IL NUOVO CIMENTO  VOL. 54 A, N. 4  21 Dicembre 1979



# Phases of $Z_N$ Lattice Gauge Theory (*).

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(ricevuto il 30 Ottobre 1979)

**Summary.** — We use the Lagrangian formulation of the $Z_N$ gauge theory in four dimensions and demonstrate its equivalence with the partition function for a Coulomb gas of electric and magnetic loops. This model is shown to be self-dual when expressed in the Villain form. It is found, as a lower bound, that for $N < 4$ there are only two phases related by duality, while for $N > 4$ three phases appear. The explicit calculation of the Wilson loop identifies them as electric confining, nonconfining and magnetic confining.

## 1. – Introduction.

It has been argued ([1]) that the centre of the group $SU_N$, *i.e.* the $Z_N$ Abelian group, plays an important role in the confinement problem. It is, therefore, interesting to analyse the phase diagram of the $Z_N$ gauge theory.

When a $Z_N$ lattice gauge model is expressed in a Lagrangian formalism using the Wilson action ([2]), the introduction of the Villain ([3]) approximation allows us to show the self-duality properties in four dimensions ([4]). This in-

(*) To speed up publication, the author of this paper has agreed to not receive the proofs for correction.
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([1]) G. H. HOOFT: *Nucl. Phys. B*, **138**, 1 (1978).
([2]) K. WILSON: *Phys. Rev. D*, **10**, 2445 (1974).
([3]) J. VILLAIN: *J. Phys. (Paris)*, **36**, 581 (1975).
([4]) A. CASHER: *Nucl. Phys. B*, **151**, 353 (1979).

dicates that, if only two phases exist, the transition occurs at the symmetry point.

However, it has been suggested (5) that for sufficiently high $N$ a third phase appears. The same conclusion has been reached using the Hamiltonian formalism (6). Also the spin system in two dimensions (7) shows a third phase, which is consistent with the four-dimensional gauge result (8).

The purpose of this work is to identify the three phases through the explicit analysis of the Wilson loop behaviour and to establish a lower bound for $N$ in order that the third phase can exist.

In sect. 2 we give a self-contained description of the Lagrangian formalism leading to a Coulomb gas of electric- and magnetic-current loops and showing the self-duality property.

In sect. 3 we study the different limits of the coupling constant identifying an electric-confining, a magnetic-confining and a nonconfining phase. The electric-confining phase shows an area law for the Wilson loop in the direct lattice and a perimeter law in the dual lattice. The magnetic-confining phase has just the dual behaviour, while the nonconfining phase presents a perimeter behaviour in both lattices.

Section 4 is devoted to establish a lower bound to the critical $N$ for the existence of the third phase using an entropy argument (9), and to draw the phase diagram of the model.

Short conclusions are given in sect. 5.

## 2. – Coulomb gas analogy.

We work with a Lagrangian version of the $Z_N$ gauge theory in a path-integral formulation.

The partition function of the system is defined by

$$Z = \sum_{\{n_\mu\}} \exp\left[-K \sum_{\text{plaquettes}} \cos\left(\frac{2\pi}{N} N_{\mu\nu}\right)\right], \tag{1}$$

where $N_{\mu\nu} = \Delta_\mu n_\nu - \Delta_\nu n_\mu$ with $\Delta_\mu$ finite-difference operator. $n_\mu$ are link variables $0 \leqslant n_\mu \leqslant N-1$ and $\mu = 1, \ldots, 4$.

In the Villain approximation (3) eq. (1) is rewritten as

$$Z = \sum_{\{n_\mu\}} \sum_{m_{\mu\nu}=-\infty}^{\infty} \exp\left[-\frac{K}{2} \sum_{r,\mu<\nu} \left(\frac{2\pi}{N} N_{\mu\nu} - 2\pi m_{\mu\nu}\right)^2\right], \tag{2}$$

(5) S. Elitzur, J. Shigemitsu and R. Pearson: Princeton preprint (1979).
(6) M. Weinstein, D. Horn and S. Yankielowicz: Tel-Aviv preprint TAUP 723-79.
(7) J. L. Cardy: Santa Barbara preprint UCSB TH-30 (1978).
(8) A. A. Migdal: Ž. Èksp. Teor. Fiz., 69, 810, 1457 (1975).
(9) T. Banks, R. Myerson and J. Kogut: Nucl. Phys. B, 129, 493 (1977).

$r$ being the site of the lattice. This approximation keeps the same periodicity of eq. (1) and clearly has the same behaviour for $K \to \infty$. It is argued [10] that the topological properties of eq. (2) are the same as those of eq. (1). In eq. (2) $m_{\mu\nu}$ is a plaquette variable with $m_{\mu\nu} = - m_{\nu\mu}$ because $\mu \leftrightarrow \nu$ means just a reversal of the plaquette orientation.

A gauge transformation is given by $n_\mu \to n_\mu + \Delta_\mu \varrho$, where $\varrho$ is a scalar variable at each site, $0 \leqslant \varrho \leqslant N-1$.

Using Poisson's sum rule, we may write eq. (2) as

$$(3) \qquad Z \propto \int_0^{2\pi} \mathrm{D}\varphi_\mu \sum_{p_\mu=-\infty}^{\infty} \sum_{m_{\mu\nu}=-\infty}^{\infty} \exp\left[-\frac{K}{2} \sum_{r,\mu<\nu} (F_{\mu\nu} - 2\pi m_{\mu\nu})^2 + iN \sum_{r,\mu} p_\mu \varphi_\mu\right],$$

where

$$F_{\mu\nu} = \Delta_\mu \varphi_\nu - \Delta_\nu \varphi_\mu .$$

Choosing a suitable gauge we keep only 3 independent $\varphi_\mu$. In appendix A we show how it is possible to sum three $m_{\mu\nu}$ to the independent $\varphi_\mu$ to extend the range of integration to $\pm\infty$. The remaining $m_{\mu\nu}$ may be written as

$$(4) \qquad \begin{cases} m_{\mu\nu} = \varepsilon_{\mu\nu\alpha\beta} n_\alpha (n \cdot \Delta)^{-1} M_\beta \,, \\ M_\mu = \varepsilon_{\mu\nu\varrho\sigma} \Delta_\nu m_{\varrho\sigma} \,, \end{cases}$$

where $M_\mu$ is a dual-link variable and $n_\alpha = \delta_{\alpha 1}$. Thus $m_{1\nu} = 0$ and $M_1 = 0$.

Finally we may write

$$(5) \qquad Z \propto \int_{-\infty}^{\infty} \mathrm{D}\varphi_\mu \sum_{p_\mu=-\infty}^{\infty} \sum_{m_{\mu\nu}=-\infty}^{\infty} \exp\left[-\frac{K}{2} \sum_{r,\mu<\nu} (F_{\mu\nu} - 2\pi m_{\mu\nu})^2 + iN \sum_{r,\mu} p_\mu \varphi_\mu\right].$$

In the same way we may define the generating functional

$$(6) \qquad Z[J] \propto \int_{-\infty}^{\infty} \mathrm{D}\varphi_\mu \sum_{p_\mu=-\infty}^{\infty} \sum_{m_{\mu\nu}=-\infty}^{\infty} \exp\left[-\frac{K}{2} \sum_{r,\mu<\nu} (F_{\mu\nu} - 2\pi m_{\mu\nu})^2 + i \sum_{r,\mu} P_\mu \varphi_\mu\right]$$

with $P_\mu = N p_\mu + J_\mu$.

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[10] M. EINHORN and R. SAVIT: *Phys. Rev. D*, **19**, 1198 (1979).

Now we are able to perform the Gaussian integral (see appendix B) and end up with the following generating functional [5]:

$$Z[J] = Z_0 \sum_{p_\mu, M_\mu = -\infty}^{\infty} \exp\left[-\frac{1}{2K} \sum_{r,r'} J_\mu(r) G(r, r') J_\mu(r')\right] \delta(\Delta_\mu M_\mu)\, \delta(\Delta_\mu p_\mu) \cdot \exp\left[\sum_{r,r'}\left[-\frac{N^2}{2K} p_\mu(r) G(r, r') p_\mu(r') - \frac{N}{K} p_\mu(r) G(r, r') J_\mu(r') + 2\pi N i p_\mu(r) G(r, r') \Delta'_\alpha m_{\mu\alpha}(r') + 2\pi i J_\mu(r) G(r, r') \Delta'_\nu m_{\mu\nu}(r') - 2\pi^2 K M_\varrho(r) G(r, r') M_\varrho(r')\right]\right], \tag{7}$$

where

$$Z_0 = \int_{-\infty}^{\infty} \mathrm{D}\varphi_\mu \exp\left[-\frac{K}{2} \sum_{r,\mu<\nu} (\Delta_\mu \varphi_\nu - \Delta_\nu \varphi_\mu)^2\right] \delta(\Delta_\mu \varphi_\mu).$$

and $G(r, r')$ is the Green's function of the 4-dimensional Laplace operator $\Delta_\mu \Delta_\mu G(r, r') = -\delta_{r,r'}$, *i.e.* $G(r, r') = (1/4\pi)(1/|\mathbf{r} - \mathbf{r}'|^2)$ + gauge-dependent terms.

The Wilson loop can be calculated as

$$\langle \exp[-w(F)] \rangle = \frac{Z[J]}{Z[J = 0]} \tag{8}$$

with

$$J_\mu(r) = \begin{cases} 1, & \text{if the link } r \to r + \hat{\mu} \in \Gamma, \\ -1, & \text{if the link } r + \hat{\mu} \to r \in \Gamma, \\ 0, & \text{otherwise.} \end{cases}$$

We note that $\Delta_\mu J_\mu = 0$, because the Wilson loop is closed.

In appendix C we show that

$$\sum_{r,r'} J_\mu(r) G(r, r') J_\mu(r') \approx \frac{\pi R}{a} + \frac{1}{4} \ln \frac{2R}{a}, \tag{9}$$

where $2\pi R$ is the perimeter of the loop and $a$ is the lattice parameter.

We may recognize the « changes » appearing in eq. (7) writing out the Euler-Langrange equations from eq. (5):

$$\Delta_\mu(F_{\mu\nu} - 2\pi m_{\mu\nu}) = i p_\nu N .$$

Thus we identify $p_\nu$ as « electric » currents and from eq. (4) $M_\mu$ as « magnetic » currents. This is because eq. (4) corresponds to the « dual » electromagnetic equation

$$\Delta_\nu \tilde{m}_{\mu\nu} = M_\mu \qquad \text{with} \qquad \tilde{m}_{\mu\nu} = \varepsilon_{\mu\nu\varrho\sigma} m_{\varrho\sigma} .$$

Then, because of the constraints in eq. (7), our partition function describes current loops of electric and magnetic charges interacting through a Coulomb potential (5). It is known (4) that the Villain form of the $Z_N$ gauge model is self-dual in four dimensions. We can ask ourselves what does it mean in terms of these charges. We start from $Z[J=0]$ and perform a dual transformation

$$\text{(10)} \qquad \exp\left[-\frac{K}{2}(F_{\mu\nu} - 2\pi m_{\mu\nu})^2\right] \propto \int_{-\infty}^{\infty} \mathrm{D}s_{\mu\nu} \left\{\exp\left[-\frac{1}{2K}(s_{\mu\nu})^2 + is_{\mu\nu}(F_{\mu\nu} - 2\pi m_{\mu\nu})\right]\right\},$$

where $s_{\mu\nu}$ is a plaquette variable.

Then eq. (5) becomes

$$\text{(11)} \qquad Z \propto \int_{-\infty}^{\infty} \mathrm{D}\varphi_\mu \int_{-\infty}^{\infty} \mathrm{D}s_{\mu\nu} \cdot \sum_{p_\mu, m_{\mu\nu} = -\infty}^{\infty} \exp\left[\sum_{r,\mu<\nu} \left\{-\frac{1}{2K}(s_{\mu\nu})^2 + is_{\mu\nu}(F_{\mu\nu} - 2\pi m_{\mu\nu}) + iNp_\mu\varphi_\mu\right\}\right],$$

and, integrating over a particular $\varphi_\mu$ for a certain link, we obtain

$$\text{(12)} \qquad \int_{-\infty}^{\infty} \mathrm{d}\varphi_\mu \exp\left[i\varphi_\mu\left({\textstyle\sum'} s_{\mu\nu} + Np_\mu\right)\right] \propto \delta(\Delta_\nu s_{\mu\nu} + Np_\mu),$$

where $\sum'$ means a sum over plaquettes sharing the same link.

Solving eq. (12), we have

$$\text{(13)} \qquad s_{\mu\nu} = N\left(n_\mu(n\cdot\Delta)^{-1}p_\nu - n_\nu(n\cdot\Delta)^{-1}p_\mu\right) + \frac{N}{2\pi}\varepsilon_{\mu\nu\varrho\sigma}\Delta_\varrho l_\sigma ,$$

where the operator $(n\cdot\Delta)^{-1}$ has been defined in appendix A, and $l_\sigma$ is a link variable defined on the dual lattice (the link $\sigma$ is normal to the plaquette $\mu-\nu$). Then inserting eq. (13) in eq. (11), we get

$$\text{(14)} \qquad Z \propto \int_{-\infty}^{\infty} \mathrm{D}l_\mu \sum_{p_\mu, m_{\mu\nu} = -\infty}^{\infty} \exp\left[\sum_{r,\mu<\nu}\left\{-\frac{1}{2K}\left(\frac{N}{2\pi}L_{\mu\nu} - N\varepsilon_{\mu\nu\varrho\sigma}n_\varrho(n\cdot\Delta)^{-1}p_\sigma\right)^2\right\}\right] \cdot \exp\left[\sum_{r,\mu<\nu}\left\{\left(-2\pi iN\left(n_\mu(n\cdot\Delta)^{-1}p_\nu - n_\nu(n\cdot\Delta)^{-1}p_\mu\right)m_{\mu\nu} - iN\varepsilon_{\mu\nu\varrho\sigma}\Delta_\varrho l_\sigma m_{\mu\nu}\right\}\right],$$

where $L_{\mu\nu} = \Delta_\mu l_\nu - \Delta_\nu l_\mu$.

But, since $(n_\mu(n\cdot\Delta)^{-1}p_\nu - n_\nu(n\cdot\Delta)^{-1}p_\mu)m_{\mu\nu}$ is an integer number, we are led to the partition function

$$Z \propto \int_{-\infty}^{\infty} Dl_\mu \sum_{p_\mu, m_{\mu\nu}=-\infty}^{\infty} \exp\left[\sum_{r,\mu<\nu}\left\{-\frac{1}{2K}\left(\frac{N}{2\pi}\right)^2 (L_{\mu\nu} - 2\pi\varepsilon_{\mu\nu\varrho\sigma}n_\varrho(n\cdot\Delta)^{-1}p_\sigma)^2\right\}\right]\cdot \exp\left[\sum iNl_\sigma\varepsilon_{\mu\nu\varrho\sigma}\Delta_\varrho m_{\mu\nu}\right]. \tag{15}$$

From eqs. (5) and (15) we see that a duality transformation relates $p_\mu \leftrightarrow M_\mu$ and $K/2 \leftrightarrow (1/2K)(N/2\pi)^2$. Since the exchange $p_\mu \leftrightarrow M_\mu$ is just electric-magnetic duality, we learn that the partition function eq. (5) is exactly self-dual in terms of electric and magnetic variables changing $K \leftrightarrow (1/K)(N/2\pi)^2$.

It is easy to prove that in the presence of external currents the self-duality property is not valid.

## 3. – Limiting cases for the coupling constant.

Now we will study the behaviour of eq. (7) in different limiting cases.

*a*) $\lim K \to 0$. In this limit only the magnetic loops are relevant to the partition function, since, to have a nonvanishing contribution, one must take $p_\mu = 0$. The contribution we must evaluate is

$$\exp\left[2\pi i \sum_{r,r'} J_\mu(r)\, G(r, r')\, \Delta'_\nu m_{\mu\nu}(r')\right].$$

Because $\Delta_\mu J_\mu = 0$, we may write

$$J_\sigma = \varepsilon_{\sigma\mu\nu\varrho}\Delta_\mu Q_{\nu\varrho}\,. \tag{16}$$

If we take the loop in the plane 3-4 we may express $Q_{\nu\varrho} = n_\nu L_\varrho$, $n$ being a unit vector such that $n_\nu = \delta_{\nu 1}$ and $L_\varrho$ a vector (defined on the dual lattice)

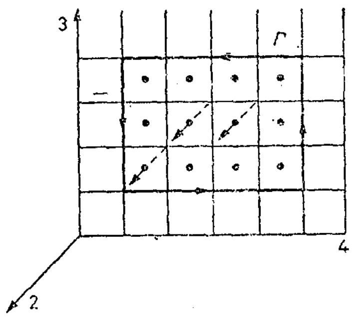


Fig. 1. – Definition of $L_\varrho$. $L_\varrho$ is a vector whose values is 1, if it goes through a plaquette enclosed by the Wilson loop $\Gamma$, and 0 otherwise.

normal to each plaquette in the plane 3-4 (see fig. 1) given by

$$L_\varrho = \delta^{\perp}_{\varrho 2} = \begin{cases} 1\,, & \text{if the plaquette through which } L_\varrho \text{ passes is enclosed by the Wilson loop for } \varrho = 2\,, \\ 0\,, & \text{otherwise}. \end{cases}$$

Using eqs. (4) and (16), we find (see appendix D)

$$\exp\left[2\pi i \sum_{r,r'} J_\mu(r)\, G(r, r')\, \Delta'_\nu m_{\mu\nu}(r')\right] = \exp\left[-2\pi i \sum_{r \in S_\Gamma} (n\cdot\Delta)^{-1} M_\varrho(r)\, L_\varrho(r)\right], \tag{17}$$

where $S_\Gamma$ means the surface enclosed by the loop.

Equation (17) has the following physical meaning. Because $L_\varrho$ is a vector normal to each plaquette enclosed by the loop, we must sum over magnetic loops that intersect normally the Wilson loop. If it is totally enclosed by $S_\Gamma$ (*i.e.* all its intersections are within the surface of the gauge loop), its contribution to eq. (17) vanishes, while, if the loop is large enough such that it is not totally enclosed by $S_\Gamma$, it contributes to eq. (17) at those plaquettes which are intersected.

From fig. 2*a*) we can clearly see that small loops can only contribute to a perimeter behaviour [11] (because the Wilson loop is very large), while fig. 2*b*) suggests that a condensate of large loops will lead to an area behaviour.

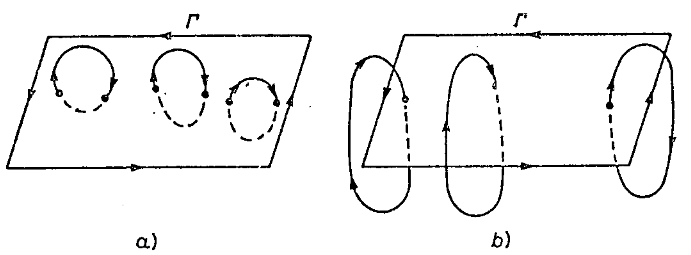


Fig. 2. – *a*) Small magnetic loops totally enclosed by the Wilson loop. *b*) Large magnetic loops intersecting the surface of $\Gamma$ once.

*b*) $\lim N^2/2K \to 0$, $K \to \infty$. In this case the magnetic loops are not relevant to the partition function because one must take $M_\varrho = 0$. The term we must compute is

$$\exp -\left[\frac{N}{K} \sum_{r,r'} p_\mu(r)\, G(r, r')\, J_\mu(r')\right]$$

[11] T. Yoneya: *Nucl. Phys. B*, **144**, 195 (1978).

From eq. (16) and «integrating» by parts, the exponent becomes

$$(18)\qquad -\frac{N}{K}\sum_{r,r'} L_\varrho(r')\varepsilon_{\sigma\mu\nu\varrho} n_\nu p_\sigma(r)\Delta_\mu G(r,r') = \frac{N}{K}\varepsilon_{\mu\sigma 12}\sum_{r,r'\in S_\Gamma}\Delta_\mu G(r,r')p_\sigma(r)\,,$$

where only $r'$ is on $S_\Gamma$. This expression can be rewritten in a more familiar way. Because $p_\sigma$ is an electric-current loop, the continuum limit of eq. (18) is

$$\frac{N}{K}\int_{S_\Gamma} \mathrm{d}r'\,\varepsilon_{\sigma\mu 12}\Delta'_\mu \oint_{\substack{\text{electric}\\ \text{loop}}} \frac{p_\sigma(r)\,\mathrm{d}r}{|r-r'|^2}\,,$$

and we recognize $\oint p_0(r)\,\mathrm{d}r/|r-r'|^2$ as the vector potential of electromagnetism in $4+1$ dimensions,

$$B_{12} = \varepsilon_{\sigma\mu 12}\Delta'_\mu \oint \frac{p_\sigma(r)\,\mathrm{d}r}{|r-r'|^2}$$

being the corresponding magnetic-field component. Thus $\int_{S_\Gamma} B_{\mu\nu}\,\mathrm{d}s_{\mu\nu}$ is the total magnetic flux going through the surface $S_\Gamma$. The magnetic-field lines are closed and are on the plane perpendicular to the electric current generating them.

From this analysis we conclude that the electric loops enclosed by the Wilson loop cannot contribute to an area behaviour, because the total flux which goes through the surface can only have contributions from loops which are near the boundary of the gauge loop, as is shown in fig. 3. Figure 3 strongly suggests the effect of electric loops, which are far and near, respectively, from the boundary of the gauge loop. Thus it is clear that a condensate of electric loops will at most contribute to a perimeter behaviour.

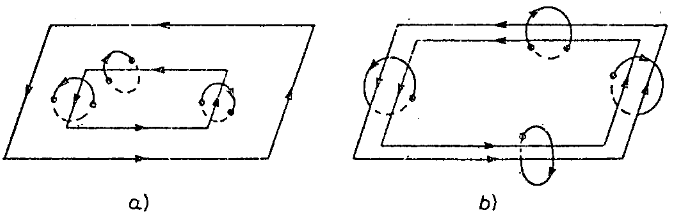


Fig. 3. – $a$) Small electric loop and its magnetic-field lines inside $S_\Gamma$. $b$) Large electric loop near the boundary of $S_\Gamma$.

In appendix E we calculate the effect of electric loops of definite charge inside the Wilson loop, its contribution being of order $\ln(2R/a)$. In the situation in which there is a condensate of these loops, large loops will be present. Then we have to sum over all possible lenghts of loops inside $\Gamma$ finding a total contribution $\approx \pi R/a + \frac{1}{4}\ln(2R/a)$ (see appendix E). So that in this limit, where only electric loops are relevant to the partition function, the Wilson loop behaves with a perimeter law. If we sum a term $\exp\left[i\sum_r C_\mu l_\mu\right]$ to eq. (15), regarding $C_\mu$ as a current defined on the dual lattice (magnetic currents), and associate with it a «dual Wilson loop» for magnetic charges, after following the steps which led to eq. (7), one gets the same equation but with the exchanges $M_\mu \leftrightarrow p_\mu$ and $K \leftrightarrow (1/K)(N/2\pi)^2$. Therefore, because of duality, in the limit $N^2/2K \to 0$, $K \to \infty$, the dual Wilson loop behaves with an area law equivalent to magnetic confinement. In the other limit, $K \to 0$, the dual Wilson loop has a perimeter behaviour.

Summarizing, the behaviour of the theory in the limiting cases is the following. $\lim K \to 0$: magnetic loops are relevant, the Wilson loop decays with an area law and there is confinement of electric charges. $\lim N^2/2K \to 0$, $K \to \infty$: the electric loops become important to $Z[J]$ and the Wilson loop behaves like the perimeter. There is no confinement of electric charges, but there is confinement of magnetic ones.

We expect another behaviour in the middle of these two limiting cases. $\lim K \to \infty$, $\lim N^2/2K \to \infty$. In this situation neither the electric loops nor the magnetic ones contribute in a relevant way to the partition function. The main contribution to $Z[J]$ comes from the term $\exp\left[-(1/2K)\sum J_\mu(r)\cdot G(r, r')J_\mu(r')\right]$ and the Wilson loop decays as the perimeter in both the direct and the dual lattice. There is no confinement either of electric or of magnetic charges. We suggest that this phase takes place for $N > N_c$.

## 4. – Phase diagram.

We will give a naive argument (9) to roughly determine the value of $N_c$ and the coupling constant at which a phase transition takes place. We have a gas of electric- and magnetic-current loops interacting via a Coulomb potential in four dimensions. In the limit in which a class of them is condensed, we expect a great amount of loops of this kind to be of infinite length and the vacuum will be filled with them. They look like a random walk and we may argue that the forces between neighbouring links are cancelled on the average (both for links belonging to the same loop or to neighbouring loops). In this crude approximation only the self-energy of each link survives, and a loop contributes as $G(0)L$, $L$ being its length. Then we have to compute, *e.g.*, the following term:

$$S = \sum_{M_e=-\infty}^{\infty} \delta(\Delta_\mu M_\mu)\exp[-2\pi^2 KG(0)LM_e^2]. \tag{19}$$

The sum is over closed non-back-tracking (to avoid double counting) random walks. But, since they are not too well known, we will take them simply as random closed walks.

It has been found ([12]) that the number of possible configurations of closed random walks of length $L$ is roughly

$$P(L) = \mu^L f_1(L)\,, \tag{20}$$

$ft(L)$ being a function such that $\lim_{L\to\infty} [f_1(L)]^{1/L} \to 1$ and ([13]) $\mu \simeq 2D$, where $D$ is the dimensionality of the system.

Then from eq. (19) the free energy of a single loop carrying a unit charge is given by

$$F_m \simeq \left(2\pi^2 K G(0) - \ln \mu\right) L\,, \tag{21a}$$

and the same holds for the electric loops

$$F_e \simeq \left(\frac{N^2}{2K} G(0) - \ln \mu\right) L\,. \tag{21b}$$

From eq. (21$a$) we see that when $K$ is small enough $\left(2\pi^2 K G(0) < \ln \mu\right)$ large loops are more important to the partition function than the small ones. Considering eq. (21$b$) too, it is clear that when $K$ increases magnetic loops become irrelevant and for $N$ small enough $\left((N^2/2K)G(0) < \ln \mu\right)$ electric loops are important. Thus we expect a phase transition for $K_c$ ($N < N_c$) given by the duality relation

$$K_c = \frac{N}{2\pi}\,. \tag{22}$$

For $K < K_c$ the vacuum is filled with magnetic loops which lead to electric confinement, and for $K > K_c$ ($N < N_c$) the electric loops condense giving magnetic confinement.

As was suggested above, we expect for $N > N_c$ a phase with neither electric nor magnetic confinement. From eq. (21) this takes place when

$$N^2 > \frac{2K \ln \mu}{G(0)} \quad \text{and} \quad K > \frac{\ln \mu}{2\pi^2 G(0)} = K'_c\,.$$

Then a lower bound is given by

$$N^2 > 4\pi^2 K_c'^2 = N_c^2\,. \tag{23}$$

([12]) J. M. Hammersley: *Proc. Cambridge Philos. Soc.*, **53**, 642 (1957); **57**, 516 (1961); *Adv. Chem. Phys.*, **15**, 229 (1969).
([13]) M. Stone and P. Thomas: *Phys. Rev. Lett.*, **41**, 351 (1978).

To calculate $K'_c$ we use the value (⁹) of the Green's function regularized at the origin $G(0) \approx 0.19$ for the lattice in four dimensions (*). Then

$$K'_c \simeq 0.56\,, \qquad N_c \simeq 3.6\,. \tag{24}$$

For $N > N_c \simeq 4$ we expect three different phases: electric confining for $K < K'_c$, magnetic confining for $K > N^2/4\pi^2 K'_c$ and an intermediate nonconfining phase which is characterized by massless ([14]) photons and is analogous to that of QED. This suggests the phase diagram shown in fig. 4.

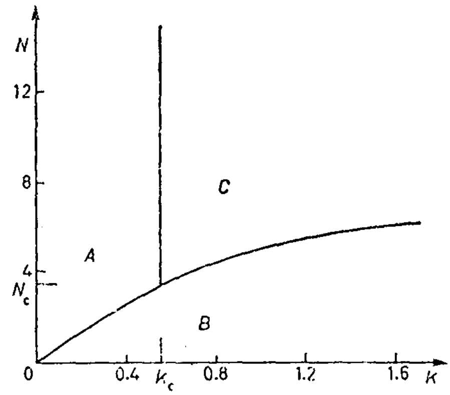


Fig. 4. – $Z_N$ phase diagram. The phase $A$ corresponds to electric confinement, phase $B$ to magnetic confinement and $C$ to nonconfinement. Phases $A$ and $B$ are separated by $N = 2\pi K$, $A$ and $C$ by $K = K'_c$ and $B$ and $C$ by $N = 2\pi\sqrt{KK'_c}$.

## 5. – Concluding remarks.

We have analysed the behaviour of the Wilson loop for different regions of the coupling constant of the $Z_N$ lattice gauge theory in four dimensions.

As expected we found three phases characterized by the area behaviour (confinement of static electric charges), perimeter behaviour (confinement of static magnetic charges) and perimeter law without any confinement.

Using an entropy argument we found the phase diagram for this theory establishing a lower bound for the critical $N$ necessary to expect three phases.

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(*) This value has been taken by comparing the numerical coefficients of the last term in the exponent of our eq. (7) with the second exponent of eq. (17) of ref. ([9]). This was also done in ref. ([5]). Nevertheless, since there is an apparent difference of a factor of 2 between both expressions, if the value $G(0) = 0.38$, as quoted in ref. ([9]) is introduced in our formulae, a too low bound $N_c = 2$ would be obtained.

([14]) T. BANKS and E. RABINOVICI: Princeton preprint (1979).

This value is consistent with the one obtained by comparing with the $U_1$ model ([5]) or analysing the spin system ([7]), and lower than estimates coming from the Hamiltonian version of the theory ([5,6]). Despite this uncertainty, we believe that the features of the phase diagram are correct.

Having characterized the three possible phases of the $Z_N$ gauge theory, one should apply more refined methods, *e.g.* the renormalization group techniques, to determine the phase diagram with greater accuracy.

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The author is deeply indebted to L. MASPERI for encouraging comments and for reading the manuscript. He also thanks J. CARDY, whose visit was supported by the Organization of American States through the Multinational Program in Physics, for very useful suggestions which stimulated this work, and to the Particle Group of La Plata University for interesting discussions.

## APPENDIX A

We show the method to extend the integration range from eq. (3) to eq. (5). Using Poisson's sum rules

$$\sum_{n_\mu=0}^{N-0} f\left(\frac{2\pi}{N}\, n_\mu\right) = \frac{N}{2\pi} \sum_{p_\mu=-\infty}^{\infty} \int_0^{2\pi} \mathrm{d}\varphi_\mu \exp\left[i\varphi_\mu p_\mu N\right] f(\varphi_\mu) \,, \tag{A.1}$$

we have arrived at eq. (3).

There are six independent $m_{\mu\nu}$ and three independent $\varphi_\mu$.

We sum three $m_{\mu\nu}$ to the independent $\varphi_\mu$, *i.e.* we sum $m_{12}$ $m_{13}$, $m_{14}$ to $\varphi_2$, $\varphi_3$, $\varphi_4$, respectively, and define $\varphi'_\mu = \varphi_\mu - \pi M_\mu$. Because

$$(\Delta_\mu \varphi_\nu - \Delta_\nu \varphi_\mu - 2\pi m_{\mu\nu})^2 = \varepsilon_{\varrho\sigma\mu\nu}(\Delta_\mu \varphi_\nu - \pi m_{\mu\nu}) \,,$$

in the term $\varepsilon_{\varrho\sigma\mu\nu}(\Delta_\mu \varphi'_\nu + \pi \Delta_\mu M_\nu - \pi m_{\mu\nu})$ we impose

$$\Delta_\mu M_\nu - m_{\mu\nu} = a_{\mu\nu} = \begin{cases} 0\,, & \text{if } \mu = 1, 2, 3, 4\,, \\ m_{\mu\nu}\,, & \text{if } \mu = 2, 3, 4\,,\ \nu = 2, 3, 4\,. \end{cases} \tag{A.2}$$

The solution of eq. (A.2) is

$$a_{\mu\nu} = \varepsilon_{\mu\nu\alpha\beta} n_\alpha (n \cdot \Delta)^{-1} M_\beta \tag{A.3}$$

with $n$ a unit vector such that $n_\alpha = \delta_{\alpha 1}$ and $(n \cdot \Delta)^{-1}$ the inverse of the finite-difference operator $(n \cdot \Delta)^{-1} M_\beta = \sum_{x'_1=-\infty}^{x_1} M_\beta(x'_1, x_2, x_3, x_4)$.

In fact, $a_{12} = a_{13} = a_{14} = 0$ and

$$a_{23} = \quad n_1(n\cdot\Delta)^{-1}M_4 = m_{23}\,,$$
$$a_{24} = -n_1(n\cdot\Delta)^{-1}M_3 = m_{24}\,,$$
$$a_{34} = \quad n_1(n\cdot\Delta)^{-1}M_2 = m_{34}\,,$$

so that

$$M_4 = \Delta_1 m_{23}\,,\qquad M_3 = -\Delta_1 m_{24}\,,\qquad M_2 = \Delta_1 m_{34}\,,\qquad M_1 = 0\,. \tag{A.4}$$

Defining

$$m'_{\mu\nu} = \begin{cases} 0\,, & \text{if } \mu = 1\,,\ \nu = 2, 3, 4\,,\\ m_{\mu\nu}\,, & \text{if } \mu = 2, 3, 4\,,\ \nu = 2, 3, 4\,,\end{cases}$$

we see that

$$\begin{cases} M_\mu = \varepsilon_{\mu\nu\varrho\sigma}\Delta_\nu m'_{\sigma\varrho}\,,\\ m'_{\mu\nu} = \varepsilon_{\mu\nu\alpha\beta}n_\alpha(n\cdot\Delta)^{-1}M_\beta\,,\end{cases} \tag{A.4'}$$

and we finally obtain eq. (5), dropping the primes.

## Appendix B'

We perform the Gaussian integral to get eq. (7).
The exponent of eq. (6) is

$$E = iP_\mu\varphi_\mu - \frac{K}{2}\sum_{\mu<\nu}(\Delta_\mu\varphi_\nu - \Delta_\nu\varphi_\mu - 2\pi m_{\mu\nu})^2\,. \tag{B.1}$$

Making the shift $\varphi_\mu \to \varphi_\mu + Q_\mu$ and imposing the gauge $\Delta_\mu\varphi_\mu = 0$, $\Delta_\mu Q_\mu = 0$, we choose $Q_\mu$ such that

$$iP_\mu\varphi_\mu + K\varphi_\mu\Delta_\nu\Delta_\nu Q_\mu + 2\pi K\varphi_\mu\Delta_\nu m_{\mu\nu} = 0\,. \tag{B.2}$$

We see that

$$Q_\mu = \sum_{r'}\left[\frac{i}{K}P_\mu(r') + 2\pi\Delta'_\nu m_{\mu\nu}(r')\right]G(r, r') \tag{B.3}$$

with $\Delta_\nu\Delta_\nu G(r, r') = -\delta_{r,r'}$.
From $\Delta_\mu Q_\mu = 0$ it follows that $\Delta_\mu p_\mu = 0$, because $\Delta_\mu J_\mu = 0$ and $\Delta_\mu\Delta_\nu m_{\mu\nu} = 0$.

The term which goes out of the integral, if one uses eq. (B.2), is

$$\text{(B.4)}\qquad iP_\mu Q_\mu - \frac{K}{4}\sum_{\mu,\nu}(\Delta_\mu Q_\nu - \Delta_\mu Q_\mu - 2\pi m_{\mu\nu})^2 = \frac{-P_\mu(r)}{2K}\sum G(r,r')P_\mu(r') + \pi i P_\mu(r)\sum_{r'} G(r,r')\Delta'_\alpha m_{\mu\alpha}(r') + \pi i\Delta_\nu m_{\mu\nu}(r)\sum_{r'} G(r,r')P_\mu(r') - \pi^2 K m^2_{\mu\nu} + 2\pi^2 K\Delta_\nu m_{\mu\nu}(r)\sum_{r'}\Delta_\alpha m_{\mu\alpha}(r').$$

Because of eq. (A.4′), the last two terms in eq. (B.4) become

$$\text{(B.5)}\qquad F = 2K\pi^2\varepsilon_{\sigma\mu\nu\varrho}\Delta_\mu n_\nu(n\cdot\Delta)^{-1}M_\varrho(r)\sum_{r'}\varepsilon_{\sigma\alpha\beta\gamma}\Delta'_\alpha n_\beta(n\cdot\Delta)^{-1}M_\gamma(r')$$

and

$$\text{(B.6)}\qquad G = -\pi^2 K\varepsilon_{\mu\nu\alpha\beta}n_\alpha(n\cdot\Delta)^{-1}M_\beta\varepsilon_{\mu\nu\varrho\sigma}n_\varrho(n\cdot\Delta)^{-1}M_\sigma,$$

respectively.

Using the formulae [15]

$$\text{(B.7)}\qquad \begin{cases} \varepsilon_{\mu\nu\varrho\sigma}\varepsilon_{\alpha\beta\gamma\sigma} = \delta_{\mu\alpha}\delta_{\nu\beta}\delta_{\varrho\gamma} - \delta_{\mu\alpha}\delta_{\nu\gamma}\delta_{\varrho\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}\delta_{\varrho\gamma} + \delta_{\mu\beta}\delta_{\nu\gamma}\delta_{\varrho\alpha} + \delta_{\mu\gamma}\delta_{\nu\alpha}\delta_{\varrho\beta} - \delta_{\mu\gamma}\delta_{\nu\beta}\delta_{\varrho\alpha}, \\ \varepsilon_{\mu\nu\alpha\beta}\varepsilon_{\mu\nu\varrho\sigma} = 2(\delta_{\alpha\varrho}\delta_{\beta\sigma} - \delta_{\alpha\sigma}\delta_{\beta\varrho}), \end{cases}$$

and the fact that $n_\nu M_\nu = 0$ and $\Delta_\mu M_\mu = 0$, one obtains

$$\text{(B.8)}\qquad F = 2\pi^2 K\left\{[(n\cdot\Delta)^{-1}M_\varrho(r)]^2 - M_\varrho(r)\sum_{r'}G(r,r')M_\varrho(r')\right\}.$$

In the same way

$$G = 2\pi^2 K[(n\cdot\Delta)^{-1}M_\varrho(r)]^2.$$

Finally the generating functional equation (7) is obtained.

## Appendix C

We calculate the behaviour of the term $\sum_{r,r'} J_u(r)G(r,r')J_u(r')$.

Approximating the Wilson loop by a circular one [16] (fig. 5), in the continuum limit we have

$$\text{(C.1)}\qquad \oint_\Gamma\oint \frac{J_{\theta_1}(r_1)J_{\theta_2}(r_2)\,\mathrm{d}r_1\,\mathrm{d}r_2}{|r_1-r_2|} = \frac{1}{4}\oint_\Gamma\oint\frac{\cos(\theta_1-\theta_2)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2}{\sin^2((\theta_1-\theta_2)/2)},$$

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(15) P. Roman: *Theory of Elementary Particles* (Amsterdam, 1960), p. 57.
(16) E. Fradkin and L. Susskind: *Phys. Rev. D*, 17, 2637 (1978).

having written $J_{\theta_1}(r_1) = \exp[i\theta_1]$, $J_{\theta_2}(r_2) = \exp[-i\theta_2]$, $r_i = R\exp[i\theta_i]$, $i = 1, 2$, and $|r_1 - r_2|^2 = 2R^2(1-\cos(\theta_1-\theta_2))$. In order to preserve the convergence of the integral, we introduce a cut-off $a/R$.

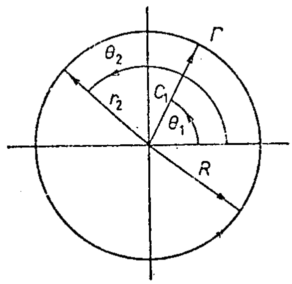


Fig. 5. – Wilson loop and integration variables.

The range of integration of $\theta_1$ is $a/R < \theta_1 \leqslant \theta_2 - a/R$, and $a/2R < \theta_2 \leqslant \pi - a/R$. In this way angles and angular differences are greater than $a/R$, so that

$$I(\theta_2) = \int \frac{\cos(\theta_1-\theta_2)\,\mathrm{d}\theta_1}{\sin^2\frac{1}{2}(\theta_1-\theta_2)} = \operatorname{ctg}\frac{a}{2R} + \frac{a}{R} + \operatorname{ctg}\left(\frac{a}{2R}-\frac{\theta_2}{2}\right) - \left(\frac{a}{R}-\theta_2\right),$$

and

$$2\int_{a/2R}^{\pi-a/R} I(\theta_2)\,\mathrm{d}\theta_2 \underset{a/R\to 0}{\approx} \frac{4\pi R}{a} + \ln\frac{2R}{a}. \tag{C.2}$$

Therefore eq. (9) holds.

## Appendix D

Here we want to show eq. (17).
From eqs. (4) and (16) with $L_\varrho = \delta^\perp_{\varrho,2}$ we express

$$\exp\left[2\pi i \sum_{r,r'} J_\mu(r) G(r,r')\, \Delta'_\nu m_{\mu\nu}(r')\right] = \exp[2\pi i(e_1 - e_2 - e_3 + e_4 + e_5 - e_6)], \tag{D.1}$$

where we have used eq. (B.7), and

$$e_1 = \sum_{r,r'} \Delta_\mu n_\nu L_\varrho(r)\, \Delta'_\mu n_\nu (n\cdot\Delta')^{-1} M_\varrho(r') G(r,r') = -\sum L_\varrho(r)(n\cdot\Delta)^{-1} M_\varrho(r),$$

$$e_2 = \sum_{r,r'} \Delta_\mu n_\nu L_\varrho(r)\, \Delta'_\mu n_\varrho (n\cdot\Delta')^{-1} M_\nu G(r,r') = 0, \quad \text{by } L_\varrho n_\varrho = 0 \text{ or } n_\nu M_\nu = 0,$$

$$e_3 = \sum_{r,r'} \Delta_\mu n_\nu L_\varrho(r)\, \Delta'_\nu n_\mu (n\cdot\Delta')^{-1} M_\varrho G(r,r') = \sum_{r,r'\in S_r} M_2(r')\, \Delta_1 G(r,r') = 0,$$

because $r$ must be on the plane 3-4 so that $\Delta_1 G(r, r') = 0$,

$$e_4 = \sum_{r,r'} \Delta_\mu n_\nu L_\varrho(r) \Delta'_\varrho n_\mu (n\cdot\Delta')^{-1} M_\nu(r') G(r, r') = 0\,, \quad \text{by } n_\nu M_\nu = 0 = M_1\,,$$

$$e_5 = \sum_{r,r'} \Delta_\mu n_\nu L_\varrho(r) \Delta'_\nu n_\varrho (n\cdot\Delta')^{-1} M_\mu(r') G(r, r') = 0\,, \quad \text{by } L_\varrho n_\varrho = 0\,,$$

$$e_6 = \sum_{r,r'} \Delta_\mu n_\nu L_\varrho(r) \Delta'_\varrho n_\nu (n\cdot\Delta')^{-1} M_\mu(r') G(r, r') = 0\,, \quad \text{by } M_1 = 0 \text{ and } \Delta_\mu M_\mu = 0\,.$$

Therefore the only term which survives is $e_1$ and eq. (17) holds.

## Appendix E

We calculate the term $\sum_{r,r'} p_\mu(r') G(r, r') J_\mu(r)$ for current loops of a definite charge $p$.

The continuum limit becomes

$$A = \oint_\Gamma \int \frac{p_\mu(r')\,\mathrm{d}r'\,\mathrm{d}r}{(r-r')^2}\,. \tag{E.1}$$

We remark that the component of $p_\mu$ which contributes is the one parallel to the Wilson loop. We calculate the effect of current loops in the case in which they are condensed and fill up the surfaee of the gauge loop. We will suppose they are rings and calculate the effect of those which are inside $\Gamma$ (fig. 6).

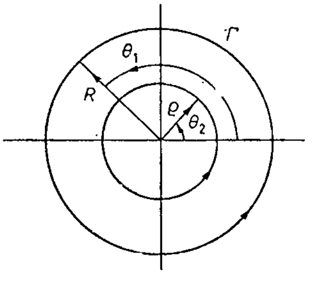


Fig. 6. – Electric loop inside $\Gamma$ and integration variables.

Calling $r = R\exp[i\theta_1]$, $r' = \varrho\exp[-i\theta_2]$, $J_{\theta_1} = \exp[i\theta_1]$, $p_{\theta_2} = p\exp[-i\theta_2]$, we have

$$A = pR\int \varrho\,\mathrm{d}\varrho \oint_{\Gamma}\oint_{\text{elec loop}} \frac{\cos(\theta_1 - \theta_2)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2}{R^2 + \varrho^2 - 2R\varrho\cos(\theta_1 - \theta_2)}\,, \tag{E.2}$$

where $R$ is the Wilson loop radius and $\varrho$ the current loop one. Keeping, as in appendix C, the integration range

$\frac{a}{R} < \theta_1 \leqslant \theta_2 - \frac{a}{R}$, $\frac{a}{2R} < \theta_2 \leqslant \pi - \frac{a}{R}$ and, integrating for $\theta_1$, $R > \varrho$, we have

$$A = 2pR\int \varrho\, d\varrho \int \left\{\frac{x}{b} - \frac{2d}{b\sqrt{d^2-b^2}} \operatorname{arctg}\left(\frac{\sqrt{d^2-b^2}}{d+b}\operatorname{tg}\frac{x}{2}\right)\right\}_{x=a/R}^{x=\theta_2-a/R} d\theta_2 \tag{E.3}$$

with $d = R^2 + \varrho^2$ and $b = -2\varrho R$.

A term we must compute is

$$B = \int_{a/2R}^{\pi-a/R} \operatorname{arctg}\left\{\frac{R+\varrho}{R-\varrho}\operatorname{tg}\left(\frac{\theta_2}{2}-\frac{a}{2R}\right)\right\} d\theta_2\,,$$

which does not diverge in the integration interval. Figure 7 shows that the integrand is a finite function in the considered range, so that $B$ is roughly $k\pi^2/2$ with $\frac{1}{2} \leqslant k < 1$.

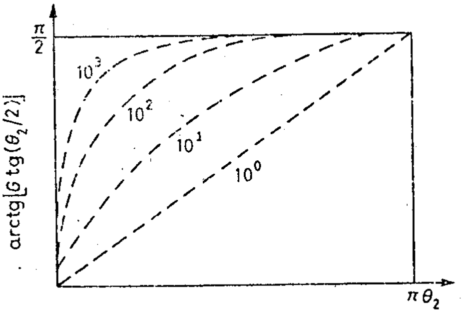


Fig. 7. – Arctg $[G\operatorname{tg}(\theta_2/2)]$ as a function of $\theta_2$ in the interval $0 \leqslant \theta_2 \leqslant \pi$, for different values of the constant $G$.

Then, as a rough estimate, the main contribution to $A$ comes from

$$D = \int_{\varrho=a}^{\varrho=R-a} d\varrho\, \frac{R^2+\varrho^2}{R^2-\varrho^2} \approx 2\operatorname{arctg}\left(1-\frac{a}{R}\right) \approx \ln\frac{2R}{a} \tag{E.4}$$

in the limit $a/R \to 0$.

From appendix C we know that, for a loop such that $\varrho = R$, the contribution is $\approx p(\pi R/a) + (p/4)\ln(2R/a)$, because to compute a loop $p_\mu$ of $\varrho = R$ is the same as

$$p\sum_{r,r'} J_\mu(r)G(r,r')J_\mu(r')\,.$$

Then our conclusion is that current loops of intensity $p$ enclosed by the Wilson loop will contribute at most to the perimeter behaviour with logarithmic corrections even when they are condensed, approximation which has been taken into account in the $\varrho$ integration.

## RIASSUNTO (*)

Si usa la formulazione lagrangiana della teoria di gauge $Z_N$ in quattro dimensioni e si dimostra la sua equivalenza con la funzione di partizione per un gas di Coulomb di cappi elettrici e magnetici. Si mostra che questo modello è autoduale quando lo si esprime nella forma di Villain. Si trova, come limite inferiore, che per $N < 4$ ci sono solo due fasi che sono in relazione attraverso la dualità mentre per $N > 4$ compaiono tre fasi. Il calcolo esplicito del cappio di Wilson li identifica come fase a confinamento elettrico, a non confinamento e a confinamento magnetico.

(*) *Traduzione a cura della Redazione.*

**Фазы $Z_N$ решеточной калибровочной теории.**

Резюме (*). — Мы используем Лагранжианную формулировку $Z_N$ калобровочной теории в четырех измерениях. Мы показываем эквивалентность этого подхода функции разделения для кулоновского газа электрических и магнитных петель. Показывается, что эта модель является самодуальной, когда записывается в форме Виллиана. Обнаружено, что на нижней границе, для $N < 4$, имеется только две фазы, связанные дуальностью, тогда как для $N > 4$ появляются три фазы. Явное вычисление петли Вильсона идентифицирует их, как электрическое удержание, неудержание и магнитное удержание.

(*) *Перввeдено редакцией.*