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11 Marzo 1969
Il Nuovo Cimento
Serie X, Vol. 60 B, pag. 8-20

C.N.E.A. Biblioteca
ARCHIVO PUBLICACIONES
Nº 1 AÑO 1969

01.69.03

An Idealized Model of a Supercurrent.

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(ricevuto il 25 Settembre 1967; manoscritto revisionato ricevuto il 20 Giugno 1968)

Summary. — It is shown that there exists a simple model which reproduces qualitatively the main characteristic features of a current flowing through a superconducting material. The model refers to a case of strong coupling which is not yet accessible to a full quantum-electrodynamical treatment, and it still ignores features depending on the intrinsic fluctuations of quantum fields.

1. – Introductory remarks.

The unprejudiced reader may take the model to be described below as a phenomenological tool, tending to summarize qualitatively the characteristic features revealed by experiments on currents in superconductors: their stability, their limited maximum value and the quantization of the magnetic flux they produce in a cylinder.

From the present author's point of view it has, however, a more specific meaning. It is meant as an introduction to a rediscussion of Faraday's experiment. In classical physics this experiment is considered to be one of the fundamental features of electrodynamics. Quantum mechanics tells us, however, more details about the behaviour of the magnet Faraday used. Indeed, if a magnetic atom is introduced into a magnetic field of a coil, it suffers a Zeeman effect, and its internal kinetic energy changes. The magnet in Faraday's experiment is by no means inert, it participates in the phenomenon just as the current in the coil does. From the point of view of present-day physics Faraday's experiment cannot be considered to be merely electromagnetic. It reveals a mechano-electromagnetic phenomenon the discussion of which requires, along with the knowledge of the theory of the Zeeman effect, a more precise knowledge of the properties of a current. The model which will be discussed below refers only to a partial aspect of the problem, to an aspect which may be called mechanomagnetic, and emphasizes the fact that we cannot speak about an « external magnetic field » without specifying the device which produces this field.

2. – The energy of a quasi-stationary current.

We consider n particles of charge ε which move in a very thin circular wire of radius ϱ. The position of the particles is determined by the components of the vector $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ in the n-dimensional configuration space. The moment of inertia of each particle shall be called $I = m \cdot \varrho^2$. The positive charges which neutralize the charge of our n particles may have fixed positions, uniformly or periodically distributed on the wire.

The energy of the system shall be assumed to be the sum of the kinetic and potential energy of the particles, $\frac{1}{2} I \sum_i \varphi_i^2 + V(\boldsymbol{\varphi})$, and of the magnetic energy of the current they form

$$\frac{1}{8\pi}\int H^2 \cdot \mathrm{d}\tau = \frac{1}{2} \cdot L \cdot J^2 ,$$

where L stands for the self-induction of the wire, and

$$J = \frac{\varepsilon}{2\pi} \sum_k \dot{\varphi}_k . \tag{1}$$

The energy of our system is then

$$E = \frac{1}{2} \cdot I \cdot \sum_i \dot{\varphi}_i^2 + \frac{1}{2} \cdot \frac{\varepsilon^2 L}{4\pi^2} \cdot \left(\sum_k \dot{\varphi}_k\right)^2 + V(\boldsymbol{\varphi}) . \tag{2}$$

$V(\boldsymbol{\varphi})$ is a static interaction potential between our particles and depends only on their relative distances. We assume $V(\boldsymbol{\varphi})$ to be a repulsive potential. According to our choice of the magnetic-energy expression, we have admitted only nonretarded magnetic fields. This choice excludes at once from our model any time-dependent radiation field and its quantum fluctuations even in the photon-free case.

The question of whether or not we have to include in (2) the energy of the magnetic field of each individual particle is irrelevant in our case. It leads only to a slight change of the meaning (renormalization) of what we have called the moment of inertia I.

The Lagrange function $\mathscr{L}$ of our system differs from (2) only by the sign of $V(\boldsymbol{\varphi})$ and does not need to be written explicitly.

Expressing canonical momenta by velocities and vice versa we obtain

$$p_i = I\cdot\dot\varphi_i + \frac{\varepsilon^2 L}{4\pi^2}\sum_k \dot\varphi_k\,, \tag{3}$$

$$I\cdot\dot\varphi_i = p_i - \frac{\varepsilon^2 L}{4\pi^2}\cdot\left(I + n\cdot\frac{\varepsilon^2 L}{4\pi^2}\right)^{-1}\cdot\left(\sum_k p_k\right). \tag{4}$$

From (3) and (4) we learn that

$$A_i = A = \frac{\varepsilon c L}{4\pi^2\varrho}\cdot\left(I + n\cdot\frac{\varepsilon^2 L}{4\pi^2}\right)^{-1}\cdot\left(\sum_k p_k\right) \tag{5}$$

has the character of a vector potential (in the φ-direction) acting on the i-th particle. Further, (4) exhibits already the action of induction in the quasi-stationary approximation: if a force acts on the *i-th* particle and modifies its momentum p_i it changes not only the velocity $\dot\varphi_i$ of this particle, all other particles feel the force and are moved slightly in the opposite direction.

Equations (1), (2) and (3) permit us now to establish the current and the Hamiltonian operator of our problem

$$J = \frac{\varepsilon}{2\pi}\cdot\left(I + n\cdot\frac{\varepsilon^2 L}{4\pi^2}\right)^{-1}\cdot\left(\sum_k p_k\right), \tag{6}$$

$$\mathscr{H} = \frac{1}{2I}\cdot\left\{\sum_i p_i^2 - \frac{\varepsilon^2 L}{4\pi^2}\cdot\left(I + n\cdot\frac{\varepsilon^2 L}{4\pi^2}\right)^{-1}\cdot\left(\sum_k p_k\right)^2\right\} + V(\boldsymbol{\varphi})\,. \tag{7}$$

Equation (7) looks more familiar if, using (5), we write

$$\mathscr{H} = \frac{1}{2I}\cdot\sum_i\left(p_i - \varrho\cdot\frac{\varepsilon}{c}\cdot A_i\right)^2 + \frac{1}{2}\cdot\frac{4\pi^2\varrho^2}{c^2 L}\cdot A^2 + V(\boldsymbol{\varphi})\,. \tag{8}$$

3. – Relative co-ordinates.

For part of our discussion it will be more convenient to use relative co-ordinates

$$\xi_r = \sum_k \alpha_r^k\cdot\varphi_k\,, \qquad \varphi_k = \sum_r \alpha_r^k\,\xi_r\,. \tag{9}$$

We require the α_r^k to represent an orthogonal transformation and we choose in particular $\alpha_1^k = 1/\sqrt{n}$. This still leaves an orthogonal transformation in the subspace $r = 2, 3, \ldots, n$ undetermined, which for the moment, needs not be specified. Our choice makes

$$\xi_1 = \frac{1}{\sqrt{n}} \cdot \sum_k \varphi_k \tag{10}$$

the « centre of gravity » or, as we have to call it here, the current co-ordinate. The remaining $n-1$ variables refer to internal relative motions of the particles.

In the variables $\boldsymbol{\xi}$ the energy expression becomes

$$E = \sum_{r=2}^{n}{}' \frac{1}{2} \cdot I \cdot \dot{\xi}_r^2 + \frac{1}{2}\left(I + n \cdot \frac{\varepsilon^2 L}{4\pi^2}\right) \cdot \dot{\xi}_1^2 + V(\boldsymbol{\xi}), \tag{11}$$

where we use the summation sign $\sum'$ for summation in the subspace $r \neq 1$.

From (11) we find the canonical momenta

$$\pi_1 = \left(I + n \cdot \frac{\varepsilon^2 L}{4\pi^2}\right) \cdot \dot{\xi}_1^2; \qquad \pi_{r\neq 1} = I \cdot \dot{\xi}_r$$

and the current and Hamilton operators

$$J = \frac{\varepsilon\sqrt{n}}{2\pi} \cdot \left(I + n \cdot \frac{\varepsilon^2 L}{4\pi^2}\right)^{-1} \cdot \pi_1 \tag{12}$$

$$\mathscr{H} = \frac{1}{2I} \cdot \sum_r{}' \pi_r^2 + \frac{1}{2} \cdot \left(I + n \cdot \frac{\varepsilon^2 L}{4\pi^2}\right)^{-1} \cdot \pi_1^2 + V(\boldsymbol{\xi}) . \tag{13}$$

Though expressions (12) and (13) seem to be simpler than the corresponding expressions (6) and (7), they are not suitable to be solved directly, because their eigenfunctions do not obey simple periodicity conditions in the variables $\boldsymbol{\xi}$.

4. – Solution of the idealized case.

We shall call the idealized model the one, in which we put

$$V(\boldsymbol{\varphi}) = 0 . \tag{14}$$

The reasons for this—from the point of view of physics rather unrealistic—assumption are the following:

a) the idealized model can be solved rigorously,

b) it contains already the main characteristic features of of our problem,

c) it permits us, afterwards, to study the modifications which are introduced by a nonvanishing potential energy.

The solution of (6) and (7) with (14) is immediate. It is given by the Schrödinger function

$$\Psi_m = \exp[i \cdot \boldsymbol{m} \cdot \boldsymbol{\varphi}] , \tag{15}$$

where the components of the quantum-number vector are restricted by the periodicity condition to integer values

$$\boldsymbol{m} = (m_1, m_2, \ldots, m_n) , \qquad m_i = 0, \pm 1, \pm 2 \ldots . \tag{16}$$

(6) and (7) with (14) and (15) furnish immediately the eigenvalues of current and energy of our system, as functions of the quantum-number vector $\boldsymbol{m}$:

$$J_m = \frac{\varepsilon \hbar}{2\pi} \cdot \left(I + n \cdot \frac{\varepsilon^2 L}{4\pi^2} \right)^{-1} \cdot \left(\sum_k m_k \right) , \tag{17}$$

$$E_m = \frac{\hbar^2}{2I} \cdot \left\{ \sum_k (m_k)^2 - \frac{\varepsilon^2 L}{4\pi^2} \cdot \left(I + n \cdot \frac{\varepsilon^2 L}{4\pi^2} \right)^{-1} \cdot \left(\sum_k m_k \right)^2 \right\} . \tag{18}$$

Transforming, now, our expressions to the co-ordinates $\boldsymbol{\xi}$ in (9) we obtain

$$\Psi_\mu = \exp[i \cdot \boldsymbol{\mu} \cdot \boldsymbol{\xi}] , \tag{19}$$

where the quantum-number vector $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$ has the components

$$\mu_r = \sum_k \alpha_r^k \cdot m_k ; \qquad \mu_1 = \frac{1}{\sqrt{n}} \cdot \sum_k m_k .$$

The components μ_r, however, have no longer integral values and depend on the particular choice of the co-ordinate system. They permit us to write the eigenvalues of current and energy in the form

$$J_\mu = \frac{\varepsilon \sqrt{n} \hbar}{2\pi} \cdot \left(I + n \cdot \frac{\varepsilon^2 L}{4\pi^2} \right)^{-1} \cdot \mu_1 , \tag{20}$$

$$E_\mu = \frac{\hbar^2}{2I} \cdot \sum_r{}' (\mu_r)^2 + \frac{1}{2} \cdot \hbar^2 \cdot \left(I + n \cdot \frac{\varepsilon^2 L}{4\pi^2} \right)^{-1} \cdot (\mu_1)^2 . \tag{21}$$

5. – The principal values of the quantum-number vector.

Among the many quantum number vectors permitted by (16) we shall consider separately those of which the components are all equal among themselves,

$$\boldsymbol{M} = (M, M, \ldots, M) . \tag{22}$$

We shall call them principal quantum-number vectors. They are characterized by one single integer M. The corresponding eigenvalues of current and energy shall be called principal eigenvalues. We observe, that the current variable (10) has a direction parallel to the principal quantum number vectors, so that in the co-ordinate system $\boldsymbol{\xi}$ a principal vector $\boldsymbol{M}$ assumes the components

$$\boldsymbol{M} = (\sqrt{n}\cdot M, 0, \dots, 0)\,. \tag{23}$$

The principal eigenvalues of current and energy become

$$J_M = M\cdot n\cdot\frac{\varepsilon\hbar}{2\pi}\cdot\left(I + n\cdot\frac{\varepsilon^2 L}{4\pi^2}\right)^{-1}, \tag{24}$$

$$E_M = \frac{1}{2}\cdot M^2\cdot n\cdot\hbar^2\cdot\left(I + n\cdot\frac{\varepsilon^2 L}{4\pi^2}\right)^{-1}. \tag{25}$$

We have, now, to compare the principal energy eigenvalues, (25), with the eigenvalues of the many nonprincipal neighbouring states (18). Let $\boldsymbol{s}$ be an arbitrary, nonprincipal quantum-number vector and let us form the energy difference

$$\Delta E(\boldsymbol{M}, \boldsymbol{s}) = E_{\boldsymbol{M}+\boldsymbol{s}} - E_{\boldsymbol{M}}\,. \tag{26}$$

We find

$$\Delta E(\boldsymbol{M}, \boldsymbol{s}) = \frac{\hbar^2}{2I}\cdot\left\{\sum_k (s_k)^2 - \frac{\varepsilon^2 L}{4\pi^2}\cdot\left(I + n\cdot\frac{\varepsilon^2 L}{4\pi^2}\right)^{-1}\cdot\left(\sum_k s_k\right)^2\right\} + M\cdot\hbar^2\cdot\left(I + n\cdot\frac{\varepsilon^2 L}{4\pi^2}\right)^{-1}\cdot\left(\sum_k s_k\right). \tag{27}$$

We shall call a state $\boldsymbol{M}$ energetically stable against a transition $\boldsymbol{s}$ if

$$\Delta E(\boldsymbol{M}, \boldsymbol{s}) > 0\,. \tag{28}$$

Before we can proceed, however, to the discussion of the stability condition, we have to mention that our solution (15) is symmetric in all particle co-ordinates in the case of a principal state $\boldsymbol{M}$. Our formulae apply, therefore, immediately to Bose particles and to strongly bound systems which behave like molecules of a Bose gas. They apply, however, also to certain states of a Fermi gas, if we understand our co-ordinates to refer to the centers of gravity of appropriately chosen pairs (Cooper pairs), even if the partners of those pairs are only weakly bound together or not bound at all (*).

(*) See: A. BOHR and B. R. MOTTELSON: *Phys. Rev.*, **125**, 495 (1962).

6. – The stability condition.

By means of the inequality

$$n\cdot\sum_k (s_k)^2 \geqslant \Big(\sum_k s_k\Big)^2 \tag{29}$$

one finds easily that the first term on the right-hand side of (27) is always positive. Supposing $M > 0$ it is, therefore, a sufficient condition for stability that

$$\sum_k s_k > 0\,. \tag{30}$$

Instability can occur only if

$$\sum_k s_k < 0\,. \tag{31}$$

In order to obtain stability in the case (31), the condition

$$M < -\frac{1}{2}\cdot\left\{\left(1 + n\cdot\frac{\varepsilon^2 L}{4\pi^2 I}\right)\cdot\sum_k (s_k)^2 - \frac{\varepsilon^2 L}{4\pi^2 I}\cdot\Big(\sum_k s_k\Big)^2\right\}\cdot\Big(\sum_k s_k\Big)^{-1} \tag{32}$$

has to be satisfied.

In order to see the meaning of (32), let us consider the case

$$s_1 = s_2 = \dots = s_{n'} = -f\,,\qquad s_{n'+1} = \dots = s_n = 0\,.$$

This choice corresponds to the case that n' of our particles make individual transitions, changing their quantum numbers by $-f$, while the remaining particles are not affected. (32) furnishes, in this case

$$M < \frac{1}{2}\cdot f\cdot\left\{1 + (n - n')\cdot\frac{\varepsilon^2 L}{4\pi^2 I}\right\}\,.$$

If $n' \ll n$ and if $1 \ll n\cdot\varepsilon^2\cdot L/(4\pi^2\cdot I)$, as is the case of superconductors, there exists a wide range of M-values and of principal states which are stable with respect to transitions to neighbouring states, even if those states belong to lower current values, according to (17) and (31). This feature disappears, as we can see from (32), as soon as we omit the terms due to induction.

Only if we consider transitions for which n' is comparable to n do we find again instability. Such transitions require processes in which most of the particles are affected simultaneously. In thermal motion, where individual processes occur independently, those transitions are extremely improbable. They occur, however, in macroscopic phenomena, when an external field acts simultaneously on all particles.

We consider, finally, our processes in the co-ordinates $\boldsymbol{\xi}$. According to (23) a principal state is characterized by the fact that all relative motions of the particles are in the ground state. A transition $\boldsymbol{s}$ leads, in general, to states in which the relative motions become excited. Our stability condition means that, in the case considered, the excitation of the relative motions costs more energy than can be gained by lowering the current. As can be seen from (13), the energy of the relative motions depends on the low mechanical moment of inertia I, while the current state is determined by a very much increased moment of inertia $I + n\cdot(\varepsilon^2 L/4\pi^2)$ the origin of which is mainly due to induction.

7. – The static interaction between the particles.

Before being able to compare our results with the experimental evidence, we have still to make sure that the features which we have found do not disappear if we take the interaction $V(\boldsymbol{\varphi})$ in (8) and (13) into account.

We suppose that the interaction $V(\boldsymbol{\xi}) = V(\xi_{r\neq 1})$ is mainly a screened Coulomb repulsion. It depends only on the relative co-ordinates and represents, written in the φ's, a periodic function on the circle. If the repulsion is strong enough, it will fix the n particles at equidistant points of the circle, but it will permit them to perform oscillations of small amplitude and correspondingly high frequency around their relative rest positions. This will be the plasma vibrations of our system. The current variable (10), however, will not be affected by the interaction. The whole, more or less rigid electron system can still flow, as a whole, on the ring.

The eigenfunctions of (7) and (13) have to be periodic functions in φ. Since the interaction does not affect the ξ_1-dependence of (19), periodicity requires the eigenfunctions to be of the form

$$\Psi_{\mu} = v_{\mu}(\xi_2, \xi_3, \ldots, \xi_n)\cdot\exp\left[i\cdot\boldsymbol{\mu}\cdot\boldsymbol{\xi}\right], \tag{33}$$

where the components μ_r are the same as the ones given by (16) and (19). The functions $v_{\mu}(\xi_{r\neq 1})$ are periodic in φ. In absence of interaction the functions v_{μ} tend to a constant value, in the limit of strong repulsion they tend to a product of Hermite polynomials, corresponding to the $n-1$ plasma oscillations of our system.

It is now easy to see, that the eigenvalues of the current, (20) and (24), are not changed by the interaction, and the eigenvalues (25) of the principal states only are shifted by a constant value, *i.e.* by the zero-point energy of the relative motions. Principal states are, therefore, not essentially altered by the interaction. The energy differences between neighbouring states of the relative motions become increased, for strong repulsive interactions even greatly increased. This increase, however, enhances our stability condition (32); it tends to make the range of stable M values even wider. Still, our lack of knowledge on the interaction energy $V(\varphi)$ bars us from making quantitative statements on the number of stable principal states.

The intervention of the interaction is also required from another point of view. Since the number n can be very appreciable, and since the energy spacing of the levels of relative motions is very dense in the case of a macroscopic ring, relative motions, according to our idealized model, would give a sensible contribution to the specified heat of the system, following the law of a free Bose gas. Experimental evidence hardly favours this point of view. We have, therefore, rather to assume, that the interaction impedes such a contribution to the specific heat.

8. – Flux quantization.

As we have seen, our model accounts for the existence of stable currents in a superconductor by its principal states. The model tells us, in addition, that there should, in a superconductor, exist other, intermediary, currents which are not stable, but can become decreased or increased by microscopic thermal processes. This result may, eventually, one day become verifiable.

The most direct comparison with experiment refers to the current eigenvalues (24). Supposing,

$$n\cdot\frac{\varepsilon^2 L}{4\pi^2}\gg I \tag{34}$$

(24) becomes

$$J_M = M\cdot\frac{2\pi\hbar}{\varepsilon L}\,. \tag{35}$$

For a cylinder of radius ϱ and length l we have $L=4\pi^2\varrho^2/l\cdot c^2$ and $H_M = 4\pi\cdot J_M/(lc)$ and therefore

$$\pi\varrho^2 H_M = M\cdot\frac{2\pi\hbar c}{\varepsilon}\,. \tag{36}$$

Equation (36) coincides with the experimental values found by DEAVER and FAIRBANKS and by DOLL and NÄHBAUER if we put

$$\varepsilon = 2\cdot e\,. \tag{37}$$

We have, therefore, to identify our n particles with the electron pairs which have been assumed first by SCHAFROTH as forming a condensing Bose gas and which are considered by BARDEEN, COOPER and SCHRIFFER to be bound together by lattice forces.

Our formulae (24) and (25) can be, therefore, considered to be generalizations of the empirical formulae (36) and (37).

9. – External magnetic field.

Before leaving the subject we shall briefly have a look at the mechanism according to which our model reacts if we place it in an external magnetic field, which may be determined by its vector potential $\mathfrak{A}_\varphi(\varrho)$.

Applying an external magnetic field means to consider a system which consists of our ring of kinetic energy T_1 and magnetic field H_1 and of a second device of kinetic energy T_2 and magnetic field H_2. H_2 represents, then, the external field applied to our ring. The total energy of the system becomes

$$E = T_1 + \frac{1}{8\pi}\int H_1^2 \cdot \mathrm{d}\tau + \frac{1}{4\pi}\int \boldsymbol{H}_1 \cdot \boldsymbol{H}_2 \cdot \mathrm{d}\tau + \frac{1}{8\pi}\int H_2^2 \cdot \mathrm{d}\tau + T_2 . \tag{38}$$

If the second device is taken to be another circuit, the field integrals can be evaluated in nonretarded approximation and furnish

$$E = T_1 + \tfrac{1}{2} \cdot L_{11} \cdot J_1^2 + L_{12} \cdot J_1 \cdot J_2 + \tfrac{1}{2} \cdot L_{22} \cdot J_2^2 + T_2 , \tag{39}$$

where L_{12} represents the mutual induction of the two circuits. This leads us to a problem of the same kind as the one considered above and can be treated in a similar way.

We shall, however, assume that the second device is an « ideal, rigid magnet », which does not change its magnetic moment during the operations which we will have to perform. An electron spin in a nonrelativistic approximation oran atomic system with magnetic moment in a weak field represent such « ideal, rigid magnets ».

In the case considered

$$\frac{1}{8\pi}\int H_2^2 \cdot \mathrm{d}\tau = \text{const}$$

is the self-energy of the magnetic field of the magnet, an irrelevant constant which can be left out of consideration.

$$\frac{1}{4\pi}\int \boldsymbol{H}_1 \cdot \boldsymbol{H}_2 \cdot \mathrm{d}\tau = \int \boldsymbol{A}_1 \cdot \boldsymbol{j}_2 \cdot \mathrm{d}\tau = \boldsymbol{\mu} \cdot \boldsymbol{H}_1(r_2)$$

is the superposition energy between the two magnetic fields, while

$$T_2 = -\boldsymbol{\mu}\cdot\boldsymbol{H}_1(r_2)$$

is the kinetic Zeeman energy of our magnet of moment $\boldsymbol{\mu}$ in the field $\boldsymbol{H}_1$ produced at the place $\boldsymbol{r}_2$ of the magnet, produced by the current J_1 flowing in our ring.

The fact, that—in our approximation—the terms T_2 and $(1/4\pi)\int \boldsymbol{H}_1\cdot\boldsymbol{H}_2\cdot \mathrm{d}\tau$ cancel is of decisive importance for our way of speaking about external magnetic fields. We cannot say, if we move a magnet with respect to a circuit, that the magnet is inert, that it does not participate in the dynamics of the phenomenon. We can only say that its participation does not produce any change of energy, or rather, that its energy change is of higher order and can be neglected.

Since

$$\frac{1}{8\pi}\int H_1^2\cdot \mathrm{d}\tau = \frac{1}{2}\cdot L\cdot J_1^2\,,$$

the total Hamiltonian, which replaces (7) and (8) becomes now

$$(7')\qquad \mathscr{H}' = \frac{1}{2I}\cdot\left\{\sum_i \left(p_i - \frac{\varepsilon}{c}\varrho\mathfrak{A}\right)^2 - \frac{\varepsilon^2 L}{4\pi^2}\cdot\left(I + n\cdot\frac{\varepsilon^2 L}{4\pi^2}\right)^{-1}\cdot\left(\sum_k\left(p_k - \frac{\varepsilon}{c}\varrho\mathfrak{A}\right)\right)^2\right\} + V(\boldsymbol{\varphi})$$

or

$$(8')\qquad \mathscr{H}' = \frac{1}{2I}\cdot\sum_i\left(p_i - \frac{\varepsilon}{c}\varrho\mathfrak{A} - \frac{\varepsilon}{c}\varrho A\right)^2 + \frac{1}{2}\cdot\frac{4\pi^2\varrho^2}{c^2 L}\cdot A^2 + V(\varphi)\,.$$

Comparing (8′) with (8) one sees that the influence of the external magnet appears only as an additional external field $\mathfrak{A}$ in the kinetic energy term of the Hamiltonian operator, while the magnetic energy refers only to the internal field A.

The expression in the form (7′) can be solved immediately. The eigenfunctions (15), (19) and (33) remain unaffected by the field $\mathfrak{A}$. What changes are the eigenvalues of current and energy. The change of the eigenvalues consists simply in the replacements

$$(40)\qquad m_k \to m_k - N_k\,,$$

with

$$(41)\qquad N_k = \frac{\varepsilon}{\hbar c}\varrho\mathfrak{A}\,,$$

where the N_k can now vary continuously and need no longer to be whole numbers. The N_k form—in the simple case we consider—a principal vector. The eigenvalues of current and energy become

$$J'_m = J_{m-N} = J_m - J_N \,, \tag{42}$$

$$E'_m = E_{m-N} \,. \tag{43}$$

J_N in (42) represents, simply, the induction current induced in our circuit by the field $\mathfrak{A}$ of the magnet. Our circuit exhibits, therefore, the diamagnetic behaviour expected by LONDON.

If we place our ring in a magnetic field $\mathfrak{A}$ and if we suppose it to be currentless, we obtain from (42) $m_k = N_k$. The principal state $\boldsymbol{M} = \boldsymbol{N}$ is, therefore, already prepared, but its current is cancelled by the corresponding induction current. If we remove adiabatically the field $\mathfrak{A}$ the supercurrent J_M appears.

10. – Conclusions.

The simple model we have studied above represents a straightforward extension of quantum mechanics, though it refers largely to field phenomena. The Schrödinger function (15) describes in our case magnetic field energy, just as it describes in an atom electrostatic energy. Still it is somewhat amazing to see how it describes strong stationary magnetic fields which interact strongly with the particles. It represents a principal current as a mechanical system of large, mainly inductive mass.

As a schematic model to be compared with known phenomena if may be sufficient. It is, however, far from being satisfactory from the point of view of the consistency of the picture we apply. It does not furnish us any information about nonadiabatic processes which involve the radiation field which we have omitted. Contrarily to the atomic case, the transitions between the states we have found refer to strong changes of the magnetic field, and it is not to be expected that those changes can be described as simply as in the atomic case. We should like to see the complete state vector of the system which describes the strong deformation of the vacuum state of the field. The Schrödinger function represents only one part of this state vector.

The transition from the full Maxwell theory to the elementary theory of quasi-stationary phenomena is a very delicate one, even in classical theory, as we know from antenna theory. In quantum theory this transition becomes even more complex. The methods we know to solve the equations of quantum electrodynamics do not apply to our strong-coupling case. We may expect, that even in very slow changes photons of necessarily very low frequency and of very little energy, but possibly in very large number will participate. This can already be seen in the infra-red difficulties for a single electron.

As long as these difficulties are not overcome, we can hardly claim to be able to describe with the desirable precision what happens if we move a magnet slowly in the neigbourhood of a coil.

RIASSUNTO (*)

Si dimostra l'esistenza di un semplice modello che riproduce qualitativamente i principali aspetti caratteristici di una corrente che fluisce attraverso un materiale superconduttore. Il modello si riferisce ad un caso di accoppiamento forte non ancora suscettibile di un completo trattamento quantoelettrodinamico e che ancora non presenta acratteristiche dipendenti dalle fluttuazioni intrinseche dei campi quantici.

(*) *Traduzione a cura della Redazione.*

Идеализироваяная модель тока в сверхпроводнике.

Резюме (*) — Показывается, что существует простая модель, которая воспроизводит качественно осиовные характеристические особеиности тока, текчщего через сверхпроводящий материал. Модель относится к случаю сильной связи, которая является еще недоступной в полной трактовке квантовой электродинамике, и модель все же игнорирует свойства, зависащие от собственных флуктуаций квантовых полей.

(*) *Переведено редакцией.*

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11 Marzo 1969
Il Nuovo Cimento
Serie X, Vol. 60 B, pag. 8-20