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Moderation of Neutrons by Elastic Collisions in a Heavy Monatomic Gas, allowing for Thermal Agitation

By Juan U. Koppel*

During the process of neutron moderation, the mean logarithmic loss of energy due to collisions decreases as the neutrons drop to energies comparable to the thermal energy of the moderating nuclei. This loss of energy continues until it falls to zero and the neutrons achieve thermodynamic equilibrium with the moderating nuclei.

The purpose of this paper is to investigate the range of energies of the moderating process using a monatomic gas of high atomic weight. This is a relatively simple case because it is not necessary to allow for the chemical bonds of the scattering nuclei, and also because simplifying approximations are applicable. The same case was reviewed by Wigner and Wilkins,[1,2] who computed the $n(E)$ spectrum for absorption according to a $1/v$ law. In this paper, we are describing the thermalization process *per se*, independently of the $\Sigma_a(E)$ function, but under the assumption that the following condition is fulfilled:

$$\Sigma_a/\xi\Sigma_s \ll 1 \tag{1}$$

which is characteristic of all good moderators.

It has proved to be quite useful, in order to attain the end envisioned, to use the length x covered in a broken line by a neutron from the point of its release. The first problem which arises is that of finding the spectral distribution of all neutrons which have covered this same distance x.

Let us suppose we are dealing with an infinite medium with a uniformly distributed source, not considering spatial dependency.

Let us call $n(x, E)\,dxdE$ the number of neutrons per unit volume, the energy of which is between E and $E+dE$, the length of travel of which is somewhere between x and $x+dx$. Then let us write:

$$q(x, E)\,dxdE = n(x, E)v\,dxdE \tag{2}$$

for the corresponding flux. $q(x, E)\,dE$ will thus be the number of neutrons per unit volume having an energy comprised between E and $E+dE$, which pass, in each unit of time, through such a point of their trajectory that their broken-line travel to this point is x. On the other hand, $n(x, E)\,dE$ is the "density" of the distribution along an axis x, of the neutrons having an energy of between E and $E+dE$.

Let us now try to determine the magnitudes so defined. To this end, we can consider that $q(x, E)$ will include only neutrons which have suffered at least one collision, so we can start from the following equilibrium equation:

$$\frac{\partial q(x, E)}{\partial x} = -q(x, E)[\Sigma_s(E)+\Sigma_a(E)] + N\int_0^\infty q(x, E')\sigma(E'\to E)\,dE' + Q\exp[-\Sigma_s(E_0)x - \Sigma_a(E_0)x]N\sigma(E_0\to E).$$

The last term on the right of this equation corresponds to the source neutrons having energy E_0, which have not suffered any collision as yet. This term is different from zero only for energies very close to E_0. Assuming $E_0 \gg kT$, we can leave it out a few mean paths from the source.

Now let us write:

$$q(x, E) = M(E)\phi(x, E) \tag{4}$$

with:

$$M(E) = E\exp[-E/kT] \text{ (Maxwellian distribution).} \tag{5}$$

Therefore:

$$M(E)\frac{\partial\phi(x, E)}{\partial x} = -M(E)\phi(x, E)[\Sigma_s(E)+\Sigma_a(E)] + N\int_0^\infty M(E')\phi(x, E')\sigma(E'\to E)\,dE'. \tag{6}$$

In view of the principle of partial balance, since $M(E)$ is the equilibrium distribution of the neutrons in a moderator at an absolute temperature T, we can write:

$$M(E')\sigma(E'\to E) = M(E)\sigma(E\to E') \tag{7}$$

from which:

$$\frac{\partial\phi(x, E)}{\partial x} = -\phi(x, E)[\Sigma_s(E)+\Sigma_a(E)] + N\int_0^\infty \phi(x, E')\sigma(E\to E')\,dE'. \tag{8}$$

Let us now take the Laplacian transform of this

* Comisión Nacional de Energía Atómica.

equation with respect to x, allowing for $\phi(0, E)=0$:

$$s\bar{\phi}(s, E) = -\bar{\phi}(s, E)[\Sigma_s(E)+\Sigma_a(E)] + N\int_0^\infty \bar{\phi}(s, E')\sigma(E\to E')\,dE'. \qquad (9)$$

If, in the integral found in this expression, we develop function $\bar{\phi}(s, E')$ as a Taylor series, we are left (leaving out the terms of order greater than the second) with:

$$\bar{\phi}(s, E)[s+\Sigma_a(E)+\Sigma_s(E)] = \bar{\phi}(s, E)\Sigma_s(E) - \frac{\partial\bar{\phi}(s, E)}{\partial E}N\int_0^\infty (E-E')\sigma(E\to E')\,dE' + \frac{\partial^2\bar{\phi}(s, E)}{\partial E^2}\frac{N}{2}\int_0^\infty (E-E')^2\sigma(E\to E')\,dE'. \qquad (10)$$

But when the atomic weight of the moderator is high, we can prove[3] that the following apply:

$$N\int_0^\infty (E-E')\sigma(E\to E')\,dE' \simeq \xi\Sigma_s(E-2kT), \qquad (11)$$

$$N\int_0^\infty (E-E')^2\sigma(E\to E')\,dE' \simeq \xi\Sigma_s E 2kT. \qquad (12)$$

Replacing in (10):

$$[s+\Sigma_a(E)]\bar{\phi}(s, E) + \frac{\partial\bar{\phi}(s,E)}{\partial E}\xi\Sigma_s(E-2kT) - \frac{\partial^2\bar{\phi}(s,E)}{\partial E^2}\xi\Sigma_s EkT = 0. \qquad (13)$$

Making the change of variable: $\epsilon = E/kT$, (14), we get finally the equation:

$$\epsilon\frac{\partial^2\bar{\phi}(s,\epsilon)}{\partial\epsilon^2} + (2-\epsilon)\frac{\partial\bar{\phi}(s,\epsilon)}{\partial\epsilon} - \frac{s+\Sigma_a(\epsilon)}{\xi\Sigma_s}\bar{\phi}(s, \epsilon) = 0. \qquad (15)$$

For Σ_s and Σ_a constant, and calling:

$$\beta = \frac{s+\Sigma_a}{\xi\Sigma_s}, \qquad (16)$$

we have the solution:

$$\bar{\phi}(s, \epsilon) = c(s)F(\beta, 2, \epsilon), \qquad (17)$$

Where:

$$F(\beta, 2, \epsilon) = 1 + \frac{\beta}{2}\epsilon + \frac{\beta(\beta+1)}{2.3}\cdot\frac{\epsilon^2}{2!} + \frac{\beta(\beta+1)(\beta+2)}{2.3.4}\cdot\frac{\epsilon^3}{3!} + \dots \qquad (18)$$

is the confluent hypergeometric series having parameters β and 2.

Therefore, where $\bar{q}(s, E) = M(\epsilon)\bar{\phi}(s, \epsilon)$ is the transform of $q(x, E)$:

$$\bar{q}(s, \epsilon) = c(s)M(\epsilon)F(\beta, 2, \epsilon). \qquad (19)$$

The parameter $c(s)$ can be determined by the condition:

$$\int_0^{\epsilon_0} q(x, \epsilon)\,d\epsilon = Q\exp[-\Sigma_a x]. \qquad (20)$$

In effect, since we assume that $E_0 \gg kT$, $\sigma(E_0\to E)=0$ holds if $E>E_0$ and, as a consequence thereof, there are no neutrons of energy greater than E_0.

If we apply the Laplace transform to Eq. (20), and assume that it is proper to invert the order of the operations, we get:

$$\int_0^{\epsilon_0}\bar{q}(s, \epsilon)\,d\epsilon = \frac{Q}{s+\Sigma_a}, \qquad (21)$$

and, introducing (19)

$$c(s) = \frac{1}{\int_0^{\epsilon_0} M(\epsilon)F(\beta, 2, \epsilon)\,d\epsilon}\cdot\frac{Q}{s+\Sigma_a} \qquad (22)$$

so that, finally:

$$\bar{q}(s, \epsilon) = \frac{Q}{s+\Sigma_a}\cdot\frac{e^{-\epsilon}\epsilon F(\beta, 2, \epsilon)}{\int_0^{\epsilon_0} e^{-\epsilon}\epsilon F(\beta, 2, \epsilon)\,d\epsilon}. \qquad (23)$$

We shall now endeavour to transform this expression by means of the known relations:

$$\frac{\epsilon}{\gamma}F(\beta+1, \gamma+1, \epsilon) = F(\beta+1, \gamma, \epsilon) - F(\beta, \gamma, \epsilon), \qquad (24)$$

$$F(\beta, \gamma, \epsilon) = e^{\epsilon}F(\gamma-\beta, \gamma, -\epsilon), \qquad (25)$$

$$(1-\beta)\int_0^{\epsilon_0} F(\beta, \gamma, \epsilon)\,d\epsilon = (1-\gamma)F(\beta-1, \gamma-1, \epsilon_0) + \gamma - \text{(26)}\ 1.$$

With Eqs. (24) and (25), we obtain:

$$e^{-\epsilon}\epsilon F(\beta, 2, \epsilon) = F(1-\beta, 1, -\epsilon) - F(2-\beta, 1, -\epsilon) \qquad (27)$$

and applying (26) to the boundary case in which γ tends to 1:

$$\lim_{\gamma\to 1}\int_0^{\epsilon_0} F(2-\beta, \gamma, -\epsilon)\,d\epsilon = \epsilon_0 F(2-\beta, 2, -\epsilon_0) \qquad (28)$$

and:

$$\lim_{\gamma\to 1}\int_0^{\epsilon_0} F(1-\beta, \gamma, -\epsilon)\,d\epsilon = \epsilon_0 F(1-\beta, 2, -\epsilon_0). \qquad (29)$$

Summing up:

$$\int_0^{\epsilon_0} e^{-\epsilon}\epsilon F(\beta, 2, \epsilon)\,d\epsilon = \epsilon_0[F(1-\beta, 2, -\epsilon_0) - F(2-\beta, 2, -\epsilon_0)] = \frac{\epsilon_0^2}{2}F(2-\beta, 3, -\epsilon_0) \qquad (30)$$

from which:

$$\bar{q}(s, \epsilon) = \frac{Q}{s+\Sigma_a}\cdot\frac{2\,e^{-\epsilon}\epsilon F(\beta, 2, \epsilon)}{\epsilon_0^2 F(2-\beta, 3, -\epsilon_0)}. \qquad (31)$$

Since we have supposed that ϵ_0 is very large, we can apply the asymptotic development of $F(2-\beta, 3, -\epsilon_0)$.

$$F(2-\beta, 3, -\epsilon_0) \sim \epsilon_0^{\beta-2}\frac{2}{\Gamma(1+\beta)} \qquad (32)$$

and, therefore,

$$\bar{q}(s, \epsilon) = \frac{Q}{s+\Sigma_a}e^{-\epsilon}\epsilon F(\beta, 2, \epsilon)\Gamma(1+\beta)\epsilon_0^{-\beta}. \qquad (33)$$

In order to obtain the back-transform of this expression, we shall resort to an integral representation[4] of the hypergeometric confluent functions:

$$F(\beta, 2, \epsilon) = \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-t} t^{\beta-1} I_1(2\{\epsilon t\}^{\frac{1}{2}}) \frac{dt}{\{\epsilon t\}^{\frac{1}{2}}}. \quad (34)$$

Therefore:

$$\bar{q}(s, \epsilon) = \frac{Q}{s+\Sigma_a} e^{-\epsilon} \epsilon \frac{\Gamma(1+\beta)}{\Gamma(\beta)} \epsilon_0^{-\beta} \times \int_0^\infty e^{-t} t^{\beta-1} \frac{I_1(2\{\epsilon t\}^{\frac{1}{2}})}{\{\epsilon t\}^{\frac{1}{2}}} dt, \quad (35)$$

or else:

$$\bar{q}(s, \epsilon) = \frac{Q}{s+\Sigma_a} e^{-\epsilon} \epsilon^{\frac{1}{2}} \beta \int_0^\infty e^{-t} \left(\frac{t}{\epsilon_0}\right)^\beta \frac{I_1(2\{\epsilon t\}^{\frac{1}{2}})}{t^{3/2}} dt. \quad (36)$$

Where ϵ_0 is very large, we can leave out the contribution to the integral between ϵ_0 and infinity so:

$$\bar{q}(s, \epsilon) = \frac{Q}{\xi\Sigma_s} e^{-\epsilon} \epsilon^{\frac{1}{2}} \int_0^{\epsilon_0} \exp\left[-\frac{s+\Sigma_a}{\xi\Sigma_s} \ln\frac{\epsilon_0}{t}\right] \times e^{-t} \frac{I_1(2\{\epsilon t\}^{\frac{1}{2}})}{t^{3/2}} dt. \quad (37)$$

Let us now change variables:

$$\frac{1}{\xi\Sigma_s} \ln\frac{\epsilon_0}{t} = u, \qquad \frac{1}{\xi\Sigma_s} \frac{dt}{t} = -du. \quad (38)$$

Therefore:

$$\bar{q}(s, \epsilon) = Q\, e^{-\epsilon} \epsilon^{\frac{1}{2}} \int_0^\infty e^{-(s+\Sigma_a)u} \exp[-\epsilon_0\, e^{-\xi\Sigma_s u}] \times \frac{I_1(2\{\epsilon\epsilon_0 \exp[-\xi\Sigma_s u]\}^{\frac{1}{2}})}{\epsilon_0^{\frac{1}{2}}\, e^{-\xi\Sigma_s u/2}} du. \quad (39)$$

The back-transform of this expression, assuming again that we may invert the order of operations, is:

$$q(x, \epsilon) = Q\, e^{-\Sigma_a x}\, e^{-\epsilon}\, \epsilon^{\frac{1}{2}} \int_0^\infty \delta(x-u) \times \exp[-\epsilon_0\, e^{-\xi\Sigma_s u}] \frac{I_1(2\{\epsilon\epsilon_0 \exp[-\xi\Sigma_s u]\}^{\frac{1}{2}})}{\epsilon_0^{\frac{1}{2}} \exp[-\xi\Sigma_s u]} du \quad (40)$$

in which δ is Dirac's function. Finally, we can write:

$$q(x, \epsilon) = Q\, e^{-\Sigma_a x}\, e^{-\epsilon} \epsilon^{\frac{1}{2}} I_1(2\{\epsilon\epsilon_0\, e^{-\xi\Sigma_s x}\}^{\frac{1}{2}}) \times \frac{\exp[-\epsilon_0\, e^{-\xi\Sigma_s x}]}{\epsilon_0^{\frac{1}{2}} \exp[-\xi\epsilon_s x/2]} \quad (41)$$

which is the solution we are seeking. Let us see, now, how this function behaves when variable x becomes large or small in comparison with $\frac{1}{\xi\Sigma_s} \ln \epsilon_0$.

(1) Large values of x: in this case, the argument of function I_1 will be small compared to 1, and we can replace this function by the first term of its development in series:

$$I_1(2\{\epsilon\epsilon_0\, e^{-\xi\Sigma_s x}\}^{\frac{1}{2}}) \simeq (\epsilon\epsilon_0\, e^{-\xi\Sigma_s x})^{\frac{1}{2}}. \quad (42)$$

In addition, factor $\exp[-\epsilon_0\, e^{-\xi\Sigma_s x}]$ will be practically equal to 1. Therefore:

$$q(x, \epsilon) \simeq Q\, e^{-\Sigma_a x}\, e^{-\epsilon} \epsilon \quad (43)$$

for

$$x \gg \frac{1}{\xi\Sigma_s} \ln \epsilon_0$$

which is the Maxwellian spectrum.

(2) Small values of x: in this case the argument of I_1 will be very large and we can use its asymptotic expansion. The predominant term in this will be:

$$I_1(2\{\epsilon\epsilon_0\, e^{-\xi\Sigma_s x}\}^{\frac{1}{2}}) \sim \frac{\exp[2\{\epsilon\epsilon_0\, e^{-\xi\Sigma_s x}\}^{\frac{1}{2}}]}{2\pi^{\frac{1}{2}}(\epsilon\epsilon_0\, e^{-\xi\Sigma_s x})^{\frac{1}{4}}}. \quad (44)$$

Therefore:

$$q(x, \epsilon) \simeq Q\, e^{-\Sigma_a x} \frac{\exp[-(\epsilon^{\frac{1}{2}} - \epsilon_0^{\frac{1}{2}}\, e^{-\xi\Sigma_s x/2})^2]}{2\pi^{\frac{1}{2}}(\epsilon_0\, e^{-\xi\Sigma_s x})^{\frac{1}{4}}} \epsilon^{\frac{1}{4}} \quad (45)$$

for $x \ll \frac{1}{\xi\Sigma_s} \ln \epsilon_0$

which is a Gaussian type distribution.

Moments

Starting from the spectral distribution (41) we can determine the expressions (independent of Σ_a):

$$\tilde{v^n}(x) = \frac{\int_0^{\epsilon_0} q(x, \epsilon) v^n\, d\epsilon}{\int_0^{\epsilon_0} q(x, \epsilon)\, d\epsilon} \quad (46)$$

with $n = -1, +1, 2, 3, \ldots$.
The denominator of these expressions, which is the total number of neutrons $a(x)$ which reach a point at a distance (x) per unit time and volume, is given by Eq. (20). Introducing Eqs. (20) and (41) in (46), and allowing for the relationship:[4]

$$\int_0^\infty e^{-x^2} x^{\mu-1} I_\nu(bx)\, dx = \frac{b^\nu \Gamma\left(\frac{\mu+\nu}{2}\right)}{2^{\nu+1}\Gamma(\nu+1)} \times F\left(\frac{\mu+\nu}{2}, \nu+1, \frac{b^2}{4}\right) \quad (47)$$

we get, for ϵ_0 very large:

$$\tilde{v^n}(x) = \left(\frac{2kT}{m}\right)^{\frac{n}{2}} \Gamma\left(2+\frac{n}{2}\right) \exp[-\epsilon_0\, e^{-\xi\Sigma_s x}] \times F\left(2+\frac{n}{2}, 2, \epsilon_0\, e^{-\xi\Sigma_s x}\right) \quad (48)$$

or, by virtue of Relationship (25):

$$\tilde{v^n}(x) = \left(\frac{2kT}{m}\right)^{\frac{n}{2}} \Gamma\left(2+\frac{n}{2}\right) F\left(-\frac{n}{2}, 2, -\epsilon_0\, e^{-\xi\Sigma_s x}\right). \quad (49)$$

Formulas (48) and (49) obviously can be written:

$$\tilde{v^n}(x) = v_{\text{th}}^n \exp[-\epsilon_0\, e^{-\xi\Sigma_s x}] F\left(2+\frac{n}{2}, 2, \epsilon_0\, e^{-\xi\Sigma_s x}\right) = \tilde{v_{\text{th}}^n} F\left(-\frac{n}{2}, 2, -\epsilon_0\, e^{-\xi\Sigma_s x}\right) \quad (50)$$

in which $\tilde{v}_{\text{th}}^n$ is the value toward which $\tilde{v}^n(x)$ tends when x tends to infinity, namely the value which corresponds to the Maxwellian spectrum (43). Let us now see a few special cases.

For $n=-1$, we can write:

$$\tilde{v}^{-1}(x)=\frac{\int_0^{\epsilon_0} n(x,\epsilon)\,d\epsilon}{\int_0^{\epsilon_0} q(x,\epsilon)\,d\epsilon}=\frac{1}{\bar{v}(x)}, \tag{51}$$

where $\bar{v}(x)$ is the mean velocity of the neutrons which travel a distance between x and $x+dx$, a magnitude which we shall use below. Applying Formula (50), we can write:

$$\frac{1}{\bar{v}(x)}=\frac{1}{\bar{v}_{\text{th}}}F(\tfrac{1}{2},\,2,\,-\epsilon_0\,e^{-\xi\Sigma_s x}). \tag{52}$$

For $n=2$, we have the mean quadratic velocity averaged over the flux, and, applying again Eq. (50), we can write:

$$\tilde{v}^2(x)=\tilde{v}_{\text{th}}^2F\left(-1,\,2,\,-\frac{\epsilon_0}{2}\,e^{-\xi\Sigma_s x}\right)$$
$$=\tilde{v}_{\text{th}}^2\left(1+\frac{\epsilon_0}{2}e^{-\xi\Sigma_s x}\right). \tag{53}$$

This last specific result can be obtained direct, using the following reasoning.

Let us take again the equilibrium Eq. (3), but let us eliminate the source term:

$$\frac{\partial q(x,E)}{\partial x}=-q(x,E)[\Sigma_a(E)+\Sigma_s(E)]$$
$$+N\int_0^\infty q(x,E')\sigma(E'\to E)\,dE'. \tag{3a}$$

Let us multiply both sides by v^2 and integrate in E. Let us now assume that the inversion of the order of integration is acceptable, and we shall obtain:

$$\frac{d}{dx}\int_0^\infty q(x,E)v^2\,dE=-\int_0^\infty q(x,E)v^2[\Sigma_a(E)+\Sigma_s(E)]\,dE$$
$$+N\int_0^\infty q(x,E')\,dE'\int_0^\infty \sigma(E'\to E)v^2\,dE. \tag{54}$$

Allowing for the fact that

$$\Sigma_s(E)=N\int_0^\infty \sigma(E\to E')\,dE',$$

we can write Eq. (54) in the form:

$$\frac{d[q(x)\tilde{v}^2(x)]}{dx}=-\frac{2N}{m}\int_0^\infty q(x,E')\,dE'$$
$$\times\int_0^\infty \sigma(E'\to E)\,(E'-E)\,dE-\int_0^\infty q(x,E)\,v^2\Sigma_a(E)\,dE. \tag{55}$$

If we now use relation (11), we have:

$$\frac{dq(x)}{dx}\tilde{v}^2(x)+q(x)\frac{d[\tilde{v}^2(x)]}{dx}=-\xi\Sigma_s q(x)\left[\tilde{v}^2(x)-\frac{4kT}{m}\right]$$
$$-\int_0^\infty q(x,E)\,v^2\Sigma_a(E)\,dE. \tag{56}$$

On the other hand, integrating Eq. (3a) directly with respect to E we get:

$$\frac{dq(x)}{dx}=-\int_0^\infty q(x,E)\Sigma_a(E)\,dE \tag{57}$$

and, as a consequence, Eq. (56) takes the form:

$$\frac{d[\tilde{v}^2(x)]}{dx}=-\xi\Sigma_s\left[\tilde{v}^2(x)-\frac{4kT}{m}\right]$$
$$+\frac{1}{q(x)}\int_0^\infty q(x,E)\,\Sigma_a(E)[\tilde{v}^2(x)-v^2]\,dE. \tag{58}$$

For $\Sigma_a=$ constant we have:

$$\tilde{v}^2(x)=\left(v_0^2-\frac{4kT}{m}\right)e^{-\xi\Sigma_s x}+\frac{4kT}{m} \tag{59}$$

and, therefore, neglecting $\frac{4kT}{m}$ with respect to v_0^2 in the parenthesis:

$$\frac{\tilde{v}^2(x)}{\tilde{v}_{\text{th}}^2}=\frac{mv_0^2}{4kT}e^{-\xi\Sigma_s x}+1=\frac{\epsilon_0}{2}e^{-\xi\Sigma_s x}+1. \tag{60}$$

It is of interest to note that for small values of x, that is for large values of $\epsilon_0\,e^{-\xi\Sigma_s x}$, we can apply the asymptotic development of the hypergeometric confluent functions. Equation (50) then takes the form:

$$\tilde{v}^n(x)\sim\tilde{v}_{\text{th}}^n\,\epsilon_0^{n/2}e^{-n\xi\Sigma_s x/2}\frac{1}{\Gamma\left(2+\frac{n}{2}\right)}=v_0^n\,e^{-n\xi\Sigma_s x/2}, \tag{61}$$

an expression which is accurate when we can leave aside the effect of thermal agitation of the moderator, or, for $\tilde{v}^n\gg\tilde{v}_{\text{th}}^n$.

Migration and Moderation Lengths

Until now, we have assumed, in order to deduce the expressions mentioned above, that the capture cross-section of the medium is independent of the neutron energy. However, this is not so for the moderators found in practice. Fortunately, Condition (1) is generally met:

$$\frac{1}{c}=\frac{\Sigma_a(\tilde{v}_{\text{th}})}{\xi\Sigma_s}\ll 1, \tag{1}$$

which is characteristic of any good moderator and, in the case we are now considering, Expressions (50) remain a good approximation, as can be seen from Eqs. (15) and (58). We define the migration length of the neutrons having a given energy in a certain moderator by:

$$M=(\tfrac{1}{6}\overline{r^2})^{\frac{1}{2}} \tag{62}$$

in which $\sqrt{\overline{r^2}}$ is the mean quadratic distance, in a straight line, covered in the moderator by the neutrons from the source in which they originate, until they are absorbed.

We shall define here as moderation length L_s a quantity taken from the expression commonly used in reactor computation:

$$M^2 = L_s^2 + L^2 \tag{63}$$

in which L is the thermal diffusion length. L_s^2 as defined is really a function of L^2. Nevertheless, we shall see further on that, if Condition (1) is fulfilled, dependency on L^2 is given only by terms of the order of L^2/c^2, which can be neglected.

We shall now determine these magnitudes for an absorbing moderator following the $1/v$ law. It is easy to show that:[5]

$$r^2 = 2\lambda_t \bar{x}, \tag{64}$$

in which:

$$\bar{x} = \frac{1}{Q}\int_0^\infty x\left[-\frac{dq(x)}{dx}\right]dx = \frac{1}{Q}\int_0^\infty q(x)\,dx \tag{65}$$

is the mean track length covered by the neutrons from their point of origin to their absorption.

In order to compute $\bar{x}$, let us again take Eq. (57), applicable for any function $\Sigma_a(E)$:

$$\frac{dq(x)}{dx} = -\int_0^\infty q(x, E)\Sigma_a(E)\,dE. \tag{57}$$

Since we have now assumed:

$$\Sigma_a(E) = \frac{1}{vl}, \tag{66}$$

we can write:

$$\frac{dq(x)}{dx} = -\frac{q(x)}{l\bar{v}(x)} \tag{67}$$

from which:

$$q(x) = Q\exp\left[-\frac{1}{l}\int_0^x \frac{dx}{\bar{v}(x)}\right]. \tag{68}$$

Substituting this expression for $q(x)$ in (65), we obtain:

$$\bar{x} = \int_0^\infty \exp\left[-\frac{1}{l}\int_0^x \frac{dx}{(\bar{v}x)}\right]dx \tag{69}$$

and, consequently:

$$M^2 = \frac{\lambda_t}{3}\int_0^\infty \exp\left[-\Sigma_a(\bar{v}_\infty)\int_0^x \frac{\bar{v}_\infty}{\bar{v}(x)}\,dx\right]dx \tag{70}$$

in which $\bar{v}_\infty$ is the limit toward which $\bar{v}(x)$ tends, when x grows indefinitely. It can be shown (see Appendix I and Ref. 6) that, if Condition (1) is fulfilled, and if terms of order $\frac{1}{c^2}\lambda_a(\bar{v}_{th})$ are neglected, Formula (69) reduces to:

$$\bar{x} = \frac{1}{\Sigma_a(\bar{v}_\infty)} + \int_{\bar{v}_\infty/v_0}^1 x\,d\left(\frac{\bar{v}_\infty}{\bar{v}(x)}\right). \tag{71}$$

If Condition (1) now is fulfilled, we may assume that:

$$\frac{\bar{v}_\infty}{\bar{v}(x)} = F(\tfrac{1}{2}, 2, -\epsilon_0\, e^{-\xi\Sigma_s x}) \tag{72}$$

and we get (see Appendix 2):

$$\bar{x} = \frac{1}{\Sigma_a(\bar{v}_\infty)} + \frac{1}{\xi\Sigma_s}[\ln\epsilon_0 - (1 + \ln 4 - \varepsilon_x)]$$

$$= \frac{1}{\Sigma_a(\bar{v}_\infty)} + \frac{1}{\xi\Sigma_s}(\ln\epsilon_0 - 1.81). \tag{73}$$

In determining term $1/\Sigma_a(\bar{v}_\infty)$ we must allow for the fact that $\bar{v}_\infty$ is slightly different from $\bar{v}_{th}$. In order to compute this difference, we use Equation (58). Considering that $d\widetilde{v^2}(x)/dx$ must vanish as x tends to infinity, and assuming that $q(x, E)$ then tends to have a Maxwellian spectrum, we can write (see Appendix 3):

$$\frac{1}{\Sigma_a(\bar{v}_\infty)} = \frac{1}{\Sigma_a(\bar{v}_{th})} - \frac{1}{8\xi\Sigma_s} \tag{74}$$

if we leave out the terms of the order $\frac{\lambda_a(\bar{v}_{th})}{c^2}$.

We finally obtain:

$$M^2 = L^2 + \frac{\lambda_t\lambda_s}{3\xi}(\ln\epsilon_0 - 1.68). \tag{75}$$

Applying, then, Definition (63) we find, for the moderating length:

$$L_s^2 = \frac{\lambda_t\lambda_s}{3\xi}(\ln\epsilon_0 - 1.68). \tag{76}$$

Total Flux and Total Neutron Density

By virtue of the definitions given at the beginning, these magnitudes are expressed by:

$$\phi = \int_0^\infty q(x)\,dx, \tag{77}$$

$$n = \int_0^\infty q(x)\frac{dx}{\bar{v}(x)}. \tag{78}$$

For an infinite medium, we can write:

$$\phi = Q\bar{x} = QM^2/D \tag{79}$$

as can be proved by comparing Eqs. (65) and (77). If the absorption in the medium follows the $1/v$ law, we can also write:

$$n = Ql \tag{80}$$

and the mean velocity of all the neutrons will be:

$$\bar{v} = \frac{\bar{x}}{l} = \bar{v}_{th}\left[1 + \frac{1}{c}(\ln\epsilon_0 - 1.68)\right]. \tag{81}$$

We shall now compute ϕ and n in the case of a finite medium, with a neutron source distributed over it, according to an eigenfunction of the equation

$$\Delta Q(r) + B^2 Q(r) = 0. \tag{82}$$

If we wish to reduce this case to that of a finite medium, it is sufficient to add, to the neutrons lost by absorption, those lost by diffusion to the outside (the number of which per unit time and volume is: $-D\Delta\phi(r, E)\,dE$ in the energy range E, $E + dE$). But if $Q(r)$ satisfies Eq. (82), $\phi(r, E)$ will be proportional to $Q(r)$ and the lost neutrons by diffusion would be $B^2 D\phi(r, E)\,dE$. As a consequence of this, the problem presented is equivalent to that of an infinite medium with a fictitious absorption according to the law:

$$\Sigma_a'(E) = \frac{1}{lv} + B^2 D = \Sigma_a(\bar{v}_{th})\frac{\bar{v}_{th}}{v} + B^2 D. \tag{83}$$

Injecting this fictitious capture cross-section into Eq. (57), we now can write:

$$q(x) = Q \exp\left[-\frac{1}{l}\int^{x} \frac{dx}{\bar{v}(x)} - B^2 D x\right]. \qquad (84)$$

and, by Definition (78):

$$n = Ql\int_0^{\infty} \exp\left[-\frac{1}{l}\int_0^{x} \frac{dx}{\bar{v}(x)} - B^2 D x\right]\frac{dx}{l\bar{v}(x)}. \qquad (85)$$

If we integrate this expression by parts, we get:

$$n = Ql\left\{1 - B^2 D\int_0^{\infty} \exp\left[-\frac{1}{l}\int_0^{x} \frac{dx}{\bar{v}(x)} - B^2 D x\right] dx\right\}, \qquad (86)$$

that is:

$$n = Ql\left[1 - \frac{B^2 D}{Q}\int_0^{\infty} n(x)\bar{v}(x)\,dx\right] = Ql\left[1 - B^2 D\frac{\phi}{Q}\right]. \qquad (87)$$

Let us suppose that, as usual, the following condition is fulfilled

$$B^2 M^2 \ll 1 \qquad (88)$$

in addition to Condition (1). Then, if we neglect the terms of order B^4M^4, we can replace ϕ by its value in the infinite medium, and we obtain the conventional formula:

$$n \simeq Ql(1 - B^2M^2) \simeq \frac{Ql}{1 + B^2M^2}. \qquad (89)$$

We have yet to determine ϕ in the finite medium, which, by virtue of Definition (77), is given by:

$$\phi = Q\int_0^{\infty} \exp\left[-\frac{1}{l}\int_0^{x} \frac{dx}{\bar{v}(x)} - B^2 D x\right] dx. \qquad (90)$$

For the exact $\bar{v}(x)$ Function (72), this expression can only be determined numerically. But, on the other hand, we can compute it for the two functions a and

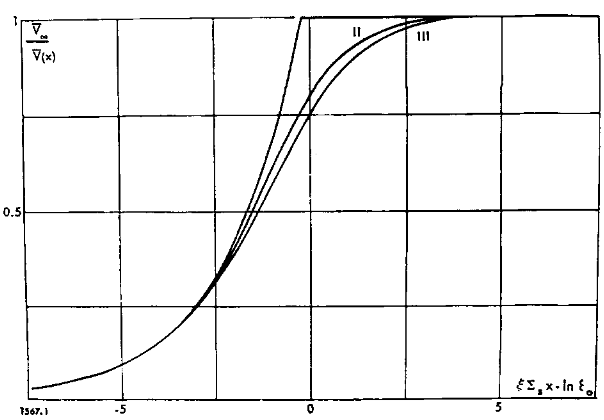

Figure 1. Velocity ratios

I; $\dfrac{\bar{v}(x)}{\bar{v}_\infty} = \dfrac{\bar{v}_0}{\bar{v}_\infty} e^{-\xi\Sigma_s x/2}$ (a)

II; $\dfrac{\bar{v}(x)}{\bar{v}_\infty} = \dfrac{1}{F(\frac{1}{2}, 2, -\epsilon_0 e^{-\xi\Sigma_s x})}$

III; $\dfrac{\bar{v}(x)}{\bar{v}_\infty} = 1 + \left(\dfrac{\bar{v}_0^2}{\bar{v}_\infty^2} - 1\right) e^{-\xi\Sigma_s x}$ (b)

b given in Appendix 1, which encompass the correct function, except in a small range of the variable about the point $x = \dfrac{1}{\xi\Sigma_s}\ln \epsilon_0$ (see Fig. 1).

The result obtained is the same for a and b, except for terms of the order of c^{-2}, c^{-1}, B^2M^2 and B^4M^4 which we can leave out. For Function (72), between a and b, we can therefore assume the same result, which is:

$$\phi = \frac{QM^2}{D[1 + B^2L^2(1 + L_s^2/2L^2M^2)]}. \qquad (91)$$

Finally, dividing ϕ by n, we find, for the mean velocity of all the neutrons in the finite medium:

$$\bar{v} = \bar{v}_{th}\left[1 + \frac{1}{c}(\ln \epsilon_0 - 1.68)\right] \times \left[1 + B^2L_s^2\left(1 - \frac{L_s^2}{2M^2}\right)\right]. \qquad (92)$$

APPENDIX 1.

Computation of the limit:

$$\lim_{\Sigma_a(\bar{v}_\infty)\to 0}\left(\bar{x} - \frac{1}{\Sigma_a \bar{v}_\infty}\right).$$

On the hypothesis that the integral $\int_{\bar{v}_\infty/v_0}^{1} x\, d\left(\dfrac{\bar{v}_\infty}{\bar{v}(x)}\right)$ converges, we have seen that:

$$\bar{x} = \int_0^{\infty} \exp\left[-\Sigma_a(\bar{v}_\infty)\int_0^{x} \frac{\bar{v}_\infty}{\bar{v}(x)}\,dx\right] dx. \qquad (69a)$$

Integrating the exponent of this expression by parts, we get:

$$\bar{x} = \int_0^{\infty} \exp\left[-\Sigma_a x\, u(x) + \Sigma_a\int_{u(0)}^{u(x)} x\, du(x)\right] dx \qquad (93)$$

in which, for brevity's sake, we set:

$$\Sigma_a(\bar{v}_\infty) = \Sigma_a$$

$$\frac{\bar{v}_\infty}{\bar{v}(x)} = u(x).$$

Let us now change variables

$$\eta = e^{-\Sigma_a x}; \quad \frac{d\eta}{\eta} = -\Sigma_a\, dx. \qquad (94)$$

Thus:

$$\bar{x} = \frac{1}{\Sigma_a}\int_0^1 \eta^{u(\Sigma_a^{-1}\ln\eta^{-1})} \exp\left[\Sigma_a\int_{u(0)}^{u(x)} x\, du(x)\right]\frac{d\eta}{\eta}. \qquad (95)$$

If we develop the exponential in series we get:

$$\bar{x} = \frac{1}{\Sigma_a}\int_0^1 \eta^{u(\Sigma_a^{-1}\ln\eta^{-1})-1} \times\left[1 + \Sigma_a\int_{u(0)}^{u(\Sigma_a^{-1}\ln\eta^{-1})} x\, du(x) + \Sigma_a^2 f(\eta)\right] d\eta. \qquad (96)$$

When Σ_a tends towards zero $u\left(\dfrac{1}{\Sigma_a}\ln\dfrac{1}{\eta}\right)$ tends towards

1 and the term $\Sigma_a^2 f(\eta)$ disappears. Accordingly:

$$\lim_{\Sigma_a\to 0}\left(\bar{x}-\frac{1}{\Sigma_a}\right)=\int_0^1 d\eta\int_{u(0)}^1 x\,du(x), \qquad (97)$$

i.e.,

$$\lim_{\Sigma_a\to 0}\left(\bar{x}-\frac{1}{\Sigma_a}\right)=\int_{\bar{v}_\infty/v_0}^1 x d\left(\frac{\bar{v}_\infty}{\bar{v}(x)}\right). \qquad (98)$$

N.B.: The present demonstration of Formula (98) was suggested by R. Scarfiello.

Let us now compare the values of $\bar{x}-\lambda_a$ obtained from Eq. (69) for various functions $\bar{v}(x)$ with the asymptotic values (98).

(a)
$$\left.\begin{aligned}\frac{\bar{v}(x)}{\bar{v}_\infty}&=\frac{\bar{v}_0}{\bar{v}_\infty}e^{-\xi\Sigma_s x/2}\ \text{if}\ x\leqslant\frac{2}{\xi\Sigma_s}\ln\frac{\bar{v}_0}{\bar{v}_\infty}\\ \frac{\bar{v}(x)}{\bar{v}_\infty}&=1\qquad\qquad \text{if}\ x\geqslant\frac{2}{\xi\Sigma_s}\ln\frac{\bar{v}_0}{\bar{v}_\infty}\end{aligned}\right\}\ \text{(Fermi's approximation)}$$

Equation (69)

$$\bar{x}-\lambda_a=\frac{1}{\xi\Sigma_s}\left[\ln\left(\frac{v_0}{\bar{v}_\infty}\right)^2-\left(2.24+\frac{2}{c}\right)\right], \qquad (99)$$

Equation (98)

$$\lim_{\lambda_a\to\infty}(\bar{x}-\lambda_a)=\frac{1}{\xi\Sigma_s}\left[\ln\left(\frac{v_0}{\bar{v}_\infty}\right)^2-2.24\right]. \qquad (100)$$

(b)
$$\frac{\bar{v}(x)}{v_\infty}=\left[1+\left(\frac{v_0^2}{\bar{v}_\infty^2}-1\right)e^{-\xi\Sigma_s x}\right]^{\frac{1}{2}}.$$

Equation (69)

$$\bar{x}-\lambda_a=\frac{1}{\xi\Sigma_s}\left[\ln\left(\frac{v_0}{\bar{v}_\infty}\right)^2-\left(1.63+\frac{3.29}{c}\right)\right], \qquad (101)$$

Equation (98)

$$\lim_{\lambda_a\to\infty}(\bar{x}-\lambda_a)=\frac{1}{\xi\Sigma_s}\left[\ln\left(\frac{v_0}{\bar{v}_\infty}\right)^2-1.63\right]. \qquad (102)$$

In both cases, the two results are equal except for terms of the order of λ_a/c^2.

For the correct function $\bar{v}_\infty/v(x)=F(\frac{1}{2},2,-\epsilon_0\,e^{-\xi\Sigma_s x})$, Expression (69) can only be computed numerically, but, since this function fits closely between functions *a* and *b* (see Fig. 1), we may assume, for it as well, the equivalence of the Formulae (69) and (98), but for terms of the order of λ_a/c^2.

APPENDIX 2.

Computation of the integral:

$$I=\int_{\bar{v}_\infty/v_0}^1 x\,d\left(\frac{\bar{v}_\infty}{\bar{v}(x)}\right)=\int_0^\infty x\,dF(\tfrac{1}{2},2,-\epsilon_0\,e^{-\xi\Sigma_s x}). \qquad (103)$$

Changing variables:

$$y=\epsilon_0\,e^{-\xi\Sigma_s x},\quad -\frac{dy}{y}=\xi\Sigma_s\,dx. \qquad (104)$$

Then:

$$I=\frac{1}{\xi\Sigma_s}\int_{\epsilon_0}^0\ln\left(\frac{\epsilon_0}{y}\right)\frac{dF(\frac{1}{2},2,-y)}{dy}dy, \qquad (105)$$

or:

$$I=\frac{1}{\xi\Sigma_s}\ln\epsilon_0[1-F(\tfrac{1}{2},2,-\epsilon_0)]-\frac{1}{\xi\Sigma_s}\int_{\epsilon_0}^0\ln y\frac{dF(\frac{1}{2},2,-y)}{dy}dy. \qquad (106)$$

Applying Formula (26) we can write:

$$\frac{1}{\xi\Sigma_s}\int_{\epsilon_0}^0\ln y\frac{dF(\frac{1}{2},2,-y)}{dy}dy=\frac{1}{4\xi\Sigma_s}\int_0^{\epsilon_0}\ln y F(\tfrac{3}{2},3,-y)\,dy=\frac{J}{4\xi\Sigma_s}. \qquad (107)$$

In order to compute J we shall resort to an integral representation[4] of $F(\frac{3}{2},2,-y)$ (Ref. 4):

$$F(\tfrac{3}{2},2,-y)=\frac{8}{\sqrt{\pi}}\int_0^\infty e^{-t^2}J_2(2t\sqrt{y})\frac{dt}{y}. \qquad (108)$$

Using this expression in J, inverting the order of integration, and assuming that ϵ_0 is very large, we can write:

$$J=\frac{8}{\sqrt{\pi}}\int_0^\infty e^{-t^2}dt\int_0^\infty\ln y J_2(2t\sqrt{y})\frac{dy}{y}. \qquad (109)$$

If we set:

$$z=2t\sqrt{y},\quad \frac{dz}{z}=\frac{dy}{2y}, \qquad (110)$$

we can decompose J into:

$$J=\frac{32}{\sqrt{\pi}}\int_0^\infty e^{-t^2}dt\int_0^\infty\ln z J_2(z)\frac{dz}{z}-\frac{32}{\sqrt{\pi}}\int_0^\infty e^{-t^2}\ln(2t)\,dt\int_0^\infty J_2(z)\frac{dz}{z}. \qquad (111)$$

According to Ref. 4, we can write:

$$\int_0^\infty\ln z J_2(z)\frac{dz}{z}=\tfrac{1}{4}[\psi(1)+\psi(2)+2\ln 2]=\tfrac{1}{4}(\ln 4+1-\varepsilon_x), \qquad (112)$$

$$\int_0^\infty J_2(z)\frac{dz}{z}=\tfrac{1}{2}, \qquad (113)$$

$$\int_0^\infty e^{-t^2}\ln(2t)\,dt=\tfrac{1}{4}\int_0^\infty e^{-u}\ln(4u)\frac{du}{\sqrt{u}}=\frac{\sqrt{\pi}}{4}[\psi(\tfrac{1}{2})+\ln 4]=-\frac{\sqrt{\pi}}{4}\varepsilon_x, \qquad (114)$$

and, therefore:

$$J=4(\ln 4+1-\varepsilon_x). \qquad (115)$$

Finally, substituting this result in (106) and (107) and considering that term $1/\xi\Sigma_s.\ln\epsilon_0 F(\frac{1}{2},2,-\epsilon_0)$ is negligible for large values of ϵ_0, we then have:

$$I=\frac{1}{\xi\Sigma_s}[\ln\epsilon_0-(1+\ln 4-\varepsilon_x)]. \qquad (116)$$

APPENDIX 3.

Computation of the difference $\lambda_a(\bar{v}_\infty)-\lambda_a(\bar{v}_{th})$, for absorption according to the $1/v$ law.

From Eq. (58) we find that, if x tends to infinity:

$$\tilde{v}_\infty^2=\frac{4kT}{m}+\frac{\Sigma_a(\bar{v}_{th})}{\xi\Sigma_s}\left(\frac{8kT}{\pi m}\right)^{\frac{1}{2}}$$
$$\times \lim_{x\to\infty}\frac{\int_0^\infty q(x, E)\tilde{v}^2(x)\frac{dE}{v}-\int_0^\infty q(x, E)v\,dE}{q(x)} \quad (117)$$

or else:

$$\tilde{v}_\infty^2\left[1-\frac{1}{c}\left(\frac{8kT}{\pi m}\right)^{\frac{1}{2}}\lim_{x\to\infty}\frac{\int_0^\infty q(x, E)\frac{dE}{v}}{q(x)}\right]$$
$$=\frac{4kT}{m}-\frac{1}{c}\left(\frac{8kT}{\pi m}\right)^{\frac{1}{2}}\lim_{x\to\infty}\frac{\int_0^\infty q(x, E)v\,dE}{q(x)}. \quad (118)$$

We shall assume that:

$$\lim_{x\to\infty}\frac{q(x, E)}{q(x)}=\frac{1}{(kT')^2}e^{-E/kT'}E'. \quad (119)$$

Thus we obtain relationships:

$$\tilde{v}_\infty^2=\frac{4kT'}{m} \quad (120)$$

$$\lim_{x\to\infty}\frac{\int_0^\infty q(x, E)\frac{dE}{v}}{q(x)}=\left(\frac{\pi m}{8kT'}\right)^{\frac{1}{2}} \quad (121)$$

$$\lim_{x\to\infty}\frac{\int_0^\infty q(x, E)v\,dE}{q(x)}=\left(\frac{q\pi kT'}{8m}\right)^{\frac{1}{2}} \quad (122)$$

and, therefore,

$$\tilde{v}_\infty^2=\frac{4kT'}{m}=\frac{\frac{4kT'}{m}-\frac{1}{c}\cdot\frac{3k}{m}(TT')^{\frac{1}{2}}}{1-\frac{1}{c}(T/T')^{\frac{1}{2}}}. \quad (123)$$

Neglecting the terms of the order of $\frac{1}{c^2}$ we can write:

$$T'=T\left[1-\frac{3}{4c}(T'/T)^{\frac{1}{2}}+\frac{1}{c}(T/T')^{\frac{1}{2}}\right] \quad (124)$$

and also, ignoring the terms of order $(T'-T)/c$, we have:

$$T'=T\left(1+\frac{1}{4c}\right). \quad (125)$$

For $\bar{v}_\infty$ we obtain:

$$\bar{v}_\infty=\left[\frac{8kT}{\pi m}\left(1+\frac{1}{4c}\right)\right]^{\frac{1}{2}} \quad (126)$$

or else:

$$\bar{v}_\infty=\bar{v}_{th}\left(1+\frac{1}{4c}\right)^{\frac{1}{2}}\simeq\bar{v}_{th}\left(1+\frac{1}{8c}\right), \quad (127)$$

and, finally:

$$\lambda_a(\bar{v}_\infty)-\lambda_a(\bar{v}_{th})=\frac{1}{8\xi\Sigma_s} \quad (128)$$

LIST OF SYMBOLS

B = buckling
$c=\xi\Sigma_s/\Sigma_a$ moderating coefficient
D = diffusion coefficient
E = energy
$\varepsilon_x=0.57721$, Euler's constant
I_ν = modified Bessel function of order ν
J_ν = Bessel's function of order ν
k = Boltzmann's constant
l = mean life of the neutrons in an infinite medium with $1/v$ absorption
L = diffusion length of thermal neutrons
L_s = moderation length
m = mass of a neutron
M = migration length
n = neutron density
N = number of atoms per cm^3
q = neutron flux
Q = source intensity
T = absolute moderator temperature
v = velocity
V = mean velocity
$\bar{v}_{th}$ = mean thermal velocity
x = broken-line travel
$\epsilon=E/kT$
ϕ = total neutron flux
λ_t = transport mean free path
λ_a = absorption mean free path
$\sigma(E\to E')$ = differential scattering cross-section, for neutrons scattered from E, $E+dE$ to E', $E'+dE'$
Σ_s = scattering cross-section
Σ_a = absorption cross-section
ξ = mean logarithmic energy loss per collision

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