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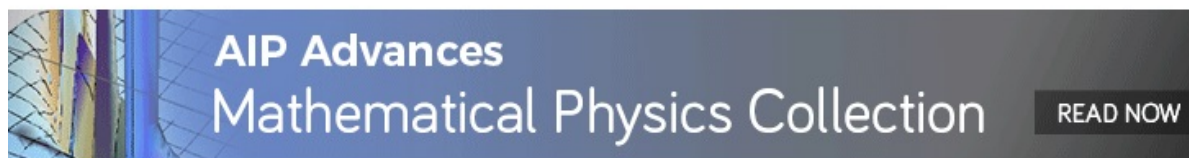
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ABSTRACT

The simplest model for the evolution of the mean-value of a surface gravity wave propagating in a random bottom has been connected with the telegrapher's equation. This analysis is based on the comparison of the mean-value solution of dispersive plane-wave modes propagating in a binary exponential-correlated disordered floor with the solution of the homogeneous telegrapher's equation. Analytical results for the exact dispersion-relation are presented. In addition, the time-dependent analysis of mean-value monochromatic waves is also shown.

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I. INTRODUCTION

Physical and biological oceanography are nowadays crucial areas of investigation in preventing degradation of the surface of the Earth because they are critical for climate systems.¹ From a physical point of view, surface gravity waves are wave motions occurring at the interface between the atmosphere and the mirror of water; the study of this complex system is a phenomenon of crucial importance for the future of Earth's ecology. Linear and non-linear approximations are of relevance in the analysis of surface gravity and capillary waves.²

Bottom topography structure is a key ingredient to be taken into account to describe the evolution of surface waves. Several approaches have been introduced to study wave motion in random media; for example, an amplitude equation was proposed to study non-linear effects and disorder,³ experimental acoustic sensing,⁴ integro-differential equations combining KdV and Burgers terms,⁵ and the analysis of surface waves at a wavy bottom.⁶

In general, wave propagation in disordered media has been an important subject of investigation, mainly for its diversity in physical and engineering applications, allowing prediction of dissipation and dispersion in the propagation of waves in random media (WRM). Most of the results presented in the analysis of WRM are based on the dispersion relation obtained from the first order

smoothing approximation.^{7,8} This dispersion relation has been used in the past for a number of applications in oceanography: waves in an irregular depth,⁹ water wave scattering,¹⁰ shear waves (on a Couette flow), internal and inertial waves, Rossby waves, and Kelvin and edge waves.⁷ These mean-value solutions are meaningful only when the perturbation is small, while it is clear that these cannot tell us how a coherent wave-motion is affected by general random inhomogeneities, which is the main goal of the theory of WRM. The failure of standard perturbation expansion is well known,⁸ so in view of these difficulties, exact solutions of *toy models* are always welcome.

In addition, other approaches have been used to tackle fundamental problems of WRM, and the most successful ones belong to the effective medium approximation.^{11–13} In the analysis of disorder in a continuous random media, we can mention sound propagation,¹⁴ diffusion in random media,¹⁵ and electromagnetic waves through disordered media.¹⁶

In a recent paper,¹⁷ it was noted that for space exponential-correlated symmetric binary disorder, the mean-value of a generic field in a random media can be found in an *exact* way. Notably, from these results, the mean-value of a surface wave propagating in a binary disordered bottom can also be studied.¹⁸

Two important results are presented in this article: first, the *exact* dispersion-relation for the simplest model of a 1D surface

gravity wave on a random bottom (a free surface in the absence of any forcing or rotational effect) and second, the connection of a mean-value gravity wave with the solution of the homogeneous telegrapher's equation (TE) as well as the characterization for the rate of absorption of energy. We have compared monochromatic surface waves in random media with plane-wave solutions of the TE. This hyperbolic TE has recently been used in many areas of research: neuroscience,^{19,20} electric transmission lines,²¹ asymptotic diffusion from Boltzmann scattering,²² electromagnetic analysis in multilayered conductor planes,²³ and the application of the TE in 2D and 3D for engineer problems.²⁴ In this manner, adopting the TE to represent the mean-value evolution of a surface wave contributes to the understanding of surface waves on a random bottom. In general, the TE is well understood while the challenging problem of surface gravity waves on a randomly irregular floor still remains to be solved in many aspects, so the present contribution is highlighted in that direction.

II. PROPAGATION OF OCEAN WAVES IN A RANDOM BOTTOM

It is well known that the propagation of small-amplitude surface gravity waves with characteristic wavelength comparable to the depth is governed by Laplace's equation together with a Neumann boundary condition at the bottom ($z = -H$). Thus, a linearized kinematic-dynamic boundary condition (at $z = 0$) leads to the evolution equation for surface gravity waves. If the bottom is irregular, $H \rightarrow H(x, y)$; this random boundary problem is much more complex to be worked out, and it is not covered by the usual (optic) theory of WRM. While long surface gravity waves over an irregular bottom are much simpler to be tackled, this evolution equation for the *free surface* ψ was proposed many years ago^{9,10} [see Eq. (1) for the 1D case]. That is to say, here, the free surface $\psi = \psi(x, t)$ is considered a one-dimensional wave propagation over a one-dimensional topography, and this simplifies the analysis but does not eliminate the effects that we want to illustrate. In the present section, we will present the *exact* mean-value solution for this evolution equation when the irregularity, at the bottom, is emulated by an exponential-correlated symmetric binary process: $\xi(x) = \pm\Delta$, $\forall x \in (-\infty, +\infty)$ (see Fig. 2 in the Appendix).

Following Refs. 7, 9, and 10, consider the 1D random surface gravity wave evolution equation (with appropriated initial conditions),

$$[\partial_t^2 - v^2 \partial_x^2] \psi(x, t) = [\partial_x \xi(x) \partial_x] \psi(x, t). \quad (1)$$

In order to study the dispersion-relation of this surface wave, it is convenient to use symmetric initial conditions,

$$\psi(x, t)|_{t=0} = \delta(x) \text{ and } \partial_t \psi(x, t)|_{t=0} = 0. \quad (2)$$

In this context, in Eq. (1), the operator $[\partial_t^2 - v^2 \partial_x^2]$ corresponds to the (ordered) Green's wave operator, and $\partial_x \xi(x) \partial_x$ corresponds to the random perturbation operator, which explicitly depends on the stationary random binary field $\xi(x)$. The total quantity $v^2 + \xi(x) \geq 0$, $\forall x \in (-\infty, +\infty)$ takes into account fluctuations from the random bottom (see the Appendix). We note that for a generic perturbation, the requirement of a space-correlated disorder bounded

from below is something that prevents the use of a Gaussian disorder in a perturbation theory, while a binary disorder^{13,15,17} or a Poisson disorder is suitable.²⁵

In particular, symmetric binary disorder with exponential-correlation allows exact calculations.²⁶ In the Laplace representation, the mean-value of the random field (1) can be solved, and its solution is¹⁸

$$\langle \psi(k, s) \rangle = \frac{s}{s^2 + (v^2 + \gamma_1(k, s))k^2}$$

with

$$\gamma_1(k, s) = \frac{\Delta^2}{v^2} \left(\frac{\frac{s}{|v|} \left(\frac{1}{\lambda} + \frac{s}{|v|} \right)}{\left(\left(\frac{1}{\lambda} + \frac{s}{|v|} \right)^2 + k^2 \right)} - 1 \right). \quad (3)$$

Here, for the stationary symmetric binary disorder $\xi(x)$, we have used the two-point correlation function,²⁸

$$\langle \xi(x_1) \xi(x_2) \rangle = \Delta^2 \exp(-|x_1 - x_2|/\lambda), \quad (4)$$

with a characteristic space correlation length λ . For $\Delta \leq v^2$, formula (3) gives the exact dispersive character of the mean-value of a wave packet propagating in a binary disordered medium.

From this result, we can study the dispersion-relation $s = s(k)$ for plane-wave-like modes as a function of the disorder's correlation length λ , intensity $0 \leq \Delta \leq v^2$, and wavelength $2\pi/k$. In what follows, we will be interested in strong disorder, while the weak disorder case can also be analyzed in a similar way. In the case of strong disorder, there are space realizations, where the quantity $v^2 + \xi(x)$ can be zero, in the random equation (1), for some particular space position x (zero point measure).

For strong disorder, $\Delta = v^2$, the dispersion-relation for the mean-value of a plane-wave-like mode follows putting $\gamma_1(k, s)|_{k=0}$ into $\langle \psi(k, s) \rangle$. That is to say, $v^2 + \gamma_1(k=0, s)$ represents the effective velocity of the wave-motion. Thus, the dispersion-relation can be studied by taking the inverse Laplace transform of the auxiliary function,

$$\overline{\psi(k, s)} = \frac{s}{s^2 + v_{\text{eff}}^2(0, s)},$$

where

$$v_{\text{eff}}^2(0, s) = \frac{s|v|}{(1/\lambda + s/|v|)}. \quad (5)$$

Due to the presence of disorder, the field $\overline{\psi(k, s)}$ can have a real part in the complex k -structure of poles $s_{\pm}(k)$, leading to damping in the wave evolution, that is,

$$\overline{\psi(k, s)} = \frac{|v|/\lambda + s}{(s - s_+)(s - s_-)}$$

with

$$s_{\pm} = \frac{|v|}{2\lambda} \left(-1 \pm \sqrt{1 - (2\lambda k)^2} \right). \quad (6)$$

After taking the inverse Laplace transform of $\overline{\psi(k, s)}$ and multiplying $\overline{\psi(k, t)}$ by e^{-ikx} , we get two dispersive plane-waves propagating in opposite directions if $2\lambda k > 1$, that is,

$$\overline{\psi(k, t)} e^{-ikx} = \frac{|v|}{\lambda} \left(\frac{e^{s_+ t} - e^{s_- t}}{s_+ - s_-} \right) e^{-ikx} + \left(\frac{s_+ e^{s_+ t} - s_- e^{s_- t}}{s_+ - s_-} \right) e^{-ikx}. \quad (7)$$

Damping evolution in this dispersive plane-wave is characterized by the rate $|v|/\lambda$. From (6), we see that *there is* oscillatory propagation only if $|2\lambda k| > 1$, that is, when the wavelength is shorter than twice the correlation length of the disorder. For times $t \gg 2\lambda/|v|$, the wave has completely been damped. In addition, from (7), and for $k \in [-\frac{1}{2\lambda}, \frac{1}{2\lambda}]$, we see that for strong disorder, there is a gap in the k -axis where the dispersive wave is non-propagating, and this fact is similar to Anderson's localization approach. While our result is similar to that of Anderson's localization, we noted that Eq. (1) is different from the evolution equation for the propagation of electromagnetic waves in a medium with a random dielectric constant and unit permeability.⁸

In the limit $\lambda \rightarrow \infty$, the local disorder $\xi(x)$ turns to be a constant; in this limit, the localized wave disappears, and we recover a free wave evolution $[\lim_{\lambda \rightarrow \infty} s_{\pm}(k) \rightarrow \pm i v k]$ showing two plane-waves moving in opposite directions, $e^{ik(x \pm vt)}$, due to the initial condition (2).

For a wave propagating in a weak disordered medium, $0 < \Delta < v^2$, there can be two little twin localized gaps (symmetric) around the origin in the k -axis (see Ref. 29). We note that in the strong disorder limit, $\Delta \rightarrow v^2$, these gaps collapse to only one for $|2k\lambda| < 1$.

In the weak disorder regime and for an intensity such that $0 < 1 - (\Delta/v^2)^2 < 1/9$, oscillatory damped waves can exist outside the twin localized gaps.

For very weak disorder $1/9 < 1 - (\Delta/v^2)^2 < 1$, the twin localized gaps disappear, approaching the ordered case without damping (limit $\Delta \rightarrow 0$). These results can be seen from the analysis of poles of the general solution (3) (see Ref. 29).

III. PLANE-WAVE MODES IN THE TELEGRAPHER'S EQUATION

The TE $[\partial_t^2 + \frac{1}{\tau} \partial_t - v^2 \partial_x^2] \phi(x, t) = 0$ has been used, in the past, for many purposes, for example, in ballistic-diffusion problems;^{31–33} here, we will present an application of the TE in the context of propagation of surface gravity waves in a random bottom.

In Laplace's representation, the TE looks like

$$\left[s^2 + \frac{s}{\tau} - v^2 \partial_x^2 \right] \phi(x, s) = s(s + 1/\tau) \delta(x),$$

with

$$\phi(x, t)|_{t=0} = \delta(x), \quad \partial_t \phi(x, t)|_{t=0} = 0, \quad (8)$$

these initial conditions represent a symmetric pulse propagating in both directions. In the Fourier–Laplace representation, the solution of the telegrapher's field is

$$\begin{aligned} \phi(k, s) &= \int_0^\infty dt e^{-st} \int_{-\infty}^{+\infty} dx e^{ikx} \phi(x, t) \\ &= \frac{s(s + 1/\tau)}{s(s + 1/\tau) + v^2 k^2} \\ &= \frac{s(s + 1/\tau)}{(s - s_-)(s - s_+)} \end{aligned} \quad (9)$$

with

$$s_{\pm} = \frac{1}{2\tau} \left(-1 \pm \sqrt{1 - (2v\tau k)^2} \right). \quad (10)$$

A dispersive plane-wave can be studied by taking the inverse Laplace transform of $\phi(k, s)$ and multiplying this time-dependent function by e^{-ikx} . From these considerations, we see that only for $2v\tau k > 1$, an oscillatory evolution can be obtained. That is to say, dispersive plane-wave modes from the TE have the following form:

$$\phi(k, t) e^{-ikx} = \frac{1}{\tau} \left(\frac{e^{s_+ t} - e^{s_- t}}{s_+ - s_-} \right) e^{-ikx} + \left(\frac{s_+ e^{s_+ t} - s_- e^{s_- t}}{s_+ - s_-} \right) e^{-ikx}. \quad (11)$$

A plane-wave mode can only exist for short wavelength, $k^{-1} < 2v\tau$, before the damping completely eliminates the wave. The dissipation of power per unit volume is characterized by the telegrapher's parameter τ^{-1} (see, for example, Ref. 30). As expected for $\tau \rightarrow \infty$, we recover a free monochromatic evolution.

In addition, for time scales $s^{-1} \gg \tau$, the solution of the TE goes to $\phi(k, s) \rightarrow \frac{s/\tau}{s/\tau + v^2 k^2}$; that is, it approaches the expected diffusion-like behavior at long times with the diffusion coefficient $D = \tau v^2$. In this last regime, the solution of the TE does not present any wave-motion.

IV. ATTENUATION OF GRAVITY WAVES ON A RANDOM BOTTOM

A. Surface waves on a random bottom and the solution of the telegrapher's equation

For short time scales $t < \tau$, and comparing the dispersive plane-wave of the TE (11) against the mean-value of a dispersive plane-wave in a random bottom (7), we see that both results have the same solution if we identify

$$\tau \rightarrow \lambda/|v|. \quad (12)$$

That is, the rate of energy absorption, proportional to τ^{-1} in the TE, is equal to the rate of energy dissipated by a surface wave propagating at velocity v in a strong disordered random bottom with a correlation length λ .

On the other hand, as it is well known, for long time scales $t \gg \tau$, the solution of the TE $\phi(x, t)$ approaches a diffusion packet, indicating the demolition of the surface gravity wave. This behavior can be seen by taking the inverse Fourier–Laplace of (9), in the limit $\tau s \ll 1$.

We end by commenting that for very weak disorder with intensity such that $1/9 < 1 - (\Delta/v^2)^2 < 1$, the twin localized gaps disappear;²⁹ thus, in this regime, a one-to-one correspondence between the mean-values of a surface gravity wave in a random bottom and the solution of the TE is not there.

B. Time-evolution of gravity waves on an irregular floor

Plane-wave modes associated with the problem of dispersive gravity waves can be easily studied in terms of the TE relation (12). That is, the time-dependent analysis of the amplitude $\psi(k, t)$ is given by (7) for any wave number k . In particular, at short times $t \ll \lambda/|v|$, where λ is the correlation length of the disorder in the depth and v is a characteristic velocity given in terms of parameters of the problem, $v^2 = gH_0$ (here, g is the gravity and H_0 is the mean depth); we expect

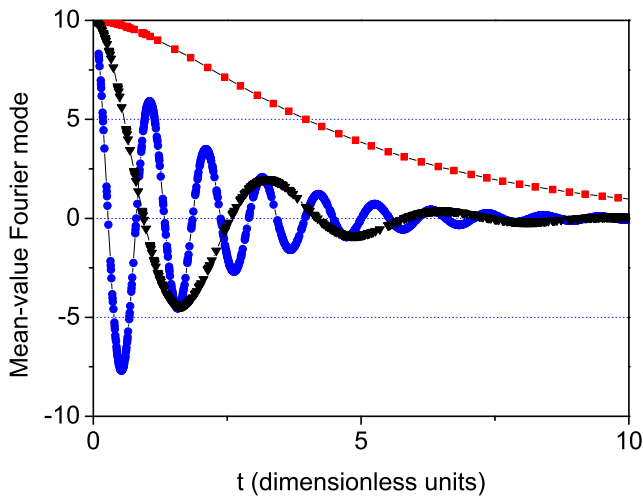


FIG. 1. Plot of the time-dependent amplitude $\overline{\psi(k, t)} \times 10$ from (7) for $v = \lambda = 1$, $k = 0.45$ (red squares), $k = 2$ (black triangle), and $k = 6$ (blue circles). A localized mode can be seen for any $|k| < 1/2$. As expected by the relation (12) at times scales $t \gg 1$ (in this case, $\tau = 1$), the propagation of any mean-value monochromatic wave has been attenuated.

that for wave numbers outside the localized gap $|k| > 1/2\lambda$, there will be attenuation and propagation. However, if $|k| < 1/2\lambda$, gravity waves are non-propagating. In addition to these results, using the relationship with the TE (11), we expect that at long times $t \gg \lambda/|v|$, when the telegrapher's field $\phi(x, t)$ is diffusive, a mean-value surface wave $\langle \psi(x, t) \rangle$ is completely attenuated.

Using dimensionless units ($t' \rightarrow tv/\lambda$ and $x' \rightarrow x/\lambda$), we have plotted in Fig. 1, from (7), the time-dependent behavior of amplitude $\overline{\psi(k, t)}$ for several values of k inside and outside of the localized gap for the strong disorder case.

V. CONCLUSIONS

We have studied, in the absence of any forcing or rotational effect, the propagation mean-value of a 1D surface gravity wave on a random floor. Calculations have been carried out by introducing projector operators and Terwiel's cumulants in (1). This corresponds to working out a linear kinematic approximation for long surface gravity waves over an irregular bottom.⁷ In a recent paper, the solution for the mean-value of a generic field was presented as a series expansion in Terwiel's cumulants, and it has been proven that this series cuts when the disorder is binary with exponential-correlation.¹⁸ Then, we have applied this result to tackle the exact propagation and attenuation of surface waves on a randomly irregular floor. Thus, all conclusions in the present paper have been arrived on the basis that the random bottom has been emulated by a space-stationary exponential-correlated symmetric binary disorder $\xi(x)$, fulfilling $v^2 + \xi(x) \geq 0, \forall x \in (-\infty, +\infty)$ in (1).

The analysis of propagation and attenuation has been carried out, calculating the exact dispersion-relation for plane-wave modes from (1). Then, we have shown the occurrence of non-propagating

surface waves, similar to Anderson's localization phenomena in solid state physics.

In addition, in the present communication, the connection between surface gravity waves propagating in a random bottom and the time-evolution of the solution of the homogeneous TE has been presented. This fact indicates that the rate τ^{-1} in the TE can be associated with the dissipation of energy of a surface wave propagation in a random floor with a space correlation length λ , that is, the relation $\tau = \lambda/|v| = \lambda/\sqrt{gH_0}$ presented in Eq. (12). We have shown that (for short time scales: $t < \tau$) dispersive plane-wave modes from the TE corresponds to dispersive monochromatic modes of a mean-value surface wave on a random bottom.

An interesting open suggestion is that the present approach be extended to a higher dimension, for example, to study wave attenuation in 2D with independent binary disorder in each axis direction.

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APPENDIX: FREE SURFACE WAVE MOTION ON A RANDOM BOTTOM

In the absence of any forcing or rotational effects, the free surface ψ obeys the wave equation,⁷

$$g\nabla \cdot (H\nabla\psi) - \partial_t^2\psi = 0, \quad (\text{A1})$$

where g is the gravity, ∇ is the two-dimensional gradient operator, and

$$H = H_0(1 + \xi(x, y)). \quad (\text{A2})$$

Here, we have taken the ocean floor to be located at $z = -H(x, y)$, where $H(x, y) = H_0[1 + \xi(x, y)]$. Thus, H_0 is a constant (the mean depth), and $\xi(x, y)$ is a centered 2D random field with x and y being the horizontal coordinates.¹⁰ In Fig. 2, we present a sketch of the 1D motion of a free surface propagating on an irregular bottom.

In 1D, we can write Eq. (1) with the constraint $v^2 + \xi(x) \geq 0$, where $v^2 = gH_0$ and the random function $\xi(x)$ emulates the fluctuations on the bottom. In particular, in order to find the exact solution

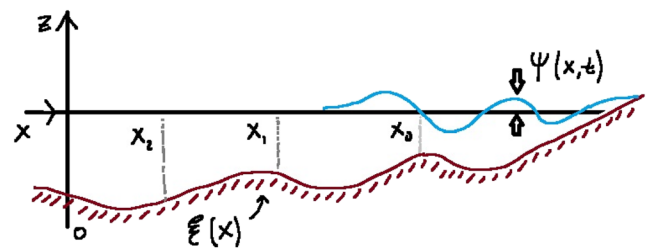


FIG. 2. Representation of a free surface $\psi(x, t)$ wave propagating on a randomly irregular floor with space exponential-correlated fluctuations $\xi(x)$.

of the 1D version of (A1),¹⁸ we assume a symmetric stationary (in space²⁸) binary stochastic process $\xi(x) = \pm\Delta$, $\forall x \in [-\infty, +\infty]$. This is the so-called (Markov in space) a dichotomic process,¹² which is characterized by the correlation function (4); therefore, λ is the space correlation length, and Δ is its intensity. If the binary disorder has not an exponential correlation function (for example as in the case of an intermittent disorder²⁷) the problem cannot be solved in an exact way and it must be worked out invoking a perturbation theory.¹³

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

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- Any m -fold integral appearing in the series expansion for the solution of the random field Eq. (4) in Ref. 18 can be reordered in space, leading to the conclusion that for the particular case of exponential-correlated symmetric binary disorder all Terwiel's cumulants $\langle \xi(y_1) \cdots \xi(y_n) \rangle_T$ are zero for $n > 1$. This is so because the space sequence $\dots, y_j, y_{j+1}, \dots$ can be ordered in each corresponding domain of integration, and odd cumulants are zero by symmetry. We noted that any Terwiel cumulant can be written using projector operators $\mathcal{P}$ as $\langle \xi(y_1) \cdots \xi(y_n) \rangle_T = \mathcal{P} \xi(y_1) (1 - \mathcal{P}) \xi(y_2) \cdots (1 - \mathcal{P}) \xi(y_n)$ (see Refs. 12 and 17). If the disorder had a non-exponential correlation structure, for example, for an intermittent binary process,²⁷ this series expansion would be the starting point to introduce the effective medium approximation to solve the mean-value of the field.¹³ Here, we are interested in the study of an exact mean-value wave propagation, which is why we only use a space-stationary exponential-correlated symmetric binary disorder $\xi(x)$ of intensity $0 \leq \Delta \leq v^2$.
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- Space stationary in the sense that any n -moment of the random field $\xi(x)$ fulfills translational invariance: $\langle \xi(x_1 - X) \xi(x_2 - X) \cdots \xi(x_{n-1} - X) \xi(x_n - X) \rangle = \langle \xi(x_1) \xi(x_2) \cdots \xi(x_{n-1}) \xi(x_n) \rangle$, $\forall X$. We note here that for the present symmetric binary disorder any $2n$ -moment breaks in the form $\langle \xi(x_1) \xi(x_2) \rangle \cdots \langle \xi(x_{2n-1}) \xi(x_{2n}) \rangle$ if the sequence is ordered: $x^2 \geq \dots \geq x^{2n-1} \geq x^{2n}$. This is the key ingredient to prove that Terwiel's cumulants, for the present binary disorder, cancel for order higher than the second one (see Ref. 18).
- The two twin localized gaps follows from the analysis of positivity of the discriminant of a plane-wave mode associated to the general solution (3) for any value of the disorder intensity Δ , that is, $K^4 > (1 + (18 - 27\chi)\chi)K^2 - 16\chi$, $\chi \in (0, 1)$ with $K \equiv 2k\lambda$, $\chi \equiv 1 - (\Delta/v^2)^2$.
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