THE USE OF COHERENT STATES IN THE THEORY OF SUPERFLUIDITY

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It is shown how the matrix elements of the pseudo-unitary operator producing the Bogoliubov transformation for bosons and the corresponding states with sharp particle number, can be rather simply evaluated using eigenstates of the destruction operators.

There are a number of model Hamiltonians for a real bose system, which are diagonalized by the transformation

$$\alpha_k(\varphi) = \Lambda(\varphi) a_k \Lambda^{-1}(\varphi) = u_k a_k + v_k a^+_{-k}, \tag{1}$$

where

$$\Lambda(\varphi) = \exp\{-\sum_k \theta_k (a^+_k a^+_{-k}\, e^{2i\varphi} - a_k a_{-k}\, e^{-2i\varphi})\}$$

and $u_k = \cosh\theta_k$, $v_k = e^{-2i\varphi}\sinh\theta_k$. The model Hamiltonians we refer to above, are approximated by some "reduced Hamiltonian", which is not gauge invariant.

Eigenstates of a_k have been used in the past [1] to express matrix elements of Λ, using Schwinger's variational principle. The use of the properties shown by Glauber [2-4] makes the derivation more transparent. The Schrödinger coherent states are defined by

$$a_k|\{a'_k\}\rangle = a'_k|\{a'_k\}\rangle, \qquad \alpha_k|\{\alpha'_k\}\rangle = \alpha'_k|\{\alpha'_k\}\rangle. \tag{2}$$

Here $\{a'_k\} \equiv a'$ and $\{\alpha'_k\} \equiv \alpha'$ are two sets of complex numbers. Relation (1) implies $|\alpha'\rangle = \Lambda|a'\rangle$, $a' = \alpha'$. We need further the properties

$$\langle a'|a''\rangle = \exp\sum_k a'^*_k a''_k, \qquad \frac{\partial}{\partial a'_k}|a'\rangle = a^+_k|a'\rangle,$$
$$1 = \int |a'\rangle\langle a'|\,d\mu(a'), \qquad d\mu(a') = \prod_k \frac{1}{\pi}\, d^2a'_k \exp(-|a'_k|^2) \tag{3}$$

Which are likewise valid for $|\alpha'\rangle$. To evaluate $\langle a'|\Lambda|a''\rangle = \langle a'|\alpha''\rangle$, we differentiate it with respect to the eigenvalues

$$\frac{\partial\langle a'|\alpha''\rangle}{\partial a'^*_k} = \langle a'|a_k|\alpha''\rangle = (B_k\alpha''_k + \Gamma_k a'^*_{-k})\langle a'|\alpha''\rangle \qquad \frac{\partial\langle a'|\alpha''\rangle}{\partial\alpha''_k} = \langle a'|\alpha^+_k|\alpha''\rangle = (B_k a'^*_k + A_k\alpha''_{-k})\langle a'|\alpha''\rangle \tag{4}$$

where $A_k = v_k/u_k$, $B_k = 1/u_k$, $\Gamma_k = -v^*_k/u_k$. Eqs. (4) have the common solution

$$\langle a'|\alpha''\rangle = C\exp\{\sum_k(\tfrac{1}{2}\alpha''_k A_k \alpha''_{-k} + \alpha''_k B_k a'^*_k + \tfrac{1}{2}a'^*_k\Gamma_k a'^*_{-k})\} \tag{5}$$

The constant C can be determined using eq. (5) and the completeness relation for $|\alpha'\rangle$, to evaluate $\langle a'|a'\rangle$. After elementary integrations we arrive at the desired result [1]

$$|C|^2 = \prod_k (u_k u_{-k})^{-1} = (\det u)^{-1} \tag{6}$$

Now let $|\Omega_\varphi\rangle$ be the vector obtained setting all $\alpha'_k = 0$. This is the ground state for $\alpha_k(\varphi)$. We show that

$$|\Omega_N\rangle = \int_0^{2\pi} (\mathrm{d}\varphi/2\pi)\, \exp(-2\mathrm{i}N\varphi)\, |\Omega_4\rangle$$

N integer, is an eigenstate of the number operator [5] $\hat{N} = \sum_k a^+{}_k a_k$, i. e.

$$\hat{N}\,|\Omega_N\rangle = N\,|\Omega_N\rangle \tag{7}$$

To evaluate the left hand side of this equation, we expand $\hat{N}$ in terms of coherent states

$$\hat{N} = \iint \mathrm{d}\mu(a')\mathrm{d}\mu(b')\mathcal{N}(a'^*, b')\; |a'\rangle\langle b'|$$

with

$$\mathcal{N}(a'^*, b') = \langle a'|\hat{N}|b'\rangle = \sum_q a'^*{}_q b'{}_q \cdot\ \exp\sum_k a'^*{}_k b'{}_k$$

Using the auxiliary relation

$$\int \mathrm{d}\mu(b')\, \exp\{\lambda a'^*{}_k b'{}_k + \tfrac{1}{2} b'^*{}_k \Gamma_k b'^*{}_{-k}\} = \exp\{\tfrac{1}{2}\lambda^2 \sum_k a'^*{}_k \Gamma_k a'^*{}_{-k}\}$$

we obtain

$$\hat{N}|\Omega_N\rangle = \int_0^{2\pi} \frac{\mathrm{d}\varphi}{2\pi} \int \mathrm{d}\mu(a') \int \mathrm{d}\mu(b')\, \exp(-2\mathrm{i}N\varphi)\; |a'\rangle\langle b'|\Omega_\varphi\rangle \sum_q a'^*_q b'_q\; \exp\{\sum_k a'^*{}_k b'{}_k\} =$$

$$= C\int_0^{2\pi} \frac{\mathrm{d}\varphi}{2\pi} \int \mathrm{d}\mu(a')\, \exp(-2\mathrm{i}N\varphi)\; |a'\rangle\, N\, \exp\sum_k \tfrac{1}{2} a'^*{}_k \Gamma_k a'^*{}_{-k} = N\; |\Omega_N\rangle$$

The physical implications of coherent states for the theory of superfluidity have been recently discussed in ref. 6.

The results reported in this letter are part of a work on the mathematical properties of the pair Hamiltonian model [7]. The author is indebted to Prof. W. Thirring for suggesting this work to him and for valuable comments.

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