On the Non-linear Damping of Mechanical Oscillators in Flows of 4He

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Abstract In the studies of both classical and quantum turbulence, significant attention is devoted to the investigation of the behavior of various submerged resonators. Upon entering the turbulent regime, the oscillators start to experience a significant drag force, which varies non-linearly with velocity. We present an empirical way of modeling such systems, and calculate the expected resonant response of such oscillators near the *fundamental frequency* as a function of the applied driving force. We apply the model to the crossover from linear to non-linear drag forces and compare with previous models as well as selected experimental data on the transition to turbulence in oscillatory flow of ^{4}He.
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1 Introduction

The problem of laminar and turbulent flows and the associated drag and lift forces acting on submerged bodies may be regarded as one of the cornerstones of fluid dynamical research, dating back to the works of Carl Oseen, Claude-Louis Navier, Sir George Stokes and Sir Isaac Newton, among others. Today, it is known that the drag forces are highly sensitive to various types of flow instabilities, details of the geometry of the submerged body including surface roughness, and to the behavior of the fluid at boundaries. In the case of oscillatory flow, one also has to consider drag forces out-of-phase with the velocity and effects due to a finite viscous penetration depth. Obviously, one can hardly attempt to handle the underlying laws of physics behind all these complex phenomena. Therefore, we take a purely empir-

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ical stance in the hope that the results of our simple models will be of use to other investigators working in the areas of classical and quantum turbulence.

In this report, we investigate several ways of modeling these drag forces empirically based on experimental results obtained in both normal and superfluid ^{4}He, focusing on the crossover from linear to non-linear drag force, which is regarded indicative of the first instabilities occurring and of vortices appearing in the wake. We will revisit the experimental results and suggested forms of the drag force in superfluid ^{4}He obtained at low temperatures for a vibrating sphere [1,2], for tuning forks in normal and superfluid helium [3,4,5,6] and for a double paddle oscillator [7].

For the purposes of the discussion below, we specifically exclude from our considerations all hysteretic effects observed experimentally near the transition, focusing instead on the problems of describing the onset of non-linear drag, modeling the drag force in developed turbulence, and accounting for the observed frequency shifts upon entering the non-linear dissipation regime.

2 Models and results

In steady flow of classical fluids, the drag force consists of two main components: the viscous friction, which is linear with velocity and dominates at low Reynolds numbers, and the pressure drag, which is related to different flow patterns emerging upstream and downstream from the submerged body, i.e., the formation of a wake. The pressure drag is usually described as scaling with the velocity squared, which is a reasonable approximation at Reynolds numbers of order 10^3 or 10^4. Nevertheless, a finite pressure drag exists even before complex vortical structures emerge in developed turbulence and is not usually associated with a well-defined critical velocity.

In oscillatory flows, the situation is different as the body moves through its own wake periodically and the in-line force, $F_{\|}$, may be expected to have components not only in-phase with the object's velocity, but also with its acceleration, as can be seen from the Morison equation [8]:

$$F_{\|} = C_m \rho V \ddot{x} + \frac{1}{2} C_d \rho A \left|\dot{x}\right| \dot{x}, \tag{1}$$

where C_m, and C_d are the inertia and drag coefficients, respectively. V is the volume of the body, A its area perpendicular to the flow, ρ is the fluid density, the displacement of the body is given by x, and the dot represents differentiation with respect to time. The common practice is to assume that C_m and C_d are constant and we use this model to describe the forces acting in normal He I.

The equation of motion for a harmonic oscillator with a vacuum effective mass m_{eff}, spring constant k_0, driven by a force of amplitude F, which experiences only linear dissipation (incorporating both both viscous drag and internal friction) with a proportionality coefficient Γ, and further is subject to the forces given by the Morison equation (1), is of the form:

$$m_{\mathrm{eff}} \ddot{x} + C_m \rho V \ddot{x} + \frac{1}{2} C_d \rho A \left|\dot{x}\right| \dot{x} + \Gamma \dot{x} + k_0 x = F \cos(\omega t), \tag{2}$$

where the product $C_m \rho V$ in the second term is often referred to as the hydrodynamic added mass. This equation is of course difficult to solve, as the non-linear term introduces higher harmonics of the driving force frequency. Nevertheless, it is possible to examine the behavior at the *fundamental* frequency under the assumption that the Fourier components of the displacement at higher frequencies are significantly lower than at the fundamental. This assumption is well-justified for a high-Q resonator such as the tuning fork, whose natural harmonics occur at frequencies other than integral multiples of the fundamental[9,10]. Under these assumptions, we may average part of the non-linear term over one period to obtain:

$$\left\langle \frac{1}{2} C_d \rho A \left|\dot{x}\right| \right\rangle_T \dot{x} = \frac{1}{\pi} C_d \rho A \omega x_1 \, \dot{x}, \tag{3}$$

where x_1 represents the Fourier component of the displacement at the fundamental frequency, arriving at an effective drag force that varies in-phase with the velocity.

In pure superfluid ^{4}He in the zero temperature limit, turbulence can be created in the form of a tangle of quantized vortices. Depending on the experimental arrangement, this tangle may be created from a remnant vortex loop attached to the body, or from colliding vortex rings (such as those emitted by a moving grid). It is believed that the tangle may be created only when a certain critical velocity is reached, up to which point the superfluid either exhibits potential flow or perhaps a quasi-laminar flow may be established at some intermediate velocity. Bearing in mind that experimental results have been obtained suggesting the existence of two distinct critical velocities as well as various hysteretic phenomena at the onset of quantum turbulence, we consider these to be beyond the scope of this publication.

For the sake of clarity, we work with the scenario that the non-linear dissipation comes into effect at velocities exceeding a critical value, v_c, effectively replacing the non-linear Morison term. We use i) the quadratic expression suggested in Refs. 1,2 based on data from a microsphere at mK temperatures, and ii) a similar form with a smoother behavior near the critical velocity:

$$\varepsilon_1 H\left(\left|\dot{x}\right| - v_c\right)\left(\dot{x}^2 - v_c^2\right)\frac{\dot{x}}{\left|\dot{x}\right|} \tag{4}$$

$$\varepsilon_2 H\left(\left|\dot{x}\right| - v_c\right)\left(\dot{x} - v_c\right)^2\frac{\dot{x}}{\left|\dot{x}\right|} \tag{5}$$

where ε_1 and ε_2 are numerical constants and $H()$ represents the Heaviside step function. Upon averaging parts of the non-linear terms over one period as discussed above, we arrive at the respective expressions for effective drag:

$$\frac{2}{\pi}\varepsilon_1 v_c f_1(v_r)\,\dot{x};\;\; f_1(v_r) = \sqrt{v_r^2 - 1} - \frac{\operatorname{arccosh}(v_r)}{v_r} \tag{6}$$

$$\frac{2}{\pi}\varepsilon_2 v_c f_2(v_r)\,\dot{x};\;\; f_2(v_r) = \sqrt{v_r^2 - 1} + \frac{\operatorname{arccosh}(v_r)}{v_r} - 2\arccos\left(\frac{1}{v_r}\right), \tag{7}$$

where $v_r = \dot{x}/v_c$ is a rescaled dimensionless velocity. The expressions are valid only for $v_r > 1$, as the non-linear drag is only present for peak velocities above v_c.

In the following, we analyze the model suitable for classical fluids, consisting of Eq. (2) with an effective drag according to Eq. (3) and consider only motion at the fundamental frequency, $x = x_1\cos(\omega t + p)$. We eliminate the phase shift p, and arrive at a quartic equation for the amplitude of motion in the form:

$$\left[\omega^2\left(\frac{c_d \rho A \omega x_1}{\pi} + \gamma\right)^2 + \left(\omega^2 - \omega_0^2 + c_m \rho V \omega^2\right)^2\right] x_1^2 = f^2, \tag{8}$$

where $\omega_0^2 = k_0/m_{\mathrm{eff}}$, $c_d = C_d/m_{\mathrm{eff}}$, $c_m = C_m/m_{\mathrm{eff}}$, $\gamma = \Gamma/m_{\mathrm{eff}}$, and $f = F/m_{\mathrm{eff}}$.

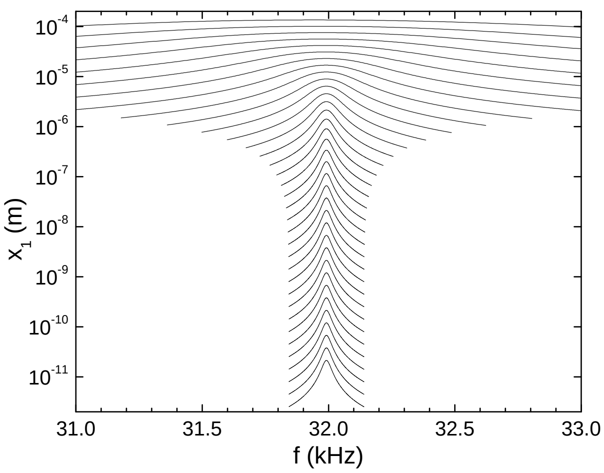

Fig. 1 Resonance responses of an oscillator with non-linear dissipation at various drives calculated using the model outlined in the text, considering Eq. (2) and the effective drag from Eq. (3) plotted versus the frequency, f. The model parameters include the dimensions of a 32 kHz tuning fork[3] and variables adjusted to match experimental data obtained in He I. A transition from linear to non-linear damping regime is clearly observed at amplitudes of order 10^{-7} m, where the responses become non-Lorentzian. Properties of these curves are further analyzed in Fig. 2.

From Eq. (8), we see that the effect of the non-linear drag is to introduce an amplitude-dependent dissipation, and that the term with c_m causes a *constant shift* in the resonance frequency. This quartic equation can be solved analytically, and two of its solutions (real-valued at complementary intervals of ω) will reconstruct the resonance curve of the non-linear oscillator, as shown in Fig. 1.

Examining the properties of these curves further and comparing them with experimental data on tuning fork A1 from Ref. 3 in Fig. 2, one finds that while the drag force is described rather well, it is no longer the case with the frequency shift in the non-linear regime. A constant inertia coefficient, C_m, within the model means that the only frequency shift possible is due to growing effective dissipation. This is, however, insufficient to describe the experimental data and we hypothesize that an additional shift in frequency will occur due to a part of the quadratic force acting in-phase with the acceleration, rather than with the velocity. This proposition is, however, beyond the scope of this publication and will be analyzed and tested elsewhere.

To investigate the behavior of oscillators submerged in He II, we will analyze a model with a well-defined threshold velocity, above which additional dissipation is likely to occur. For this purpose, we will again use Eq. (2), but with the terms (6) or (7) replacing the Morison term. In Fig. 4, we compare the solutions with results of experiments on bodies oscillating in the turbulent regime in superfluid ^{4}He. They include a 32 kHz tuning fork (see Fig. 3) in the low temperature limit[6], where the normal fluid is virtually non-existent, as well as at 1.32 K and 2.16 K (measured with fork L1 in Ref. 4), where the normal fractions ρ_N/ρ are around 5 % and 90 %, respectively. Data on a double paddle oscillator[7], with a

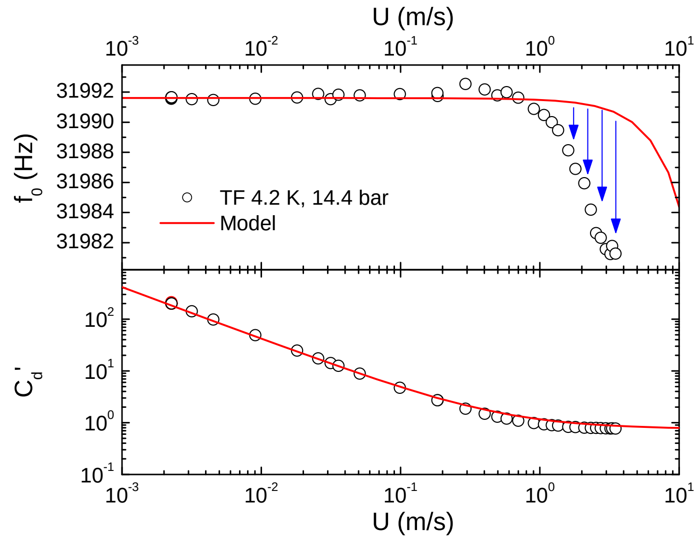

Fig. 2 (color online) Resonance frequencies, f_0, and drag coefficients as defined from experimentally accessible quantities, $C_d' = 2F/(\rho AU^2) = 2C_d/\pi$ obtained using a tuning fork in He I. In comparison, we plot the results of the model using Eq. (2) incorporating the effective drag from Eq. (3). While the dissipation-related drag coefficient is reproduced rather well, this model with a constant inertia coefficient, C_m, is clearly insufficient to describe the observed frequency shift upon entering the non-linear regime. Within the model, the same absolute shift, which is only due to increased dissipation, could only be obtained at much higher velocities.

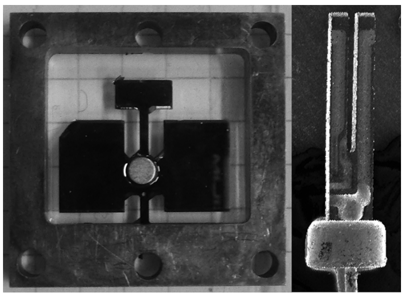

Fig. 3 Left: Photograph of the double paddle oscillator. Right: An electron micrograph of a 32 kHz tuning fork. The two images are not to scale – the size of the frame of the double paddle is 30 mm, whilst the length of the tuning fork tines is 3.7 mm.

resonance frequency of 350 Hz, are shown in the inset of the figure, at 1.55 K ($\rho_N/\rho = 30$ %) and 2.08 K ($\rho_N/\rho = 86.5$ %). We show both the drag coefficient C_d' in the upper panel and the ratio of the non-linear force to the total force defined as $(F_T - F_L)/F_T$ in the lower panel, where F_T is the force supplied to the oscillator by the excitation mechanism and F_L is an extrapolation of the force supplied in the low amplitude (linear) part of the amplitude versus excitation curves.

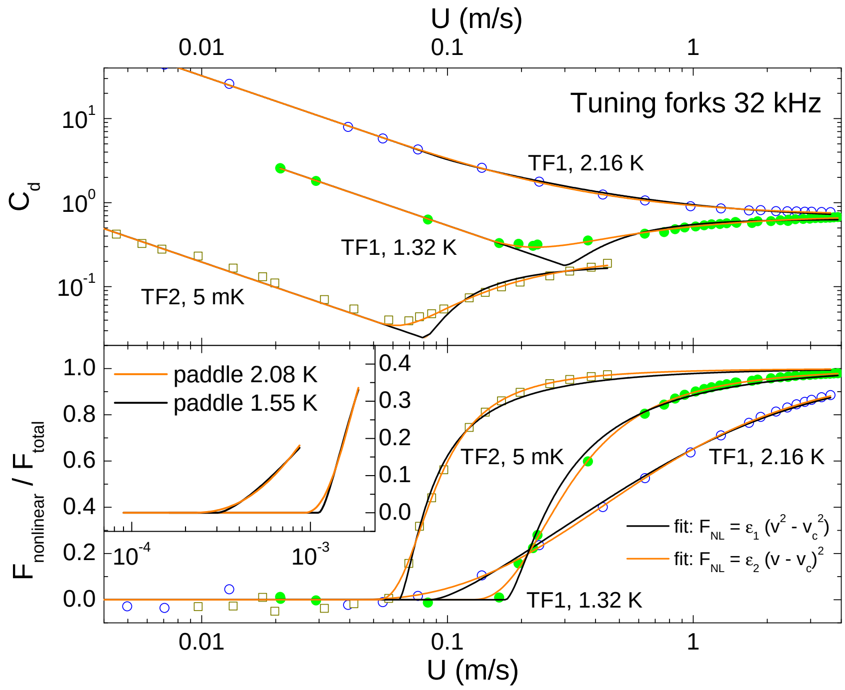

Fig. 4 (color online) Upper panel: Drag coefficient, C_d' for tuning forks vibrating in superfluid helium[4,6] at various temperatures, with fits to Eq. (2) with the term (7) in gray (orange) color and with the term (6) in black. It can be seen that the term (7) gives a slightly better fit for all the selected data ranging from the low temperature limit to temperatures near the lambda point. Lower panel: Non-linear part of the friction force, divided by the total force supplied to the oscillating structure, symbols correspond to the data on the upper panel. Inset: Data from the double paddle oscillator[7]. In Ref. 7 the data have been fitted using an expression that represents a cubic drag force. This well-describes the data over the accessible range of velocities, but upon comparison with the tuning fork data, the cubic law is not expected to hold at higher velocities. The critical velocities at different temperatures seem to decrease with the temperature in agreement with Ref. 4.

The agreement is found to be reasonable, and both models give an adequate description of the data. It is difficult to distinguish between the models based only on the analyzed data (without discussing the temperature dependence of the fitted parameters), but Eq. (2) with the term (7) proportional to $(v - v_c)^2$, seems to yield a better fit overall. In the drag coefficients, shown in the upper panel, the difference is even more apparent and the term (7) seems to provide better fits. A more thorough analysis is under way and we hope that its results will shed more light on the different forms of non-linear drag forces.

3 Conclusions

We have tested different expressions of the non-linear force which emerges when oscillating objects produce turbulence in classical and quantum fluids, with the aim of finding an empirical mathematical description that can guide the search for a physical model of the behavior of oscillators in the turbulent regime.

We have found that for classical fluids, the drag force is described rather well by a simple quadratic law, but that a refinement of our model is needed to account for the frequency shift observed in the non-linear regime.

In superfluid He II, it seems that the solution of the equation of motion with a non-linear force $F \propto (v - v_c)^2$, may give a slightly better fit to the drag coefficient, yielding the correct quadratic limiting behavior observed at high drives, but also giving a good fit over the intermediate regime. Although only representative temperatures are shown here, the fits are reasonably good over the entire temperature range, and cover oscillators of different shapes, sizes and frequencies. Bearing this in mind, it is still clear that the microsphere data discussed in Refs. 1,2 will be better described using the term (6) suggested therein. A more systematic study over a broader range of temperatures and the analysis of scaling of the fitting parameters with temperature and frequency would be useful and will be the subject of future research.

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