_{1_m}$ to all the m-plane we can immediately see that there are no fixed poles, e.g. for $m = -1$

(9) $$a^-_{1_{-1}} = -\frac{\alpha_t}{\mu} \left.\frac{1}{(1-y)^{\alpha_t+1}}\right|_{y=1} ,$$

which vanishes for $\alpha_t < -1$ and, through analytic continuation, for all values of α_t . On the other hand, the use of Stirling asymptotic expansion for $B(\alpha_t, n+1)$ allows us to write

(10) $$a^-_{1_m}(t) = \sum_{\tau=0}^{N-1} 2\,\beta_\tau(t,m) \quad \zeta(m+\tau+1-\alpha_t) + \\ + \sum_{n=0}^{\infty}\left[a_{m_n} - \sum_{\tau=0}^{N-1} \frac{2\beta_\tau(t,m)}{(n+1)^{m+\tau+1-\alpha_t}}\right] ,$$

where ζ is the Riemann function and $2\beta_\tau$ are the coefficients of the expansion of $a_{m n}$ in inverse powers of $(n + 1)$. It is clear that the second term of RHS is regular for $\mathrm{Re}\,\alpha_t < m + N$. Therefore, the singularities of a^-_{1m} come from $\zeta(1)$, i.e. there will be poles for $m = \alpha_t - \tau$ ($\tau = 0, 1, 2 \ldots.$) with residues $\beta_\tau (t, \alpha_t - \tau)$, which correspond to parent and daughters Khuri poles. The expressions for the residues of parent, first and second daughters turn out to be (from here on we drop their explicit dependence)

$$\beta_0 = - \frac{(2\mu)^{\alpha_t}}{\Gamma(\alpha_t)} \qquad \text{(11a)}$$

$$\beta_1 = - \frac{(2\mu)^{\alpha_t - 1}}{\Gamma(\alpha_t)} \frac{\alpha_t}{2} (c - 1) , \qquad \text{(11b)}$$

$$\beta_2 = - \frac{(2\mu)^{\alpha_t - 2}}{\Gamma(\alpha_t)} \frac{\alpha_t}{4} (\alpha_t - 1) \left(\frac{c^2}{2} - c - \frac{\alpha_t}{6} + \frac{1}{3} \right) , \qquad \text{(11c)}$$

respectively.

These Khuri poles contribute to the amplitude A^t_1 when the integral along path γ of Fig. 1a

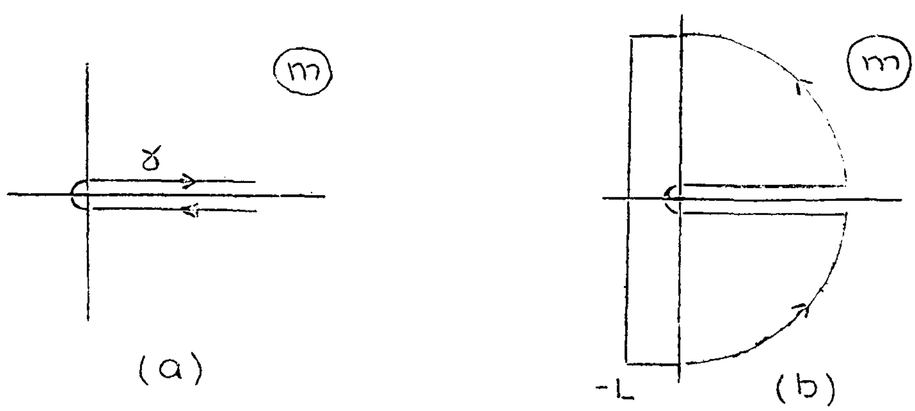

Fig. 1

$$(12)\qquad A_1^{t\,*} = -\frac{1}{2i}\int_\gamma dm\,(-\nu)^m\,\frac{1}{2}\left(a_{1m}^-\,\xi_m^- + a_{1m}^+\,\xi_m^+\right),$$

where $\xi_m^{\pm} = \left\{1 \pm \exp\left[-i\pi m\right]\right\}/\sin\pi m$ are the signature factors, is opened according to the path of Fig. 1b as is usually done in the Sommerfeld-Watson transform, giving the expression

$$(13)\qquad A_1^{t\,*} = -\pi\sum_{\tau=0}^{\infty}\beta_\tau\,(-\nu)^{\alpha_t}\,\frac{1-(-)^\tau e^{-i\pi\alpha_t}}{\sin\pi\alpha_t} + \text{Background}.$$

Regarding the amplitude with $I = 2$ in t-channel, since

$$(14)\qquad V(s,\,u) = -\frac{1}{2\mu}\sum_{n=0}^{\infty}\frac{(-)^n}{B(\alpha_t+1-c,\,n+1)}\left(\frac{1}{\nu-\nu_n}-\frac{1}{\nu+\nu_n}\right),$$

$$(15)\qquad \operatorname{Im} A_2^t = \frac{\pi}{2\mu}\sum_{n=0}^{\infty}\frac{(-)^n}{B(\alpha_t+1-c,\,n+1)}\left[\delta(\nu-\nu_n)-\delta(\nu+\nu_n)\right],$$

we obtain from (4)

$$(16a)\qquad a_{2_m}^- = 0,$$

$$(16b)\qquad a_{2_m}^+ = \frac{1}{\mu}\sum_{n=0}^{\infty}\frac{(-)^n}{B(\alpha_t+1-c,\,n+1)}\,\frac{1}{\nu_n^{m+1}}.$$

Now, the asymptotic behaviour of the terms of the sum (16b) relates a_{2m}^+ to $(1-2^{1-\sigma})\,\zeta(\sigma)$, where $\sigma = m - \alpha_t + c$, and this expression has no divergences; therefore A_2^t contains no Khuri poles. On the other hand the extension of a_{2m}^+ to negative integer values of m leads to the expression

$$(17)\qquad a_{2m}^+ = (\alpha_t+1-c)(4\mu)^m\,2^{-(\alpha_t-c)}\,\frac{d^{-m-1}}{dz^{-m-1}}\cosh z^{-(\alpha_t+2-c)}\Big|_{z=0}.$$

We see that (17) is different from zero only for odd negative integer values and therefore gives the residues of wrong-signature fixed poles of the amplitude $A_2^{t\,(+)}$.

3.- Finite Energy Sum Rules.

3.1. Amplitude A_1^t .

We begin considering FESR for $\operatorname{Im} A_1^t$. Since in this case we have no fixed poles both right and wrong-signature FESR are on an equal footing; it will appear in fact that they are qualitatively equivalent.

Let us describe the first right-signature sum rule in some detail:

$$(18) \qquad I_1^{(o)} = \int_0^{\nu_c} \operatorname{Im} A_1^t \, d\nu = -\frac{\pi}{2\mu} \sum_{n=0}^{N} \frac{1}{B(\alpha_t, n+1)} ,$$

where N is the last resonance included in the energy range defined by ν_c . The expression (18) can be summed (see also Ref. (9)) to give

$$(19) \qquad I_1^{(o)} = -\frac{\pi}{2\mu} \frac{1}{(\alpha_t+1)\,\Gamma(\alpha_t)} \frac{\Gamma(\alpha_t+N+2)}{\Gamma(N+1)} .$$

For large N the use of Stirling expansion up to second non-leading term allows us to write

$$I_1^{(o)} = I_{1,0}^{(o)} + I_{1,1}^{(0)} + I_{1,2}^{(0)} + \dots ,$$

with the explicit form

(20 a) $$I^{(0)}_{1,0} = -\frac{\pi}{2\mu}\,\frac{1}{\Gamma(\alpha_t)}\,\frac{N^{\alpha_t+1}}{\alpha_t+1},$$

(20 b) $$I^{(0)}_{1,1} = -\frac{\pi}{4\mu}\,\frac{\alpha_t+2}{\Gamma(\alpha_t)}\,N^{\alpha_t},$$

(20 c) $$I^{(0)}_{1,2} = -\frac{\pi}{48\mu}\,\frac{\alpha_t}{\Gamma(\alpha_t)}\left(3\alpha_t^2+11\alpha_t+10\right)N^{\alpha_t-1}.$$

We may now compare expansion (20) with the contribution to (18) of Khuri poles which, from (13), is

$$\pi\beta_0\,\frac{\nu_c^{\alpha_t+1}}{\alpha_t+1}+\pi\beta_1\,\frac{\nu_c^{\alpha_t}}{\alpha_t}+\pi\beta_2\,\frac{\nu_c^{\alpha_t-1}}{\alpha_t-1}.$$

For this purpose we take the cut-off as

(21) $$\nu_c = \left(N+\varepsilon+1+\frac{\alpha_t}{2}-\frac{c}{2}\right)/2\mu,$$

where $0 < \varepsilon < 1$, to place ν_c between the N^{th} and $(N+1)^{th}$ resonance. Expanding ν_c in powers of N and introducing the residues (11) we obtain the contributions of Khuri poles to $I^{(0)}_1$ (notice that e.g. not only the first daughter but also the parent trajectory contributes to the order N^{α_t}). What comes out is:

i) the leading term coincides with (20 a) for any value of ε;

ii) to get agreement with first non-leading term (20 b) one has to choose $\varepsilon = 1/2$;

iii) for the second non-leading contribution one cannot reproduce (20 c) for any choice of ε; in fact, with $\varepsilon = 1/2$ one obtains

$$-\frac{\pi}{48\mu}\,\frac{\alpha_t}{\Gamma(\alpha_t)}\left(3\alpha_t^2+11\alpha_t+11\right)N^{\alpha_t-1}$$

showing disagreement in the last term.

In a similar way the first wrong-signature sum rule is

$$I_1^{(1)} = \int_0^{\nu_c} \nu \, \mathrm{Im}\, A_1^t \, d\nu = -\frac{\pi}{2\mu} \sum_{n=0}^{N} \nu_n \frac{1}{B(\alpha_t, n+1)} = \tag{22}$$

$$= -\frac{\pi}{(2\mu)^2} \frac{\Gamma(N+\alpha_t+2)}{\Gamma(\alpha_t)\,\Gamma(N)} \left(\frac{1}{\alpha_t+2} + \frac{\frac{\alpha_t}{2}+1-\frac{c}{2}}{\alpha_t+1} \frac{1}{N} \right) .$$

Like $I_1^{(0)}$, the asymptotic expansion of (22) shows agreement with Khuri poles contribution up to the first non-leading term with the choice $\varepsilon = 1/2$. But a discrepancy appears again in the second non-leading term since, while from (22) one obtains

$$I_{1,2}^{(1)} = \tag{23}$$

$$= -\frac{\pi}{(2\mu)^2} \frac{1}{(\alpha_t+2)\,\Gamma(\alpha_t)} \left(\frac{\alpha_t^4}{8} + \frac{5}{6}\alpha_t^3 - \frac{c}{4}\alpha_t^2 - c\,\alpha_t - c + \frac{19}{8}\alpha_t^2 + \frac{41}{12}\alpha_t + 2 \right),$$

from Khuri poles contribution the last three terms in parenthesis are different, i.e. $\frac{29}{12}\alpha_t^2 + \frac{65}{24}\alpha_t + \frac{5}{12}$.

Analogous discrepancy in the second non-leading term appears in the next FESR

$$I_1^{(2)} = \int_0^{\nu_c} \nu^2 \, \mathrm{Im}\, A_1^t \, d\nu = \tag{24}$$

$$= -\frac{\pi}{(2\mu)^3} \frac{\Gamma(N+\alpha_t+2)}{\Gamma(\alpha_t)\,\Gamma(N)} \left(\frac{N}{\alpha_t+3} + \frac{\alpha_t+3-c}{\alpha_t+2} - \frac{1}{\alpha_t+3} + \frac{\frac{\alpha_t}{2}+1-\frac{c}{2}}{\alpha_t+1} \frac{1}{N} \right).$$

We turn now our attention to FESR for $\mathrm{Re}\, A_1^t$ which, from (1) and (5), can be written as

$$\mathrm{Re}\, A_1^t = \frac{1}{2\mu} \sum_{n=0}^{\infty} \frac{1}{B(\alpha_t, n+1)} \, P\left(\frac{1}{\nu-\nu_n} + \frac{1}{\nu+\nu_n} \right) . \tag{25}$$

The first wrong signature sum rule is

$$(26)\qquad R_1^{(0)} = \int_0^{\nu_c} \mathrm{Re}\, A_1^t \, d\nu = \frac{1}{2\mu} \sum_{n=0}^{\infty} \frac{1}{B(\alpha_t, n+1)} \ln \left| \frac{\nu_c^2}{\nu_n^2} - 1 \right| ,$$

which should be confronted to the Khuri poles contribution

$$(27)\qquad \pi \beta_0 \, \mathrm{tg} \frac{\pi}{2}\alpha_t \, \frac{\nu_c^{\alpha_t+1}}{\alpha_t+1} - \pi \beta_1 \, \mathrm{ctg} \frac{\pi}{2}\alpha_t \, \frac{\nu_c^{\alpha_t}}{\alpha_t} + \pi \beta_2 \, \mathrm{tg} \frac{\pi}{2}\alpha_t \, \frac{\nu_c^{\alpha_t-1}}{\alpha_t-1} .$$

It is difficult to compare (26) and (27) analytically, but it is anyhow apparent that both expressions cannot be consistent for all values of t ; e.g. for $\alpha_t = -1$ expression (26) is finite while the parent contribution to (27) diverges. We have evaluated both (26) and (27) numerically assuming $\alpha_0 = .5$ and taking the cut-off $s_c = 3\ \mathrm{GeV}^2$ half-way between 3rd and 4th resonances. The computation of (26) and (27), reported in the first two columns of Table I under $R_1^{(0)}$ and R_K respectively, was done for two values of t, i.e. t=0 and t=-1.4 . We obtain a severe disagreement for t=-1.4 , as was expected, the two first daughters being negligible in comparison with the parent contribution to R_K .

Thus we find that FESR for Re A_1^t are satisfied worse than those for Im A_1^t particularly for non-forward scattering.

3.2. <u>Amplitude A_2^t</u> .

We analyze first FESR for ImA_2^t . Using (15) , the first wrong-signature sum rule is

$$(28)\qquad I_2^{(0)} = \int_0^{\nu_c} \mathrm{Im}\, A_2^t \, d\nu = \frac{\pi}{2\mu} \sum_{n=0}^{N} \frac{(-)^n}{B(t, n+1)} ,$$

where we have taken for simplicity $\alpha_0 = .5$ and $\mu^2 = 0$. It can be seen that (28) may be summed to give

$$(29) \qquad I_2^{(0)} = \frac{\pi}{2\mu} \; \frac{(-)^N}{B(t, N+1)} \; F(-N, 1, -N-t; -1) \; ,$$

with F hypergeometric function.

On the other hand, using

$$(30) \qquad \sum_{n=0}^{\infty} \frac{\Gamma(z+n)}{\Gamma(n+1)} \, y^n = \Gamma(z) \, (1-y)^{-z} \; ,$$

valid for $y=-1$ if $z<1$ (16), (28) gives for $N \to \infty$ the residue of the fixed pole at -1

$$(31) \qquad I_2^{(0)} \to \frac{\pi}{2\mu} \, t \; 2^{-(t+1)} = \frac{\pi}{2} \, a^{+}_{2_{-1}} \; , \quad t < 0$$

as it is seen comparing with (17).

For the first right-signature sum rule we have

$$(32) \qquad \begin{aligned} I_2^{(1)} &= \int_0^{\nu_c} \nu \, \mathrm{Im}\, A_2^t \, d\nu = \\ &= \frac{\pi}{8\mu^2} \sum_{n=0}^{N} (-)^n \frac{(t+2n+1)}{B(t, n+1)} = \frac{\pi}{8\mu^2} (-)^N \frac{t+N+1}{B(t, N+1)} \; , \end{aligned}$$

and using again (30) one obtains for $N \to \infty$

$$(33) \qquad I_2^{(1)} \to 0 \quad , \qquad t < -1 \quad ,$$

so that for $t < -1$ $I_2^{(1)}$ tends to zero, in agreement with the absence of fixed pole at -2, but for $t > -1$ it shows an oscillating behaviour which increases in absolute value with N.

In the same way, the second moment sum rule is

(34) $$I_2^{(2)} = \frac{\pi}{32\mu^3} \sum_{n=0}^{N} (-)^n \frac{(t+2n+1)^2}{B(t,n+1)} ,$$

which for $N \to \infty$ gives

(35) $$I_2^{(2)} \to -\frac{\pi}{32\mu^3} t(t+1)\, 2^{-t} = \frac{\pi}{2} a^{t}_{2_{-3}} , \quad t < -2 ,$$

so that we obtain the fixed pole residue at -3 for $t < -2$ and oscillating values otherwise.

Summarizing, (31) (33) and (35) show that FESR are not related to residues of fixed poles for all the range of t .

Concluding this Section we consider the first right-signature sum rule for Re A_2^t

(36) $$R_2^{(0)} = \int_0^{\nu_c} \mathrm{Re}\, A_2^t \, d\nu = -\frac{1}{2\mu} \sum_{n=0}^{\infty} \frac{(-)^n}{B(t,n+1)} \ln \left| \frac{\nu_c - \nu_n}{\nu_c + \nu_n} \right|$$

with the same assumptions done in (28) . It is interesting to see that for $t=0$ $R_2^{(0)} = 0$ as it should because of the absence of Regge poles, but for $t < 0$ disagreement appears: e.g. for $t = -1$ $R_2^{(0)} \neq 0$ because of the contribution of the first resonance and for $t = -2$ because of the two first ones. We reproduce in Table II the values of $R_2^{(0)}$ for different values of t taking $s_c = 3\ \mathrm{GeV}^2$. It is seen that $R_2^{(0)}$ oscillates and increases in absolute value with $|t|$.

4.- Separation of Veneziano amplitude in Regge contribution and background integral.

Since in A_2^t there are no Regge poles, the whole expression (14) corresponds to background integral which is therefore responsible for the characteristics of FESR .

More complicated is the situation for A_1^t where both Regge poles and background contribute. It is easy to rewrite A_1^t as

$$(37)\qquad A_1^t = A_a + A_b \quad ,$$

with

$$(38\ a)\qquad A_a = \frac{\pi}{\Gamma(\alpha_t)}\left\{\frac{\Gamma(\alpha_s+\alpha_t)}{\Gamma(\alpha_s)}\ \frac{e^{-i\pi\alpha_t}}{\sin\pi\alpha_t} - \frac{\Gamma(\alpha_s+\alpha_t+1-c)}{\Gamma(\alpha_s+1-c)}\ \frac{1}{\sin\pi\alpha_t}\right\}$$

$$(38\ b)\qquad A_b = \frac{\pi}{\Gamma(\alpha_t)}\ \frac{\Gamma(\alpha_s+\alpha_t)}{\Gamma(\alpha_s)}\ \frac{e^{i\pi\alpha_s}}{\sin\pi\alpha_s} \ .$$

A_a contains the resonances in u and t coming from $V(u,t)$ and a part of $V(s,t)$, the rest, containing the resonances in s , being assigned to A_b ; this last part is usually neglected when considering $\mathrm{Im}\ \alpha_s \to \infty$ for $s \to \infty$. On the other hand one can verify, using Stirling expansion, that for $s \to \infty$ A_a reproduces the sum of Khuri poles contributions with residues (11) and the correct signature factors; it may be objected that (38 a) has not the correct symmetry under crossing, but this symmetry is recovered asymptotically for $s \to \infty$. Therefore, separation (37) corresponds asymptotically to Regge poles and background integral contributions. When we consider this separation at low energies the previous statement is no longer true since Stirling expansion cannot be applied.

To understand A_a and A_b better it is useful to perform Khuri analysis of both terms. We begin expressing them in terms of their poles contributions

$$(39\text{ a})\qquad A_a = \frac{1}{2\mu}\sum_{n=0}^{\infty}\frac{1}{B(\alpha_t,n+1)}\left(\frac{1}{\nu_n+\nu}-\frac{e^{-i\pi\alpha_t}}{\nu'_n+\nu}\right),$$

$$(39\text{ b})\qquad A_b = \frac{1}{2\mu}\sum_{n=0}^{\infty}\frac{1}{B(\alpha_t,n+1)}\left(\frac{e^{-i\pi\alpha_t}}{\nu'_n+\nu}-\frac{1}{\nu_n-\nu}\right),$$

where besides the poles (6) at ν_n corresponding to the resonances of s and u channels, spurious poles appear at

$$(40)\qquad \nu'_n = \frac{1}{2\mu}\left(n+\frac{\alpha_t}{2}+\frac{c}{2}\right),$$

due to $\alpha_s + \alpha_t = 0$ or negative integer, which of course do not contribute to A_1. Next we compute the discontinuities of A_a and A_b along the real axis of ν, which we notice do not coincide with the imaginary parts,

$$(41\text{ a})\qquad D\,A_a = \frac{\pi}{2\mu}\sum_{n=0}^{\infty}\frac{1}{B(\alpha_t,n+1)}\left[-\delta(\nu+\nu_n)+e^{-i\pi\alpha_t}\delta(\nu+\nu'_n)\right],$$

$$(41\text{ b})\qquad D\,A_b = \frac{\pi}{2\mu}\sum_{n=0}^{\infty}\frac{1}{B(\alpha_t,n+1)}\left[-\delta(\nu-\nu_n)-e^{-i\pi\alpha_t}\delta(\nu+\nu'_n)\right].$$

Now we may calculate the Khuri amplitude for A_a

$$(42)\qquad a^{\pm}_{a_m} = \frac{1}{\pi}\int_0^{\infty}\left[D\,A_a(\nu)\mp D A_a(-\nu)\right]\frac{1}{\nu^{m+1}}\,d\nu,$$

and a similar expression for A_b.

A procedure completely analogous to the one which lead to (11) allows us to obtain the residues of the Khuri poles, which apparently turn out to be of both signatures; e.g. the parent t-channel pole residues are

$$(43\ a)\qquad \beta^{\pm}_{a_0} = \pm \frac{(2\mu)^{\alpha_t}}{\Gamma(\alpha_t)}\left(1 - e^{-i\pi\alpha_t}\right) ,$$

$$(43\ b)\qquad \beta^{\pm}_{b_0} = - \frac{(2\mu)^{\alpha_t}}{\Gamma(\alpha_t)}\left(1 \mp e^{-i\pi\alpha_t}\right) ,$$

for A_a and A_b respectively. However, performing the Sommerfeld-Watson transform (12) (remind the conjugation of $a^{\pm}$ that one has to introduce because of crossing operation) separately for A_a and A_b it is easy to see that for the former only the odd-signatured pole with residue (11 a) appears whereas for the latter both contributions vanish. The same happens for the daughters, giving as result that A_b has no Regge content, being therefore purely background, whereas A_a contains all the Regge poles expected in the whole A^t_1. But A_a can be identified with the Regge contribution only asymptotically, through the use of Stirling expansion; at low energy there is a discrepancy which corresponds to a background content of A_a.

At this respect one may remark that in Ref.(4) separation (37) is considered equivalent to the one between Regge poles and background on the basis of the identification of A_a with the Borel sum (17) of Khuri poles. This corresponds to consider, instead of the series of Khuri poles:

$$(44)\qquad \Sigma = \sum_{n=0}^{\infty} c_n \, s^{-n} ,$$

where we have separated the factor s^{α}, its Laplace anti-transform

$$(45) \qquad f(x) = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!} ,$$

which may have a wider range of convergencethan Σ . The Borel sum of Σ is defined as

$$(46) \qquad \Sigma_B = s \int_0^{\infty} e^{-sx} f(x)\, dx ,$$

which is shown in Ref (4) to correspond to A_a . But this is no more than a definition since at low energies (44), which is essentially Stirling expansion, certainly does not coincide with A_a .

A further evidence of the background contribution to A_a is given by the numerical evaluation of the FESR for its real part

$$(47) \qquad R_e = \int_0^{\nu_c} \mathrm{Re}\, A_a \, d\nu = \frac{1}{2\mu} \sum_{n=0}^{\infty} \frac{1}{B(\alpha_t, n+1)} \left[\ln\left(1+\frac{\nu_c}{\nu_n}\right) - \cos\pi\alpha_t \cdot \ln\left(1+\frac{\nu_c}{\nu_n'}\right)\right]$$

with $t = -1.4$ (see Table I) which shows almost the same result as with the total A_1^t , alternatively proving that A_b is practically superconvergent with this sum rule.

It is clear that the discrepancy shown by R_K is mostly due to the divergent behaviour of the series of powers of ν near the origin. In fact, if we consider the expansion of the parent and first two daughters in powers of s , the resulting contribution to FESR (26) , reported as R_s in Table I shows a smaller discrepancy since the integral is now less singular as its lower limit is $s_0 \geqslant 2\mu^2$ instead of 0 . As an indication, if we require that the arguments of all Γ-functions of (38 a) are positive for all the range of integration of FESR , which is certainly a necessary though not sufficient condition for applying Stirling formula, it must be $|t| < 2 - 2\alpha_0 - 4\mu^2$. Next, for the validity of the expansion in powers of ν instead of α_s it must be $\nu > (c - \alpha_t)/2\mu$, which compared to (6) shows that for $t = 0$ only the first resonance is slightly below this limit whereas for increasing $|t|$ more resonances leave the "dual" region.

At first sight separation (37) seems to correspond to an interference model with one term which contains the Regge poles and no resonances in s-channel and the other just the opposite (see also Ref.(3)) . However it is clear from (39) that this is not the same as the traditional interference model; the two pieces are not uncorrelated, the link being given by the spurious poles ν'_n which appear in both parts.

We must remind that in Ref.(3) it is shown that the separation between the Regge and the s-resonance parts is somehow arbitrary. We may complete this proof showing that the non unique Regge part of that work gives asymptotically not only the parent behaviour but the correct daughter contribution too. In fact, including e.g. the first non leading term in the expansion of hypergeometric function

$$F(a,b,c;z) \underset{b\to\infty}{\longrightarrow} \frac{e^{\pm i\pi a}}{(bz)^a} \frac{\Gamma(c)}{\Gamma(c-a)} \left[1 - \frac{a(1+a-c)}{bz} + \frac{a(a+1)}{2b}\right] + \tag{48}$$
$$+ e^{bz}(bz)^{a-c} \frac{\Gamma(c)}{\Gamma(a)} \left[1 + \frac{(c-a)(1-a)}{bz} + \frac{(bz)^2}{2b}\right] ,$$

and taking $-\frac{\pi}{2} < \arg b\alpha_s < \frac{3}{2}\pi$ and $\mathrm{Re}\,(b\,\alpha_s) < 0$ (b is a parameter defined in Ref.(3)) , we obtain that the asymptotic behaviour for the contribution of the Regge part of Ref.(3) to $V(s,t)$ is

$$(49)\qquad (1-\alpha_s-\alpha_t)\ \frac{b^{1-\alpha_t}}{1-\alpha_t}\ F(1-\alpha_t,\alpha_s,2-\alpha_t\,;b)\ \xrightarrow[\alpha_s\to\infty]{}$$

$$\frac{\pi}{\Gamma(\alpha_t)}\ \frac{e^{-i\pi\alpha_t}}{\sin\pi\alpha_t}\ \alpha_s^{\alpha_t}\left[1+\frac{\alpha_t}{2}\,(\alpha_t-1)\,\frac{1}{\alpha_s}\right],$$

which does not depend on the parameter b and coincides with the Regge behaviour of parent and first daughter. This ambiguity in the separation is due to the fact that the correspondence with an asymptotic series (the Regge poles contributions) does not define uniquely the function (the previously called "Regge part" of $V(s,t)$). In other words in separation (37) it is possible to add to A_a a function rapidly decreasing with s modifying only its background integral content (see also Ref.(5)) . The separation of Ref.(3) is valid for $\mathrm{Re}\ s \to +\infty$. As shown in Ref.(2), such decomposition, as any other, is not possible if one wants to isolate a term with Regge behaviour for all s-plane. However from the physical pont of view it is interesting to analyze the asymptotic behaviour on positive real s-axis .

It is interesting to note that s-channel resonances appear in the explicit evaluation of the background integral. In fact, taking A_2^t as example, and performing the integral along the imaginary axis $-L+i\,b$ of m-plane between limits $-B$ and $+B$, we obtain

$$(50)\qquad \text{Background} = -\frac{1}{4\mu}\sum_n \frac{(-)^n}{B(\alpha_t+1-c,\,n+1)}\ \frac{1}{\nu_n}\left(\frac{\nu}{\nu_n}\right)^{-L}\cdot$$

$$\cdot\int_{-B}^{B} db\ e^{ib\ln\nu/\nu_n}\left(\mathrm{cotg}\,\frac{\pi}{2}\,m+i\right)\quad ;$$

choosing an integer L and being $\sin Bx/x \xrightarrow[B\to\infty]{} \pi\,\delta(x)$, we find

$$\text{(51)} \qquad \text{Im Background} = -\frac{\pi}{2\mu}\sum_{n=0}^{\infty}\frac{(-)^n}{B(\alpha_t+1-c,\,n+1)}\,\delta(\nu-\nu_n)\,,$$

which corresponds to s-channel resonances contribution to the amplitude.

We wish to make a remark about the way fixed poles appear in FESR . It is clear that Im Background (51), and consequently its contribution to FESR (28), is independent of the value of L we choose, to the right or to the left of -1 ; this happens because the signature factor kills the pole at -1 in A^t_2 . However if one considers the background integral of FESR , which is different from the FESR of background, the pole at -1 is regenerated through the divergence at the lower limit. In fact, using (18)

$$\text{(52)} \qquad \int_{-\infty}^{\infty} db\, \frac{e^{ib \ln \nu_c/\nu_n}}{1-L+ib} = \begin{cases} \begin{Bmatrix} 0 & , \nu_n > \nu_c \\ 2\pi\,(\nu_n/\nu_c)^{1-L} & , \nu_n < \nu_c \end{Bmatrix}, \; L<1 \\[2ex] \begin{Bmatrix} -2\pi\,(\nu_n/\nu_c)^{1-L} & , \nu_n > \nu_c \\ 0 & , \nu_n < \nu_c \end{Bmatrix}, \; L>1 \end{cases},$$

and reminding the fixed pole residues (17), one obtains

$$\text{(53)} \quad \mathrm{Im}\int_{-L-i\infty}^{-L+i\infty} dm\, \frac{i}{4}\int_0^{\nu_c} d\nu\, a^+_{2m}\,\nu^m\left(\mathrm{cotg}\,\frac{\pi}{2}m+i\right) = \begin{Bmatrix} -\dfrac{\pi}{2\mu}\displaystyle\sum_{n=0}^{N}\frac{(-)^n}{B(\alpha_t+1-c,\,n+1)}\,, & L<1 \\[2ex] \dfrac{\pi}{2}a^+_{2_{-1}} - \dfrac{\pi}{2\mu}\displaystyle\sum_{n=0}^{N}\frac{(-)^n}{B(\alpha_t+1-c,\,n+1)}\,, & L>1 \end{Bmatrix},$$

showing that the difference between the background of FESR for the integration on the right and left of -1 is compensated by the fixed pole. Similarly one can evaluate the background of A^t_1 , but with some complications due to the presence of Regge poles. Since series (8 b) represents $\bar{a}_{1_{-1}}$ only for $\alpha_t < -1$, the background integral with $L = 1$ gives all s-channel resonances for $t < -1.5$. Taking $t > -1.5$, one or more Regge poles cross the integration line and their contributions must be subtracted from those of s-resonances in the computation of background.

5.- Conclusions.

Summarizing, our analysis of Veneziano model with real trajectory for $\pi\pi$ amplitudes with defined isospin in the crossed channel gives as main results:

i) With regard to the amplitude A_1^t , right and wrong-signature FESR offer the same characteristics.

ii) FESR for Im A_1^t with any integer moment show agreement for parent Khuri trajectory and also for first daughter if the cut-off is half-way between two resonances. Disagreement appears for second daughter.

iii) FESR for Re A_1^t show approximate numerical agreement for $t = 0$ but severe discrepancy may occur for non-forward direction due to the non validity of Stirling expansion at low energy. The use of the expansion of the amplitude in powers of s instead of ν reduces the disagreement.

iv) Regarding A_2^t it is in general not possible to choose a finite cut-off for which FESR are satisfied.

v) The zero moment FESR for Im A_2^t tends to the fixed pole residue at -1 for $t < 0$; first moment FESR tends to zero for $t < -1$ and oscillates increasingly around zero for $t > -1$; the second moment one tends to the fixed pole residue at -3 for $t < -2$ and oscillates otherwise, etc.

vi) Right-signature FESR for Re A_2^t are again satisfied for finite cut-off only for $t = 0$.

vii) Separation (37) corresponds asymptotically for $s \to \infty$ to the division in Regge poles (A_a) and background integral (A_b) ; s-channel resonances are contained only in A_b .

viii) This separation is different from a traditional interference model since both terms contain the same additional spurious poles which correlate them.

ix) A_a contains background contribution since its correspondence to Regge poles, based on Stirling expansion, is only asymptotical. Therefore A_a does not in general satisfy FESR .

x) The computation of the background integral in Khuri plane shows that it contains the s-channel poles.

xi) The background integral of zero moment FESR for Im A_2^t is found to be different from the FESR of background integral, the fixed pole at -1 appearing explicitly only in the former.

+*+*+

We thank heartily our friend Dr. A. García for many enlightening and stimulating discussions during all the phases of this work. Very useful discussions with Professors J. Fulco and D. Wong are also aknowledged.

One of us (A.D.S.) wishes to express his gratitude to Comisión Nacional de Energía Atómica for hospitality at Centro Atómico Bariloche where this work was performed, and to Consiglio Nazionale delle Ricerche for a financial grant.

FOOTNOTES AND REFERENCES

(+) In Ref.(15) it is said that A_2^t has no wrong-signature fixed poles in Khuri plane. This statement is misleading since fixed poles at negative integers appear in the Sommerfeld-Watson transform Eq. (12) because of the factor $1/\sin\pi m$ of the signature factor as long as the residue a_{2_m} does not vanish.

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Table Captions

Table I. FESR for Re A_1^t.

$R_1^{(o)}$: Resonances contribution (26);

R_K : Khuri poles contribution (27);

R_a : A_a contribution (47);

R_s : Contribution of Khuri poles considered as powers of s .

Table II. FESR for Re A_2^t .

$R_2^{(o)}$: Resonances contribution (36).

TABLES

Table I

t (GeV^2)	$R_1^{(o)}$	R_K	R_a	R_s
0	-17.4	-18.2		
-1.4	-14.8	-72.5	-15.0	-17.7

Table II

t (GeV^2)	0.	-.5	-1.0	-2.0
$R_2^{(o)}$	0.	-.02	.12	-2.65