THE TILTING MODE IN FIELD-REVERSED CONFIGURATIONS

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Received 23 March 1981
Revised manuscript received 20 June 1981

The energy principle is applied to the case of the tilting mode in field-reversed configurations. Prolate equilibria are shown to be unstable and the analytic expression of the growth rate is given.

There is presently an increasing interest in compact toroidal configurations in which no conductors are linked with the plasma. Two main configurations can be distinguished among compact tori: the spheromak in which toroidal and poloidal magnetic fields are present and field-reversed configurations (FRC) in which only the poloidal magnetic field exists (reversed-field theta pinches, field-reversed mirrors, Astron).

In a recent study by Rosenbluth and Bussac on the MHD stability of the spheromak [1], based on the force-free model, it has been shown that prolate equilibria are unstable to the rotation of the symmetry axis inside the separatrix, while oblate equilibria are stable to this instability, called the "tilting mode". On the basis of such a study, it was suggested that also prolate FRC must be tilting unstable, but this instability has not been observed experimentally so far.

In a numerical work [2], reproducing the FRX-B experiment at Los Alamos, the tilting instability of a prolate equilibrium has been confirmed, and possible reasons for the discrepancy with the experiments were proposed.

This puzzling situation has encouraged us to study the problem analytically. Since the method used by Rosenbluth and Bussac, based on the search of minimum magnetic energy states subject to the invariance of the following integral over the volume of the plasma:

$$K = \int_{v_{\text{plasma}}} \mathrm{d}\boldsymbol{v}\, \boldsymbol{A}\cdot\boldsymbol{B} \,,$$

where $\boldsymbol{A}$ is the magnetic potential vector and $\boldsymbol{B}$ the magnetic field (throughout the paper gaussian units will be used), is not applicable to a FRC, where $K = 0$, we have treated the problem by applying the energy principle [3] to a special class of exact axisymmetric equilibria reproducing FRC. Our results confirm the prediction of Rosenbluth and Bussac and also show that the particular equilibria considered are stable to a rigid rotation around an axis orthogonal to the symmetry axis.

Let us consider an axisymmetric configuration of an ideal plasma in which the plasma reaches the separatrix having the form of a revolution ellipsoid. In cylindrical coordinates r, ϕ, z, with the z-axis coinciding with the symmetry axis, the equilibrium of a FRC may be described through the poloidal flux function ψ, which obeys the well-known Grad–Shafranov equation. We will consider a particular solution for ψ, known as Hill's vortex solution, corresponding to a pressure profile linear in ψ and presenting an ellipsoidal separatrix ($\psi = 0$):

$$\psi = \tfrac{3}{2}\pi B_0 r^2(1 - r^2/a^2 - z^2/b^2) \,, \tag{1}$$

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where a and b are the transversal and longitudinal semi-axes of the separatrix, respectively, and B_0 is a constant.

The plasma pressure and the magnetic field corresponding to this solution are:

$$p_0 = \frac{9B_0^2}{32\pi}\left(\frac{1}{b^2}+\frac{4}{a^2}\right)\left(r^2-\frac{r^4}{a^2}-\frac{z^2r^2}{b^2}\right), \tag{2}$$

$$\boldsymbol{B}_0 = -(\hat{e}_\phi \times \nabla\psi)/2\pi r. \tag{3}$$

In order to study the tilting stability of our equilibrium as a function of the elongation of the separatrix, we consider a special divergence-free perturbation that induces an elliptical rotation of the plasma, without distortion of the separatrix, around the axis $z = 0, \phi = 0, \pi$ of the type:

$$\boldsymbol{\xi} = \epsilon\left(\hat{e}_r \frac{z}{b^2}\sin\phi + \hat{e}_\phi \frac{z}{b^2}\cos\phi - \hat{e}_z \frac{r}{a^2}\sin\phi\right), \tag{4}$$

where ϵ is an infinitesimal, and look for the corresponding energy variation δW. To do so we apply the energy principle, which in this case reduces to:

$$\delta W = \frac{1}{8\pi}\int_{-b}^{b} dz \int_0^{a(1-z^2/b^2)^{1/2}} r\,dr \int_0^{2\pi} d\phi\,[|\nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}_0)|^2 - \nabla\times\boldsymbol{B}_0\cdot\nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}_0)\times\boldsymbol{\xi}]. \tag{5}$$

Using equations (1)–(4) it is straightforward to obtain the following expression for δW:

$$\delta W = (3\epsilon^2 B_0^2/40b)(a^2/b^2 - 1), \tag{6}$$

from which it results that δW is negative if $a^2 < b^2$, i.e. the prolate FRC are unstable to the tilting mode. Spherical FRC are neutrally stable and oblate FRC are stable. The growth rate of the instability may be easily calculated, under the hypothesis that the sum of the ionic and electronic temperatures is constant in the plasma, using the relation:

$$\omega^2 = \left(\frac{1}{2}\int d\boldsymbol{v}\,\xi^2 p_0 \frac{m_e + m_i}{k(T_e + T_i)}\right)^{-1}\delta W = \frac{63(a^2/b^2-1)}{2b^2(6a^2/b^2 + a^4/b^4 + 8)}\,\frac{k(T_e+T_i)}{m_e+m_i}, \tag{7}$$

where m_e, T_e and m_i, T_i are the electronic and ionic masses and temperatures, respectively, and k is the Boltzmann constant. Using parameters typical of the FRX-B experiment at Los Alamos [4], a = 5.4 cm, b = 25 cm, $T_e + T_i$ = 310 eV, expression (7) gives an e-folding time of about 10^{-6} s.

We have also considered perturbations that induce a rigid rotation of the plasma around the axis $z = 0, \phi = 0, \pi$ of the type $\boldsymbol{\xi} = \epsilon(\hat{e}_r z \sin\phi + \hat{e}_\phi r\cos\phi - \hat{e}_z r \sin\phi)$; then the energy variation results from the sum of two contributions, one equivalent to expression (5) and one arising from integration on the separatrix. The total expression is:

$$\delta W = \frac{1}{8\pi}\int_{-b}^{b} dz \int_0^{a(1-z^2/b^2)^{1/2}} dr\,r \int_0^{2\pi} d\phi\,[|\nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}_0)|^2 - \nabla\times\boldsymbol{B}_0\cdot\nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}_0)\times\boldsymbol{\xi}]$$
$$-\frac{1}{8\pi}\int_{-b}^{b} dz\left[\left(1-\frac{z^2}{b^2}+\frac{a^2}{b^4}z^2\right)\Big/\left(1-\frac{z^2}{b^2}\right)\right]^{1/2}\int_0^{2\pi} d\phi\;\hat{\boldsymbol{n}}\cdot\boldsymbol{\xi}\,[4\pi\boldsymbol{\xi}\cdot\nabla p_0 - \boldsymbol{B}_0\cdot\nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}_0)], \tag{8}$$

where $\hat{\boldsymbol{n}}$ is the outward vector normal to the separatrix. After performing all the integrations we obtain the result that δW is vanishing for every value of a and b. Then the equilibrium considered is neutrally stable to a rigid rotation around an axis orthogonal to the symmetry axis.

Sparks et al. [5] have shown that Hill's vortex solution can be stable to interchange modes if a nonvanishing constant pressure exists on the separatrix, which permits the criterion of Bernstein et al. [3] to be satisfied. It is interesting to note that the same effect, as results from eq. (5), does not stabilize the tilting mode, which is in agreement with the necessity and no sufficiency of the criterion mentioned above.

In conclusion, we have shown the correctness of Rosenbluth and Bussac's insights, and obtained analytically growth rates for the tilting mode in FRC as a function of the elongation of the separatrix. In spite of its very fast growth rate, up to now there are no experimental observations of this instability. Finite Larmor ion gyroradius, the "racetrack" shape of the separatrix and nonlinear effects have been proposed as possible reasons of this discrepancy [2]. There is also evidence that the plasma in field-reversed theta pinches rotates around the axis of symmetry, and this effect has not been taken into account in our analysis or elsewhere; perhaps the conservation of the angular momentum would give rise to restoring forces that stabilize the tilting mode.

References

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