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THE γ-DEPENDENT PART OF THE WAVE FUNCTIONS REPRESENTING γ-UNSTABLE SURFACE VIBRATIONS

DANIEL R. BÈS †

Institute for Theoretical Physics, University of Copenhagen

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Abstract: The equation for the γ-dependent part of the wave functions representing γ-unstable vibrations is solved. The general solution is a linear combination of wave functions belonging to a "basic set". The coefficients in this linear combination are expressed as polynomials in cos 3γ. Some consequences of these solutions are discussed. In the appendix, the equations for the coefficients are given in the cases $I \leqq 6$, and the explicit solutions are constructed for $\lambda \leqq 9$.

1. Introduction

A quadrupole deformation of a surface $R(\theta, \varphi)$ may be described either by the "fixed" coordinates α_μ, such that

$$R = R_0\Big(1+\sum_{\mu=-2}^{\mu=+2} \alpha_\mu Y_{2\mu}(\theta, \varphi)\Big) \tag{1}$$

or by two coordinates related to an intrinsic system and the three Eulerian angles θ_i, which fix the position of this intrinsic system with respect to a fixed frame of reference. The coordinates a_ν in the body-fixed system are related to the space-fixed system by the transformation

$$\alpha_\mu = \sum_\nu \alpha_\nu D^I_{\mu\nu}(\theta_i), \tag{2}$$

where $D^I_{\mu\nu}(\theta_i)$ are the transformation functions for the spherical harmonics of order I. The a_ν's satisfy the relations $a_2 = a_{-2}$ $a_1 = a_{-1} = 0$. One may replace a_0 and a_2 by β and γ, defined by

$$a_0 = \beta \cos \gamma, \qquad a_2 = \frac{1}{\sqrt{2}} \beta \sin \gamma. \tag{3}$$

The equation of motion for the intrinsic coordinates has been treated by several authors. The expression for the kinetic energy was given by A. Bohr [1]):

$$T = -\frac{\hbar^2}{2B}\left\{\frac{1}{\beta^4}\frac{\partial}{\partial \beta}\beta^4\frac{\partial}{\partial \beta} + \frac{1}{\beta^2 \sin 3\gamma}\frac{\partial}{\partial \gamma}\sin 3\gamma \frac{\partial}{\partial \gamma} - \frac{1}{4\beta^2}\sum_x \frac{Q_x{}^2}{\sin^2(\gamma - \frac{2}{3}\pi x)}\right\} \tag{4}$$

where Q_x are angular momentum operators in the variables θ_i.

† Permanent address: Comisión Nacional de Energía Atómica, Buenos Aires, Argentina.

In the same paper, Bohr has also given the explicit solution for the free $I = 0$ waves in terms of Legendre polynomials. The case when a strong γ-dependent potential stabilizes the motion around $\gamma = 0$ or $\gamma = \pi$ is also treated there and in the following series of papers written in collaboration with B. Mottelson [2,3,4]). Wilets and Jean [5]) have discussed the case of deformation potentials independent of γ. Rakavy [6]) has given a classification of surface oscillations by group-theory which is especially suitable for γ-unstable oscillations.

In this paper, a method is outlined for the construction of the γ-dependent part of the wave functions in the case of γ-unstable vibrations.

2. Construction of the Wave Functions

Once the β-part of the wave equation is separated, it is necessary to consider only eq. (5) [5])

$$L\phi(\gamma, \theta_i) = \left[\Lambda + \frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} - \tfrac{1}{4}\sum_{x}\frac{Q_x{}^2}{\sin^2(\gamma - \frac{2}{3}\pi x)}\right]\phi(\gamma, \theta_i) = 0, \tag{5}$$

$$\Lambda = \lambda(\lambda+3);$$

λ is an integer, called by Rakavy the "seniority". The solutions may be denoted by $\phi_{\lambda, I, M}(\gamma, \theta_i)$ when no degeneracies are present. Any eigenstate of the equation in the intrinsic coordinates is a linear combination of eigenstates of the equation in the fixed coordinates, which are of the form $e^{-\frac{1}{2}\beta^2}\Pi_{\mu=-2}^{\mu=+2}H_{n_\mu}(\alpha_\mu)$. By repeated application of the Clebsch-Gordan series, one can express this in the form $\sum_K D^I_{MK}(\theta_i)g_K(\beta, \gamma)$. It is then easily seen that only even values of K appear and, moreover,

$$g_K = (-)^I g_{-K}. \tag{6}$$

In order to solve eq. (5) it is convenient to transform it into a system of coupled differential equations in $x = \cos 3\gamma$. This may be performed as follows:

The first solution for a given even spin is

$$\phi_{\frac{1}{2}I, I, M}(\gamma, \theta_i) = \left(\frac{\alpha_2}{\beta}\right)^{\frac{1}{2}I} = \left[\cos\gamma D^2_{M0}(\theta_i) + \sin\gamma\left(\frac{D^2_{M2}(\theta_i)+D^2_{M-2}(\theta_i)}{\sqrt{2}}\right)\right]^{\frac{1}{2}I} \tag{7}$$

corresponding to the unique $\lambda = \frac{1}{2}I$ phonon state with $M = I$.

α_μ has two terms, one which creates a phonon and another one which annihilates a phonon. By acting with α_μ on $\phi_{\lambda, I, M}(\gamma, \theta_i)$, and making use of the properties of the Wigner coefficients, another function of the same spin can be obtained. In general, in order to obtain $\phi_{\lambda+1, I, M}(\gamma, \theta_i)$ it is necessary to orthogonalize the new wave function with respect to $\phi_{\lambda-1, I, M}(\gamma, \theta_i)$. In the case of $\lambda = \frac{1}{2}I$, there is no $\phi_{\lambda-1, I, M}(\gamma, \theta_i)$ and, thus, the new wave function is already proportional to $\phi_{\frac{1}{2}I+1, I, M}(\gamma, \theta_i)$. One repeats the procedure until the lowest $\frac{1}{2}I+1$ functions are obtained. In the case of odd spins, the lowest wave function may be obtained by operating with α_μ on a neighbour even-spin state already known. Only $\frac{1}{2}(I-1)$ functions are needed.

These $\frac{1}{2}I+1$[or $\frac{1}{2}(I-1)$] wave functions constitute a basic set which can be used in order to construct the general solution. The wave functions

$$\phi(\gamma, \theta_i) = w(\gamma)\phi_{\frac{1}{2}I,I,M}(\gamma, \theta_i)+v(\gamma)\phi_{\frac{1}{2}(I+1)I,M}(\gamma, \theta_i)+ \dots +u(\gamma)\phi_{I,I,M}(\gamma, \theta_i)$$

for even spin, and (8)

$$\phi(\gamma, \theta_i) = w(\gamma)\phi_{\frac{1}{2}(I+3)I,M}(\gamma, \theta_i)+v(\gamma)\phi_{\frac{1}{2}(I+5)I,M}(\gamma, \theta_i)+ \dots +u(\gamma)\phi_{I,I,M}(\gamma, \theta_i)$$

for odd spin, are substituted in eq. (5); $w, v, \dots u$ are functions of γ alone. In the following calculations, the different wave functions belonging to a basic set are taken to be independent, but not necessarily orthogonal to each other. The products $\phi^*_{\lambda IM}(\gamma, \theta_i)\, L\phi(\gamma,\theta_i)$ for $\lambda \leqq I$ are integrated over the Eulerian angles, and one observes that the coefficients in the resultant equations are functions only of $\cos 3\gamma$ and $\sin 3\gamma$. One obtains a set of $\frac{1}{2}I+1$ [or $\frac{1}{2}(I-1)$] coupled differential equations, where the only variable is $x = \cos 3\gamma$. These equations can be solved by expanding $w, v, \dots u$ as power series in x. One can work out recursion equations between the coefficients of successive terms of the series. The condition for the convergence of $\phi(\gamma, \theta_i)$ at $x = \pm 1$ makes the series terminate.

In the appendix, the equations for the coefficients are given for $I \leqq 6$, and the explicit solutions for the lowest number of phonons are constructed.

3. Special Features of the Solutions

1) There is a great similarity between the solutions for a given even I and the corresponding ones for $I+3$. This is due to the fact that both have the same number of possible K values and consequently the same number of coupled differential equations.

2) The structure of the solutions shows that each of the wave functions $\phi_{\lambda,I,M}(\gamma, \theta_i)$ with $\lambda \leqq I$ gives rise to a series of solutions with seniority $\lambda_\nu = \lambda+3\nu (\nu = 0, 1, 2, \dots)$. These solutions may be obtained by orthogonalizing the product $\cos 3\gamma \cdot \phi_{\lambda,I,M}(\gamma, \theta_i)$ with respect to the wave functions with eigenvalue $\Lambda < \Lambda_\nu = \lambda_\nu(\lambda_\nu+3)$.

This observation is easily understandable because $\cos 3\gamma$ is the simplest non-trivial scalar that can be made with creation and annihilation operators, and it contains a term with three creation operators.

The previous observation allows us to predict the degeneracies to be found. Only the even-spin cases are discussed. Everything said for them can be applied to the odd spin with the same number of coupled equations.

If $I = 0$, the "basic set" of wave functions with $\lambda \leqq I$ consists of one element only, because there is only one uncoupled equation to be satisfied.

In consequence, only one series of solutions is present. The possible values of the seniority are $\lambda_\nu = 3\nu$. Indeed, the coefficient of the last term in the recurrence relations has zeros only for values of the eigenvalue $\Lambda = 9\nu(\nu+1)$. There are no degeneracies.

If $I = 2$, the "basic set" contains two terms, corresponding to the fact that two coupled differential equations are present. $\phi_{1,2,M}(\gamma, \theta_i)$ and $\phi_{2,2,M}(\gamma, \theta_i)$ give rise to two series of solutions with seniorities $\lambda_\nu = 1+3\nu$ and $\lambda_\nu = 2+3\nu$. There are no degeneracies, and no solutions with $\lambda_\nu = 3\nu$.

Three coupled differential equations and, correspondingly, three unknowns are possible if $I = 4$. For every value of $\lambda > 1$ there is a non-degenerate wave function.

In the $I = 6$ case, 2 degenerate wave functions must be present for $\lambda = 6$. One of the wave functions is obtained by applying the scalar $\cos 3\gamma$ to $\phi_{3,6,M}(\gamma, \theta_i)$ and orthogonalizing with respect to $\phi_{4,6,M}(\gamma, \theta_i)$. This solution does not give an independent unknown in the system of coupled equations. As four unknowns are admitted, another independent solution is needed and, thus, the degeneracy appears. All the states with $\lambda_\nu = 6+3\nu$ are doubly degenerate.

In the same way, one expects double degeneracy for $\phi_{7,8,M}(\gamma, \theta_i)$ and $\phi_{8,8,M}(\gamma, \theta_i)$ and, in consequence, for $\lambda_\nu = 7+3\nu$ and $\lambda_\nu = 8+3\nu$, if $I = 8$. All the $I = 10$ states are doubly degenerate, with the exception of the three first ones. The first triple degeneracy appears in $\phi_{12,12,M}(\gamma, \theta_i)$, etc.

3) A "γ-parity", π_γ, is present, corresponding to the two sets of solutions given for each case. The γ-parity may be defined as the parity of the order of the polynomial in $\cos\gamma$ and $\sin\gamma$ in which each wave function may be expressed. It is equal to the parity of λ.

The electric quadrupole operator is proportional to $\phi_{1,2,M}(\gamma, \theta_i)$ and therefore has odd parity. Any interaction which conserves the γ-parity does preserve the selection rule of the forbiddenness of the transition from the second $I = 2$ state to the ground state in any order of approximation. This fact is suggested in ref. [5]) on the basis of a first-order perturbation calculation.

The simplest interaction of this kind (aside from a constant) is $\cos^2 3\gamma$. In second-order perturbation theory, such an interaction lowers the second 2+ level below the first 4+ level. Therefore it is suggested that this interaction may have some relation with the experimental spectra of even nuclei in the undeformed region. Calculations in this respect are in progress.

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Appendix

1. EQUATIONS FOR THE COEFFICIENTS OF SUCCESSIVE TERMS IN THE POWER SERIES EXPANSION OF w, v, . . .

$$\boxed{I = 0}$$

$$\phi(\gamma, \theta_i) = w;$$

w satisfies the same differential equation as the Legendre polynomials

$$9(1-x^2)\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} - 18x\frac{\mathrm{d}w}{\mathrm{d}x} + \Lambda w = 0.$$

1) $w = \sum_n a_n x^{2n}$, $\pi = +$.
The recurrence relation for the a_n's is the following:

$$a_{n+1}18(n+1)(2n+1) + a_n(\Lambda - 36n^2 - 18n) = 0.$$

The series terminates if $\Lambda = 36\nu^2 + 18\nu$ $(\nu = 0, 1, 2 \ldots)$.
2) $w = \sum_n a_n x^{2n+1}$, $\pi = -$.

$$a_{n+1}18(n+1)(2n+1) + a_n(\Lambda - 36n^2 - 54n - 18) = 0.$$

$$\boxed{I = 1}$$

No solution exists since the only possible value of K is zero, and $g_0(\gamma) = -g_0(\gamma) = 0$.

$$\boxed{I = 2}$$

$$\phi(\gamma, \theta_i) = (w\cos\gamma + v\cos 2\gamma)D^2_{M0}(\theta_i) + (w\sin\gamma - v\sin 2\gamma)\frac{1}{\sqrt{2}}\left(D^2_{M2}(\theta_i) + D^2_{M-2}(\theta_i)\right),$$

where w and v must satisfy the coupled differential equations

$$(\Lambda-4)w - 24x\frac{\mathrm{d}w}{\mathrm{d}x} + 9(1-x^2)\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + 12\frac{\mathrm{d}v}{\mathrm{d}x} = 0,$$

$$6\frac{\mathrm{d}w}{\mathrm{d}x} + (\Lambda-10)v - 30x\frac{\mathrm{d}v}{\mathrm{d}x} + 9(1-x^2)\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = 0.$$

1) $w = \sum_n a_n x^{2n}$, $v = \sum_n b_n x^{2n+1}$, $\pi = -$.

The coefficients are determined by the recurrence relations

$$\begin{aligned} &a_{n+2}81(2n+1)(2n+2)(2n+3)(2n+4) \\ &\quad +a_{n+1}18(2n+1)(2n+2)(\Lambda-36n^2-90n-59) \\ &\quad +a_n(\Lambda-36n^2-30n-4)(\Lambda-36n^2-78n-40) = 0, \\ &b_n = -a_n\frac{(\Lambda-36n^2-30n-4)}{12(2n+1)}-a_{n+1}\tfrac{3}{2}(n+1). \end{aligned}$$

The condition that the series terminates is equivalent to $\Lambda = 36\nu^2+30\nu+4$ or $\Lambda = 36\nu^2+78\nu+40$.

2) $w = \sum_n a_n x^{2n+1}$, $\nu = \sum_n b_n x^{2n}$, $\pi = +$.

$$\begin{aligned} &b_{n+2}81(2n+1)(2n+2)(2n+3)(2n+4) \\ &\quad +b_{n+1}18(2n+1)(2n+2)(\Lambda-36n^2-90n-62) \\ &\quad +b_n(\Lambda-36n^2-42n-10)(\Lambda-36n^2-66n-28) = 0, \\ &a_n = -b_n\frac{(\Lambda-36n^2-42n-10)}{6(2n+1)}-b_{n+1}3(n+1). \end{aligned}$$

$$\boxed{I = 3}$$

$$\phi(\gamma,\theta_i) = w\sin 3\gamma\frac{1}{\sqrt{2}}(D^3_{M2}(\theta_i)-D^3_{M-2}(\theta_i)),$$

$$9(1-x^2)\frac{\mathrm{d}^2w}{\mathrm{d}x^2}-36x\frac{\mathrm{d}w}{\mathrm{d}x}+(\Lambda-18)w = 0.$$

1) $w = \sum_n a_n x^{2n}$, $\pi = -$.

$$a_{n+1}18(n+1)(2n+1)+a_n(\Lambda-36n^2-54n-18) = 0.$$

2) $w = \sum_n a_n x^{2n+1}$, $\pi = +$.

$$a_{n+1}18(n+1)(2n+3)+a_n(\Lambda-36n^2-90n-54) = 0.$$

These solutions are the tesseral polynomials $T_\lambda{}^1(x)$ [7].

$$\boxed{I = 4}$$

$$\begin{aligned} \phi(\gamma,\theta_i) = &[w(6\cos^2\gamma+\sin^2\gamma)+v(\sin 2\gamma\sin\gamma-6\cos 2\gamma\cos\gamma) \\ &+5u\sin\gamma\sin 3\gamma]D^4_{M0}(\theta_i) \\ &+[w\sin 2\gamma+v\sin\gamma+2u\cos\gamma\sin 3\gamma](15)^{\frac{1}{2}}\frac{1}{\sqrt{2}}(D^4_{M2}(\theta_i)+D^4_{M-2}(\theta_i)) \\ &+[w\sin^2\gamma+v\sin 2\gamma\sin\gamma-u\sin 3\gamma\sin\gamma](35)^{\frac{1}{2}}\frac{1}{\sqrt{2}}(D^4_{M4}(\theta_i)+D^4_{M-4}(\theta_i)); \end{aligned}$$

w, v and u satisfy the coupled differential equations

$$-12\frac{\mathrm{d}w}{\mathrm{d}x}+(\Lambda-18)v-36x\frac{\mathrm{d}v}{\mathrm{d}x}+9(1-x^2)\frac{\mathrm{d}^2v}{\mathrm{d}x^2}+12\frac{\mathrm{d}u}{\mathrm{d}x}=0,$$

$$6\frac{\mathrm{d}v}{\mathrm{d}x}+(\Lambda-28)u-42x\frac{\mathrm{d}u}{\mathrm{d}x}+9(1-x^2)\frac{\mathrm{d}^2u}{\mathrm{d}x^2}=0,$$

$$(\Lambda-10)w-30x\frac{\mathrm{d}w}{\mathrm{d}x}+9(1-x^2)\frac{\mathrm{d}^2w}{\mathrm{d}x^2}-18\frac{\mathrm{d}v}{\mathrm{d}x}+10u+12x\frac{\mathrm{d}u}{\mathrm{d}x}=0.$$

1) $w=\sum_n a_n x^{2n},\quad v=\sum_n b_n x^{2n+1},\quad u=\sum_n c_n x^{2n},\quad \pi=+.$

$$\begin{aligned}0 &= a_n(\Lambda-36n^2-42n-10)(\Lambda-36n^2-90n-54)(\Lambda-36n^2-66n-28)\\ &+a_{n+1}3(2n+1)(2n+2)\Big[(6\Lambda-216n^2-348n-148)(\Lambda-36n^2-102n-83)\\ &-(24n+10)(\Lambda-36n^2-90n-54)+(3\Lambda-108n^2-510n-622)\\ &\cdot(\Lambda-36n^2-114n-88)\frac{(9\Lambda-180n^2-330n-167)}{(9\Lambda-180n^2-690n-677)}\Big]\\ &+a_{n+2}27(2n+1)(2n+2)(2n+3)(2n+4)\Big[(3\Lambda-108n^2-174n-74)\\ &+(6\Lambda-216n^2-1068n-1408)\frac{(9\Lambda-180n^2-330n-167)}{(9\Lambda-180n^2-690n-677)}\Big]\\ &+a_{n+3}729(2n+1)(2n+2)(2n+3)(2n+4)(2n+5)(2n+6)\frac{(9\Lambda-180n^2-330n-167)}{(9\Lambda-180n^2-690n-677)},\end{aligned}$$

$$\begin{aligned}c_n &= a_n\frac{(\Lambda-36n^2-42n-10)(3\Lambda-108n^2-294n-220)}{16(9\Lambda-180n^2-330n-167)}\\ &+a_{n+1}\frac{9(2n+1)(2n+2)(3\Lambda-108n^2-318n-278)}{8(9\Lambda-180n^2-330n-167)}\\ &+a_{n+2}\frac{243(2n+1)(2n+2)(2n+3)(2n+4)}{16(9\Lambda-180n^2-330n-167)},\end{aligned}$$

$$b_n = a_n\frac{(\Lambda-36n^2-42n-10)}{18(2n+1)}+a_{n+1}(n+1)+c_n\frac{(12n+5)}{9(2n+1)}.$$

2) $w = \sum_n a_n x^{2n+1}, \quad v = \sum_n b_n x^{2n}, \quad u = \sum_n c_n x^{2n+1}, \quad \pi = -.$

$$0 = b_n(\Lambda - 36n^2 - 54n - 18)(\Lambda - 36n^2 - 78n - 40)(\Lambda - 36n^2 - 102n - 70)$$
$$+ b_{n+1} 18(2n+1)(2n+2)\Big[(\Lambda - 36n^2 - 54n - 18)(\Lambda - 36n^2 - 126n - 116)$$
$$+ (\Lambda - 36n^2 - 174n^{-} - 208)(\Lambda - 36n^2 - 150n - 154)\frac{(12n+13)}{2(12n+25)}\Big]$$
$$+ b_{n+2} 81(2n+1)(2n+2)(2n+3)(2n+4)\Big[(\Lambda - 36n^2 - 54n - 18)$$
$$+ (\Lambda - 36n^2 - 198n - 278)\frac{(24n+26)}{(12n+25)}\Big]$$
$$+ b_{n+3} 729(2n+1)(2n+2)(2n+3)(2n+4)(2n+5)(2n+6)\frac{(12n+13)}{(12n+25)},$$

$$a_n = -b_n \frac{(\Lambda - 36n^2 - 54n - 18)(\Lambda - 36n^2 - 126n - 92)}{48(2n+1)(12n+13)}$$
$$- b_{n+1} \frac{9(2n+2)(\Lambda - 36n^2 - 126n - 116)}{24(12n+13)}$$
$$- b_{n+2} \frac{27(2n+2)(2n+3)(2n+4)}{16(12n+13)},$$

$$c_n = -b_n \frac{(\Lambda - 36n^2 - 54n - 18)}{12(2n+1)} - b_{n+1} \tfrac{3}{2}(n+1) + a_n.$$

$$\boxed{I = 5}$$

$$\phi(\gamma, \theta_i) = (w \cos\gamma \sin 3\gamma + v \cos 2\gamma \sin 3\gamma)\frac{1}{\sqrt{2}}(D^5_{M2}(\theta_i) - D^5_{M-2}(\theta_i))$$
$$+ (w \sin\gamma \sin 3\gamma - v \sin 2\gamma \sin 3\gamma)\frac{1}{\sqrt{2}}(D^5_{M4}(\theta_i) - D^5_{M-4}(\theta_i));$$

$$6\frac{\mathrm{d}w}{\mathrm{d}x} + (\Lambda - 40)v - 48x\frac{\mathrm{d}v}{\mathrm{d}x} + 9(1-x^2)\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = 0,$$

$$(\Lambda - 28)w - 42x\frac{\mathrm{d}w}{\mathrm{d}x} + 9(1-x^2)\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + 12\frac{\mathrm{d}v}{\mathrm{d}x} = 0.$$

1) $w = \sum_n a_n x^{2n}, \quad v = \sum_n b_n x^{2n+1}, \quad \pi = +.$

$$a_{n+2}\, 81(2n+1)(2n+2)(2n+3)(2n+4)$$
$$+a_{n+1}\, 18(2n+1)(2n+2)(\Lambda-36n^2-126n-113)$$
$$+a_n(\Lambda-36n^2-66n-28)(\Lambda-36n^2-114n-88) = 0$$
$$b_n = -a_n \frac{(\Lambda-36n^2-66n-28)}{12(2n+1)} - a_{n+1}\tfrac{3}{2}(n+1).$$

2) $w = \sum_n a_n x^{2n+1}, \quad v = \sum_n b_n x^{2n}, \quad \pi = -.$

$$b_{n+2}\, 81(2n+1)(2n+2)(2n+3)(2n+4)$$
$$+b_{n+1}\, 18(2n+1)(2n+2)(\Lambda-36n^2-126n-116)$$
$$+b_n(\Lambda-36n^2-102n-70)(\Lambda-36n^2-78n-40) = 0,$$
$$a_n = -b_n \frac{(\Lambda-36n^2-78n-40)}{6(2n+1)} - b_{n+1}3(n+1).$$

$$\boxed{I = 6}$$

$$\phi(\gamma,\theta_i) = [3\sqrt{5}w(2\cos^3\gamma+\cos\gamma\sin^2\gamma)+\sqrt{5}v(-6\cos^4\gamma+9\cos^2\gamma\sin^2\gamma+\sin^4\gamma)$$
$$+14\sqrt{5}u\sin\gamma\cos\gamma\sin 3\gamma+14\sqrt{5}\,t\sin\gamma\cos 2\gamma\sin 3\gamma]D^6_{M0}(\theta_i)$$
$$+[\sqrt{14}w(6\sin\gamma\cos^2\gamma+\tfrac{1}{2}\sin^3\gamma)+5\sqrt{14}\,v\sin^3\gamma\cos\gamma$$
$$+\sqrt{14}u(8\cos^2\gamma+3\sin^2\gamma)\sin 3\gamma$$
$$+\sqrt{14}t(8\cos^3\gamma-14\sin^2\gamma\cos\gamma)\sin 3\gamma]\frac{1}{\sqrt{2}}(D^6_{M2}(\theta_i)+D^6_{M-2}(\theta_i))$$
$$+[3\sqrt{35}\,w\cos\gamma\sin^2\gamma+\sqrt{35}\,v(3\sin^2\gamma\cos^2\gamma+\sin^4\gamma)$$
$$+2\sqrt{35}\,u\sin\gamma\cos\gamma\sin 3\gamma$$
$$-2\sqrt{35}t(11\sin\gamma\cos^2\gamma-3\sin^3\gamma)\sin 3\gamma]\frac{1}{\sqrt{2}}(D^6_{M4}(\theta_i)+D^6_{M-4}(\theta_i))$$
$$+[w\sin^3\gamma+2v\sin^3\gamma\cos\gamma-2u\sin^2\gamma\sin 3\gamma$$
$$+4t\sin^2\gamma\cos\gamma\sin 3\gamma]\left(\frac{11\times 35}{2}\right)^{\frac{1}{2}}\frac{1}{\sqrt{2}}(D^6_{M6}(\theta_i)+D^6_{M-6}(\theta_i));$$

$$[\Lambda(49-4x^2)+72x^2-882]w+(144x^2-954)x\frac{dw}{dx}+(441-36x^2)(1-x^2)\frac{d^2w}{dx^2}$$
$$+x[-45\Lambda+1260]v+(1440x^2-630)\frac{dv}{dx}-405x(1-x^2)\frac{d^2v}{dx^2}$$
$$+[\Lambda 91(1-x^2)+3528x^2-2268]u-3906x(1-x^2)\frac{du}{dx}+819(1-x^2)^2\frac{d^2u}{dx^2}$$
$$+(\Lambda 91(1-x^2)+4914x^2-3654]xt$$
$$+(1-x^2)(1008-4914x^2)\frac{dt}{dx}+819x(1-x^2)^2\frac{d^2t}{dx^2}=0,$$

$$[-\Lambda 18x+324x]w+(396x^2-72)\frac{dw}{dx}-162x(1-x^2)\frac{d^2w}{dx^2}$$
$$+[\Lambda(4+14x^2)-392x^2-112]v-(504x^2-180)x\frac{dv}{dx}+(36+126x^2)(1-x^2)\frac{d^2v}{dx^2}$$
$$+[-14\Lambda(1-x^2)x+56(1-10x^2)x]u$$
$$+(672x^2+84)(1-x^2)\frac{du}{dx}-126x(1-x^2)^2\frac{d^2u}{dx^2}$$
$$+[-14\Lambda(1-x^2)+644-1148x^2]t+(1-x^2)x756\frac{dt}{dx}-126(1-x^2)^2\frac{d^2t}{dx^2}=0,$$

$$(\Lambda 13-234)w-378x\frac{dw}{dx}+117(1-x^2)\frac{d^2w}{dx^2}$$
$$+(-\Lambda 5x+140x)v+(210x^2-120)\frac{dv}{dx}-45x(1-x^2)\frac{d^2v}{dx^2}$$
$$+(\Lambda 32-916)u-1152x\frac{du}{dx}+288(1-x^2)\frac{d^2u}{dx^2}$$
$$+(\Lambda 32-1588)xt+(456-1608x^2)\frac{dt}{dx}+288x(1-x^2)\frac{d^2t}{dx^2}=0,$$

$$(\Lambda 13-234)xw+(90-468x^2)\frac{dw}{dx}+117x(1-x^2)\frac{d^2w}{dx^2}$$
$$+(-\Lambda 5+140)v+90x\frac{dv}{dx}-45(1-x^2)\frac{d^2v}{dx^2}$$
$$+(\Lambda 32-916)xu+(120-1272x^2)\frac{du}{dx}+288x(1-x^2)\frac{d^2u}{dx^2}$$
$$+[\Lambda(80-48x^2)t+(2592x^2-4180)]t$$
$$+x(2592x^2-3744)\frac{dt}{dx}+(1-x^2)(720-432x^2)\frac{d^2t}{dx^2}=0$$

In principle, one can proceed in the same way as before, but the coefficients in the recurrence relations get very involved. Instead, it is possible to use the eigenvalues (taken, for instance, from the discussion in section 3) and solve for the coefficients of the wave functions. As in the other cases, two parities are present:

1) $w = \sum_n a_n x^{2n}, \quad v = \sum_n b_n x^{2n+1}, \quad u = \sum_n c_n x^{2n}, \quad t = \sum_n d_n x^{2n+1}, \quad \pi = -.$

$$\begin{aligned}
&a_{n+2}\,441(2n+3)(2n+4)+a_{n+1}(\Lambda 49-1908n^2-4770n-3744)\\
&\quad -a_n\,4(\Lambda-36n^2-54n-18)-b_{n+1}\,90(2n+3)(9n+16)\\
&\quad -b_n\,45(\Lambda-36n^2-82n-60)+c_{n+2}\,819(2n+3)(2n+4)\\
&\quad +c_{n+1}\,7(\Lambda 13-936n^2-2520n-1908)-c_n 7(\Lambda 13-468n^2-882n-504)\\
&\quad +d_{n+1}\,126(2n+3)(13n+21)+d_n\,7(\Lambda 13-936n^2-2160n-1368)\\
&\quad -91d_{n-1}(\Lambda-36n^2-54n-18) = 0,
\end{aligned}$$

$$\begin{aligned}
&\quad -a_{n+1}\,9(2n+2)(18n+13)-a_n\,9(\Lambda-36n^2-26n-18)\\
&\quad +b_{n+1}\,18(2n+2)(2n+3)+b_n\,2(\Lambda+90n^2+135n+17)\\
&\quad +b_{n-1}\,7(\Lambda-36n^2-108n+80)-c_{n+1}21(2n+2)(6n+1)\\
&\quad -c_n\,7(\Lambda-72n^2-48n-4)+c_{n-1}7(\Lambda-36n^2-6n+2)\\
&\quad -d_{n+1}\,63(2n+2)(2n+3)-d_n\,7(\Lambda-72n^2-144n-100)\\
&\quad +d_{n-1}\,7(\Lambda-36n^2-18n-82) = 0,
\end{aligned}$$

$$\begin{aligned}
&a_{n+2}\,117(2n+3)(2n+4)+a_{n+1}(\Lambda 13-468n^2-1458n-1224)\\
&\quad -b_{n+1}\,30(2n+3)(3n+7)-b_n 5(\Lambda-36n^2-102n-70)\\
&\quad +c_{n+2}\,288(2n+3)(2n+4)+c_{n+1}\,4(\Lambda 8-288n^2-1008n-949)\\
&\quad +d_{n+1}\,24(2n+3)(24n+43)+d_n\,4(\Lambda 8-288n^2-948n-799) = 0,
\end{aligned}$$

$$\begin{aligned}
&a_{n+1}\,9(2n+2)(26n+23)+a_n\,13(\Lambda-36n^2-54n-18)\\
&\quad -b_{n+1}\,45(2n+2)(2n+3)-b_n\,5(\Lambda-36n^2-54n-46)\\
&\quad +c_{n+1}\,24(2n+2)(24n+17)+c_n\,4(8\Lambda-288n^2-492n-229)\\
&\quad +d_{n+1}\,720(2n+2)(2n+3)+d_n\,4(\Lambda 20-1152n^2-2448n-1981)\\
&\quad -d_{n-1}\,48(\Lambda-36n^2-54n-18) = 0.
\end{aligned}$$

2) $w=\sum_n a_n x^{2n+1},\quad v=\sum_n b_n x^{2n},\quad u=\sum_n c_n x^{2n+1},\quad t=\sum_n d_n x^{2n}, \pi=+.$

$$
\begin{aligned}
&a_{n+2}\,441(2n+4)(2n+5)+a_{n+1}(\Lambda 49-1908n^2-6678n-6606)\\
&\quad -a_n 4(\Lambda-36n^2-90n-54)-b_{n+2}\,45(2n+4)(18n+41)\\
&\quad -b_{n+1}\,45(\Lambda-36n^2-118n-110)+c_{n+2}\,819(2n+4)(2n+5)\\
&\quad +c_{n+1}7(\Lambda 13-936n^2-3456n-3402)\\
&\quad -c_n\,7(\Lambda 13-468n^2-1350n-1062)+d_{n+2}63(2n+4)(26n+55)\\
&\quad +d_{n+1}\,7(\Lambda 13-936n^2-3096n-2682)\\
&\quad -d_n\,7(\Lambda 13-468n^2-1170n-702)=0,
\end{aligned}
$$

$$
\begin{aligned}
&-a_{n+1}\,18(18n^2+49n+33)-a_n\,9(\Lambda-36n^2-62n-40)+b_{n+2}\,18(2n+4)(2n+3)\\
&\quad +b_{n+1}\,2(\Lambda+90n^2+225n+107)+b_n 7(\Lambda-36n^2-54n-28)\\
&\quad -c_{n+1}\,42(2n+3)(3n+2)-c_n\,7(\Lambda-72n^2-120n-46)\\
&\quad +c_{n-1}\,7(\Lambda-36n^2-42n-10)-d_{n+2}\,63(2n+4)(2n+3)\\
&\quad -d_{n+1}\,7(\Lambda-72n^2-216n-190)+d_n 7(\Lambda-36n^2-90n-82)=0,
\end{aligned}
$$

$$
\begin{aligned}
&a_{n+1}\,117(2n+2)(2n+3)+a_n(\Lambda 13-468n^2-990n-612)\\
&\quad -b_{n+1}\,15(2n+2)(6n+11)-b_n\,5(\Lambda-36n^2-66n-28)\\
&\quad +c_{n+1}\,288(2n+2)(2n+3)+c_n\,4(\Lambda 8-288n^2-720n-517)\\
&\quad +d_{n+1}\,24(2n+2)(24n+31)+d_n\,4(\Lambda 8-288n^2-660n-397)=0,
\end{aligned}
$$

$$
\begin{aligned}
&a_{n+1}\,18(26n^2+75n+54)+a_n\,13(\Lambda-36n^2-90n-54)\\
&\quad -b_{n+2}\,45(2n+3)(2n+4)-b_{n+1}\,5(\Lambda-36n^2-90n-82)\\
&\quad +c_{n+1}\,24(2n+3)(24n+29)+c_n\,4(\Lambda 8-288n^2-780n-547)\\
&\quad +d_{n+2}\,720(2n+3)(2n+4)+d_{n+1}\,4(\Lambda 20-1152n^2-3600n-3493)\\
&\quad -d_n\,48(\Lambda-36n^2-90n-54)=0.
\end{aligned}
$$

These linear equations, plus the relations $a_{n+1}(n)=a_n(n+1)$, allow us to solve for the coefficients.

2. COEFFICIENTS IN THE EXPANSION OF w, v, . . ., u, FOR STATES WITH $I \leqq 6, \lambda \leqq 9$.

I	λ	π_γ	a_0	a_1	b_0	b_1	c_0	d_0	$N_{I\lambda}$
0	0	+	1						4
0	6	+	−1	3					16/5
0	3	−	1						4/3
0	9	−	−3	5					16/7
2	1	−	1						4
2	5	−	1		−3				8
2	7	−	−1	5	−2				16/3
2	2	+			1				4
2	4	+	−3		1				8
2	8	+	−2		−1	5			16/3
3	3	−	1						8/3
3	9	−	−1	5					64/21
3	6	+	1						8/15
4	2	+	1						144
4	4	+	5				−9		9360
4	6	+	6		13		−3		12376/5
4	8	+	−5	85	60		−6		32640
4	3	−			1				88
4	5	−	11		6				2640
4	7	−	11		3		−15		4560
4	9	−	60		−7	95	−30		27968
5	4	+	1						8/3
5	8	+	1		−5				32/3
5	5	−			1				8/3
5	7	−	−5		1				32/3
6	3	−	1						2288/3
6	5	−	14				−11		418880/3
6	7	−	37		65		−13		553280/3
6	9	−	127	1615	1530		−135		797070976/21
6	9	−	16290	16184	41714		−10485	19279	780413920000/3
6	4	+			1				1040/3
6	6	+	13		9				208832/15
6	6	+	196		406			251	908017600/3
6	8	+	2002		1274		−1105	117	592067840/3

The normalization factor for the resultant wave functions is $\left[\frac{2I+1}{8\pi^2 N_{I\lambda}}\right]^{\frac{1}{2}}$.

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