



Modified-Ricatti approach to partially solvable quantum Hamiltonians.

II: Morse oscillator related family.

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We extend the scope of the modified-Ricatti approach to partial solubility in quantum mechanics introduced in a previous work [L.D.Salem and R.Montemayor, Phys. Rev. A **43**, 1169 (1991)]. With the use of adequate mappings $u(x)$ we show the convenience of the modified-Ricatti approach to analyze potentials that can be written as rational functions on u . The necessary conditions for a Hamiltonian to be solvable are discussed in detail. By considering the exponential mapping $u = e^{-x}$ we construct a family of potentials related to the exactly solvable Morse oscillator. Within this family, we have identified a new three-parameter quasi-exactly solvable potential, which depending on the value of its coupling constants leads to a symmetric or asymmetric confining potential, with a single-well or a double-well structure. Explicit expression for the energies and eigenfunctions are given for particular cases. The analytic continuation of the symmetric subset gives places to a quasi-exactly solvable periodic potential.

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I. INTRODUCTION

Since the advent of quantum mechanics exactly solvable Schrödinger equations have been object of intense research. Historically however, the most fruitful formalism to obtain exact solutions for second order differential equations was stated before the quantum era by Darboux over a century ago.¹ The Darboux factorization method was rediscovered by Schrödinger himself in the quantum mechanics context,² and some years later thoroughly exploited by Infeld and Hull in a remarkable paper,³ where they classified most of the exactly solvable (ES) Hamiltonians. Two decades ago Natanzon obtained the most general form of the potential for which the Schrödinger equation reduces to hypergeometric form.⁴ Group theory techniques has been also applied to study the solubility problem providing an algebraic approach⁵ to obtain the spectrum and matrix elements for solvable potentials from the dynamical groups. Other interesting contribution to construct new exactly solvable potentials is based on the Gelfand and Levitan equation.⁶ On this basis Abraham and Moses⁷ and Pursey and collaborators⁸ developed a procedure to insert and delete some bound states from a given Hamiltonian. More recently, supersymmetric (SUSY) theories motivated the extensive use of SUSY transformations in quantum mechanics.⁹ Several authors recognized that SUSY transformations are related to the Schrödinger factorization method and therefore to the Darboux procedure. It was Gendenshtein who introduced the concept of shape invariance in SUSY quantum mechanics and showed¹⁰ that this is a sufficient condition to have exact solubility. Later on it was proved¹¹ that shape invariance and the condition of factorization of Infeld and Hull are equivalents.

In summary, exact solubility is found only for very few potentials. Most of them are contained in the Infeld and Hull classification,³ and have as a common feature that they can be written as simple rational functions in terms of $u(x)$, where $u(x)$ is an exponential, circular or hyperbolic function or, at most, a simple combination of them.

On the other hand, in recent years there has been a growing interest in a wider category of potentials, with only a solvable subset of eigenfunctions. There are some examples of such potentials,¹² but it is not obvious how to state the general conditions to guarantee this restricted solubility. Certainly the understanding of this problem would allow us to gain insight into the whys of the exactly solvable models and, what is more attractive, to construct new solvable potentials with a priori specified features.

This is the second paper of a series dealing with *partial solubility* (PS) and *quasi-exact* (QES) solubility in quantum mechanics. In a previous work¹³ (hereafter to be referred as paper I), we presented a general formalism to analyze solubility based on a modified-Ricatti equation for the regular component of the logarithmic derivative of the wave function. There,¹³ we fully developed the simplest version of the

method, which proved adequate to investigate solubility in the family of finite Laurent series type potentials. These include, among others, the Coulomb potential, one-dimensional (1D) and three-dimensional (3D) harmonic and anharmonic polynomial oscillator potentials, inverse power potentials, and the Coulomb plus polynomial terms.

Generalizing what was anticipated in paper I, the modified-Ricatti approach is also suitable for investigating a more general class of potentials, namely, those which can be written as rational functions of $u(x)$, provided that $(du/dx)^2$ is a polynomial in u . In fact, this is the only restriction we shall consider on the transformation $u(x)$. These two conditions are certainly satisfied by the exactly solvable potentials discussed by Infeld and Hull³ and Natanzon⁴ which then should emerge as particular cases in the solubility analysis considered here. An example of such a mapping was explicitly considered in paper I, where we chose $u = x^2$ [$(du/dx)^2 = 4u$], a transformation that is adequate to consider reflection symmetric potentials $V(x) = V(-x)$. In this family we recognized both the ES members, i.e., the 1D and 3D harmonic oscillators, and the QES members, i.e., the 1D and 3D sextic polynomial oscillators.¹² Besides these we also found an infinite number of partially solvable (PS) potentials. From this perspective each of the well-known ES potentials³ turns out to be a member of a family characterized by one of the simplest transformations $u(x)$ satisfying the previously mentioned restriction.

In this and forthcoming papers we shall use the generalized modified-Ricatti approach to construct the potential families related to well-known exactly solvable potentials³ identifying the corresponding QES members in each family.

This paper is organized as follows. In Section II we extend the scope of the modified-Ricatti approach for a quite general transformation $u(x)$. In Section III we discuss the conditions for solubility. In Section IV we consider the family of potentials related to the Morse oscillator, using the exponential mapping $u = e^{-x}$, and analyze in detail the ES (Morse oscillator) and QES members. Finally, in Section V we summarize and discuss our results.

II. EXTENSION OF THE MODIFIED-RICATTI APPROACH

In the following we extend the scope of the modified-Ricatti approach presented in paper I. When considering the quantum motion of a 1D particle in a potential $V(x)$, we are interested in closed form solutions of the Schrödinger equation

$$\frac{d^2\psi}{dx^2} + [E - V(x)]\psi = 0. \quad (2.1)$$

The wave function $\psi(x)$ satisfies the appropriate boundary conditions, e.g., if $V(x)$

is a potential well, the bounded eigenfunction vanishes at the boundaries $x_{\pm}$ of the x -physical domain¹⁴ $x_- < \Re(x) < x_+$.

Let us now consider a mapping $u(x)$, regular in the physical domain. We define the u -physical domain as the image of the x -physical domain, leaving outside the zeros of du/dx . Eventually such a mapping will allow us to obtain closed form solutions of the Schrödinger Eq. (2.1) for a given potential $V(x) = W(u)$, or alternatively, will allow us to construct potentials from given closed form eigenfunctions. Both goals can be easily achieved within the formalism developed below. For $\psi(x) = \Psi(u)$ the Schrödinger Eq. (2.1) reads

$$\left[\alpha(u) \frac{d^2}{du^2} + \beta(u) \frac{d}{du} + E - W(u) \right] \Psi(u) = 0, \quad (2.2)$$

where

$$\alpha(u) = \left(\frac{du}{dx} \right)^2, \quad \beta(u) = \frac{d^2 u}{dx^2} = \frac{1}{2} \frac{d\alpha}{du}. \quad (2.3)$$

We can always decompose

$$\Psi(u) \equiv C(u) \bar{\Psi}(u), \quad (2.4)$$

where the factor $\bar{\Psi}(u)$ is defined to have no zeros in the u -physical domain and satisfies the appropriate boundary conditions at $u(x_-)$ and $u(x_+)$. On the other hand, the nodal structure of the wave function is embodied in $C(u)$. For a given transformation $u(x)$, the number of nodes of $\psi(x)$ is univocally related to the number of simple zeros that $C(u)$ has in the u -physical domain.¹⁵ For instance if $V = W(u = x^2)$ and $\psi(x)$ has n nodes in the x -physical domain spanning the whole real axis $-\infty < \Re(x) < \infty$, the factor $C(u)$ has $[n/2]$ simple zeros in $0 < \Re(u) < \infty$. The node at the origin that odd parity eigenfunctions have is outside the u -physical domain and thus must be provided by the factor $\bar{\Psi}(u)$.

We shall assume that the nodal structure of the eigenfunction $\psi(x)$ considered can be represented by a finite number of simple zeros in the u -physical domain. Under such assumption one can always write the nodal factor $C(u)$ as a polynomial of degree N in u :

$$C(u) = \prod_{j=1}^N (u - u_j) \equiv C(u; N). \quad (2.5)$$

We should remark that some of the zeros of $C(u; N)$ can be outside the physical domain and thus, only $m \leq N$ of that zeros correspond to nodes of the wave function $\psi(x)$. In that sense, for a fixed value of N , the Eqs. (2.4-2.5) can represent more than one wave function depending on where the zeros u_j are located.

From equations (2.4) and (2.5),

$$\Psi(u) \propto \bar{\Psi}(u) \prod_{j=1}^N (u - u_j) = \exp \int^u du' \left[\bar{G}(u') + \sum_{j=1}^N (u' - u_j)^{-1} \right]. \quad (2.6)$$

where $\bar{G}(u)$ is the logarithmic derivative of $\bar{\Psi}(u)$ and thus is, by definition, regular in the u -physical domain. The substitution of Eq. (2.6) into Eq. (2.2) leads to the equation¹³

$$\alpha \left(\bar{G}^2 + \frac{d\bar{G}}{du} \right) + \frac{1}{2} \frac{d\alpha}{du} \bar{G} + 2\alpha F = W - E \quad (2.7)$$

where

$$F(u) \equiv \sum_{j=1}^N \frac{\bar{G}(u) + \frac{1}{4} \frac{d}{du} \ln \alpha - \Gamma_{Nj}}{u - u_j}, \quad (2.8)$$

and

$$\Gamma_{Nj} \equiv \sum_{\substack{k=1 \\ k \neq j}}^N \frac{1}{u_k - u_j} \quad j = 1, 2, \dots, N \geq 2. \quad (2.9)$$

To be consistent we define $F(u) \equiv 0$ for $N = 0$ and $\Gamma_{Nj} \equiv 0$ for $N = 0, 1$. Some useful relations involving the Γ_{Nj} are derived in the Appendix A.

Provided that $W(u)$ is regular in the u -physical domain, the left hand side of Eq. (2.7) must also be regular in this domain. Therefore, as both $\bar{G}(u)$ and $\alpha(u)$ are regular there, it must be:

$$\left[\bar{G}(u) + \frac{1}{4} \frac{d}{du} \ln \alpha(u) \right]_{u=u_j} = \Gamma_{Nj}, \quad j = 1, 2, \dots, N. \quad (2.10)$$

This set of nonlinear equations relates the zeros $\{u_j\}$ of $C(u; N)$ to the regular component $\bar{G}(u) = d[\ln \bar{\Psi}(u)]/du$.

Let us introduce the function

$$\tilde{G}(u) \equiv \bar{G}(u) + \frac{1}{4} \frac{d}{du} \ln \alpha, \quad (2.11)$$

which is also regular in the u -physical domain, as $\alpha(u)$ has no zeros there. The Eq. (2.7) now takes the form of the modified-Ricatti equation [see Eq. (2.9) in paper I]

$$\tilde{G}^2 + \frac{d\tilde{G}}{du} + 2F = \frac{W - E}{\alpha} + \gamma^2 + \frac{d\gamma}{du} \quad (2.12)$$

where $\gamma(u) = \frac{1}{4} d[\ln \alpha(u)]/du$. From Eq. (2.10)

$$\tilde{G}(u_j) = \Gamma_{Nj}, \quad j = 1, 2, \dots, N \quad (2.13)$$

and therefore, $F(u)$ becomes

$$F(u) = \sum_{j=1}^N \frac{\tilde{G}(u) - \tilde{G}(u_j)}{u - u_j}. \quad (2.14)$$

Finally, in terms of $\tilde{G}(u)$, the wave function reads

$$\psi(x) \propto [\alpha(u)]^{-1/4} C(u; N) \exp \int^u du' \tilde{G}(u'). \quad (2.15)$$

This particular form of writing the wave function is reminiscent of the uniform semi-classical approximation as proposed by Miller and Good.¹⁶ In that sense, the modified-Ricatti approach provides a clear understanding¹⁷ about the relationship between the Coulomb, Morse oscillator, and harmonic oscillator with centripetal barrier problems, and thus, sheds light on the origin of the Langer correction.¹⁸

Consistently, the Eqs. (2.12-2.15) reduces to the plain modified-Ricatti approach presented in Sec. II of paper I when $u(x) = x$.

To conclude this section it is very important to remark that, given the regular component $\tilde{G}(u)$ and N , the system of nonlinear Eqs. (2.13) can have more than one solution set $\{u_j\}$. A simple visualization of this fact relies on an electrostatic analogy. The set of Eqs. (2.13) can be read of as the equilibrium condition of a system of N particles with a potential energy

$$\begin{aligned} \mathcal{U}(u_1, u_2, \dots, u_N) &= \sum_{k \neq j} \ln |u_k - u_j| + \sum_{j=1}^N \int^{u_j} du' \tilde{G}(u') \\ &= \sum_{k \neq j} \mathcal{U}_{\text{pair}}(k, j) + \sum_{j=1}^N \mathcal{U}_{\tilde{G}}^{\text{ext}}(u_j), \end{aligned} \quad (2.16)$$

and constrained to move along a line [the image of the open interval (x_-, x_+) by the transformation $u(x)$]. The N positive 'unit charges' with coordinates $u_1, u_2, \dots, u_N$ interact through a *two-body* repulsive force in the presence of the *external* attractive field $\int^u du' \tilde{G}(u')$. Hence, the number of solution sets to Eq. (2.13) is equal to the number of equilibrium configurations of this N particle system. This very intuitive picture was first noticed by Stieljes in relation to the position of the zeros of Jacobi polynomials.¹⁹ Our Eq. (2.13), as well as the interpretation given in (2.16), generalizes such a finding for polynomials not so well-known in mathematical physics.^{13,20}

III. SOLUBILITY ANALYSIS

From now on, and generalizing the discussion in Section II of paper I, we focus our study on potentials for which a mapping $u(x)$ exists such that:

- i) $V(x) = W(u)$ is a rational function of u , and
- ii) $\alpha(u) = (du/dx)^2$ is a polynomial in u .

In such a case, the modified-Ricatti equation (2.12) is particularly well-suited to relate the functional form of $\tilde{G}(u)$ and $W(u)$ by simple inspection.

According to Eq.(2.15), given the transformation $u(x)$, the regular component $\tilde{G}(u)$, and N one can always construct a wave function $\psi(x)$. This is so because the polynomial $C(u; N)$ (or equivalently its zeros $\{u_j\}$) is specified by $\tilde{G}(u)$ and N through the set of Eqs.(2.13). The so constructed wavefunction is an eigenfunction of the Schrodinger equation with the potential $W(u)$ given by

$$W(u) = E + \alpha \left[\tilde{G}^2 + \frac{d\tilde{G}}{du} + 2F - \left(\gamma^2 + \frac{d\gamma}{du} \right) \right]. \quad (3.1)$$

As remarked at the end of Sec. II, given $\tilde{G}(u)$ and N , more than one wave function in the form of (2.15) can be constructed, provided that the nonlinear Eqs.(2.13) can have more than one solution set $\{u_j\}$. Let us suppose for the moment that, given $\tilde{G}(u)$ and N , we have already found the $\{u_j^{(m)}\}$, where the supra index m labels the m -th solution set of Eqs.(2.13). In principle each set $\{u_j^{(m)}\}$ can lead to a different function $F(u; N, \{u_j^{(m)}\}) \equiv F_m(u)$ and hence the left hand side of Eq. (3.1) can accordingly change. There are three possibilities:

- a) If the functions αF_m corresponding to distinct solution sets $\{u_j^{(m)}\}$ differ from each other in their u dependence, the resulting wave functions correspond to different potentials. These potentials belong to the class of PS Hamiltonians.
- b) Another much more interesting possibility is that the αF_m corresponding to distinct solution sets $\{u_j^{(m)}\}$ differ from each other only by an additive constant. In this case the different wave functions correspond to eigenstates of the same Hamiltonian, with different energies. We have then a QES potential. The solution sets $\{u_j^{(m)}\}$ differ from each other in the number of zeros inside the u -physical domain. Correspondingly, the different wave functions have different number of nodes. At most there can be $N + 1$ of these eigenfunctions.
- c) Finally if the difference between the αF_m (corresponding to two possible solutions sets) vanishes, both eigenfunctions would have the same energy. Being both eigenstates of the same Hamiltonian, this is in contradiction with the well-known fact that degeneracy cannot occur for 1D bound states due to time reversal symmetry. Hence, there must be only one solution set $\{u_j\}$ for each value of N . This is the case for ES potentials.

In summary, the recognition of PS, QES or ES of the Schrödinger Eq.(2.1) with $V(x) = W(u)$ derives from the analysis of αF_m . The necessary conditions for the situations just discussed require the possibility of writing

$$2\alpha F_m = \begin{cases} \mathcal{F}(u; N, \{u_j^{(m)}\}) - \mathcal{K}(N, \{u_j^{(m)}\}) & \text{for PS,} \\ \mathcal{F}(u; N) - \mathcal{K}(N, \{u_j^{(m)}\}) & \text{for QES,} \\ \mathcal{F}(u; N) - \mathcal{K}(N) & \text{for ES,} \end{cases} \quad (3.2)$$

where we have made explicit:

- i) The u -independent term of $2\alpha F_m$, namely $-\mathcal{K}$. This term is, up to an additive constant, the energy of the eigenfunction with zeros at $\{u_j^{(m)}\}$.
- ii) The dependence of the u -dependent term of $2\alpha F_m$, namely $\mathcal{F}$, on the different solution sets $\{u_j^{(m)}\}$ to Eqs.(2.13) for a given $\tilde{G}(u)$ and N .

From a computational perspective, in order to completely specify the eigenfunctions and eigenenergies there is a more convenient alternative to the nonlinear Eqs. (2.13) for the $\{u_j\}$. This is true at least for the more interesting cases, namely, the QES and ES potentials. It consists in finding the coefficients $\{c_k\}$ of the polynomial $C(u; N) \equiv \sum_{j=1}^N c_k u^k$. Replacing $\Psi(u) = C(u; N)\bar{\Psi}(u)$ in the Schrödinger equation (2.2) we obtain

$$\alpha \bar{\Psi} \frac{d^2 C}{du^2} + \left[2\alpha \frac{d\bar{\Psi}}{du} + \frac{1}{2} \frac{d\alpha}{du} \bar{\Psi} \right] \frac{dC}{du} + \left[\alpha \frac{d^2 \bar{\Psi}}{du^2} + \frac{1}{2} \frac{d\alpha}{du} \frac{d\bar{\Psi}}{du} + (E - W)\bar{\Psi} \right] C = 0. \quad (3.3)$$

Factorizing $\bar{\Psi}(u) = \exp \int^u du' \tilde{G}(u')$ and taking into account Eqs. (2.7) and (2.11), we obtain a second order differential equation for the polynomial $C(u; N)$

$$\alpha \left(\frac{d^2 C}{du^2} + 2\tilde{G} \frac{dC}{du} - 2F C \right) = 0. \quad (3.4)$$

The analysis summarized in Eq. (3.2) is pertinent here. When the potential is partially solvable, the coefficient $2F$ depends on the particular solution to be found (See appendix B in paper I), and the determination of the coefficients c_k from (3.4) is a formidable task due to the high nonlinearity of the problem, unless N is small.

On the other hand, in the QES situation the Eq. (3.4) defines a finite eigenvalue problem for the coefficients c_k [See Eq. (3.2)], and thus the determination of energy levels and wave functions is completely algebraic.

Lastly, when the potential is ES, Eq. (3.4) is one of the well-known differential equations of mathematical physics.

In the next section we shall use the modified-Ricatti Eq. (2.12) to construct families of partially solvable potentials starting from a given form for $\tilde{G}(u)$. Using the solubility analysis here presented we shall determine whether the necessary conditions for PS, QES or ES hold.

IV. MORSE OSCILLATOR RELATED FAMILY

We propose here to investigate a family of potentials that include as a special case the Morse oscillator,²¹

$$V_{\text{MO}}(x) = S^2(e^{-2x} - e^{-x}), \quad (4.1)$$

for which the transformation

$$u(x) = e^{-x} \quad (4.2)$$

provides closed form solutions.^{21,22} For the mapping (4.2) we have $\alpha(u) = u^2$, and $\gamma(u) = (2u)^{-1}$. Hence, the modified-Ricatti Eq. (2.12) [more conveniently written as in (3.1)] reads

$$W(u) = E + \frac{1}{4} + u^2 \left(\tilde{G}^2 + \frac{d\tilde{G}}{du} + 2F \right). \quad (4.3)$$

Following Sec. VI of paper I, we construct a family of potentials that has as a particular member the Morse oscillator. On the basis of the mapping (4.2) we consider

$$\tilde{G}(u) = \tilde{G}_{\text{MO}}(u) + \delta\tilde{G}(u), \quad (4.4)$$

where $\tilde{G}_{\text{MO}}(u) = a_0 + a_{-1}u^{-1}$ has the functional form that solves Eq. (4.3) for the Morse oscillator (4.1), and $\delta\tilde{G}(u)$ is a finite Laurent series in u , regular in the u -physical domain. Therefore,

$$\tilde{G}(u) = \sum_{k=-L}^M a_k u^k \quad (4.5)$$

with both $L, M \geq 0$. From Eq. (2.14) and with the particular finite Laurent type form of $\tilde{G}(u)$ given in Eq. (4.5), we find $F(u)$ to have also a finite Laurent series form.¹³ Explicitly

$$F(u) = \sum_{k=-L}^{M-1} f_k^N u^k \quad (4.6)$$

where the coefficients f_k^N are given by¹³

$$f_k^N = \begin{cases} N \sum_{m=k+1}^M a_m \langle\langle u^{m-k-1} \rangle\rangle & \text{for } 0 \leq k \leq M-1, \\ -N \sum_{m=-L}^k a_m \langle\langle u^{m-k-1} \rangle\rangle & \text{for } -L \leq k \leq -1, \end{cases} \quad (4.7)$$

and we have introduced the k -th moment of the set $\{u_j\}$

$$\langle\langle u^k \rangle\rangle \equiv \frac{1}{N} \sum_{j=1}^N (u_j)^k. \quad (4.8)$$

Using Eqs. (4.5-4.8) in (4.3) we obtain a family of potentials related to the Morse oscillator. This family comprises the finite Laurent potentials in u

$$W(u) = \sum_{k=-2L}^{2M} w_k u^{k+2} = e^{-2x} \sum_{k=-2L}^{2M} w_k e^{-kx}, \quad (4.9)$$

where the coupling constants w_k , for $k \neq -2$, are given by¹³

$$w_k = \begin{cases} \sum_{j=-L}^{k+L} a_j a_{k-j} & -2L \leq k \leq -L-2, \\ -La_{-L} + \sum_{j=-L}^{-1} a_j a_{-L-1-j} & k = -L-1; \\ 2f_k^N + (k+1)a_{k+1} + \sum_{j=-\min(L,M-k)}^{\min(L+k,M)} a_j a_{k-j} & -L \leq k \leq M-1, \\ \sum_{j=k-M}^M a_j a_{k-j} & M \leq k \leq 2M. \end{cases} \quad (4.10)$$

In principle, the constant w_{-2} can be arbitrarily chosen as only fixes the energy origin. Thus, for $k = -2$, we obtain a relation between the energy and the coefficients $\{a_k\}$ and f_{-2}^N :

$$E = w_{-2} - \frac{1}{4} - 2f_{-2}^N + a_{-1} - \sum_{j=-\min(L,M+2)}^{j=\min(L-2,M)} a_j a_{-j-2}. \quad (4.11)$$

It is clear from (4.9) that the potentials so far constructed, have no singularities in $-\infty < \Re(x) < \infty$ and thus the u -physical domain is the open interval $0 < \Re(u) < \infty$. Therefore, if $C(u; N)$ has m zeros on $\Re(u) > 0$, the wave function has m nodes and thus corresponds to the m -th excited level. Explicitly

$$\psi_m(x; N) = C(u; N) u^{a-1-1/2} \exp \sum_{\substack{k=-L \\ k \neq -1}}^M \frac{a_k}{k+1} u^{k+1}, \quad (4.12)$$

where the coefficients a_k can be chosen with the only requirement of making the wave function (4.12) normalizable.

The solubility analysis presented in Sec. III relies on the particular functional dependence of αF in terms of the zeros $\{u_j\}$. In fact, to perform such an analysis, there is no need to determine the set $\{u_j\}$ but only its moments $\langle\langle u^k \rangle\rangle$ on which the

coefficients f_k^N depend, according to Eq. (4.7). Taking the r -th moments of (2.13), one obtains

$$\langle\langle u^r \tilde{G} \rangle\rangle = \frac{1}{N} \sum_{j=1}^N (u_j)^r \tilde{G}(u_j) = \frac{1}{N} \sum_{j=1}^N (u_j)^r \Gamma_{Nj} \equiv \langle\langle u^r \Gamma \rangle\rangle, \quad r \geq 0. \quad (4.13)$$

The weighted moments $\langle\langle u^r \Gamma \rangle\rangle$ are evaluated in the appendix A. For the particular finite Laurent type form of $\tilde{G}(u)$ given in Eq. (4.5), the set of Eqs. (2.13) is equivalent to:

$$\sum_{k=-L}^M a_k \langle\langle u^{k+r} \rangle\rangle = \begin{cases} 0 & \text{for } r = 0, \\ -(N-1)/2 & \text{for } r = 1, \\ \frac{r}{2} \langle\langle u^{r-1} \rangle\rangle - \frac{N}{2} \sum_{s=0}^{r-1} \langle\langle u^s \rangle\rangle \langle\langle u^{r-1-s} \rangle\rangle & \text{for } r \geq 2. \end{cases} \quad (4.14)$$

Let us now isolate the more interesting members of the family, namely, the ES and the QES potentials.

a) If we were able to eliminate the dependence of αF on the moments $\langle\langle u^k \rangle\rangle$, then we would have satisfied the necessary conditions to have an exactly solvable potential [See Eqs. (3.2)]. From the system of Eqs. (4.14) one can solve the moments $\langle\langle u^k \rangle\rangle$ completely in terms of the coefficients a_k , only when:

$$\begin{aligned} \text{i) } & a_{-1} \neq 0, a_0 \neq 0 \text{ and } a_k = 0 \text{ otherwise,} \\ \text{ii) } & a_{-1} \neq 0, a_{-2} \neq 0 \text{ and } a_k = 0 \text{ otherwise,} \\ \text{iii) } & a_0 \neq 0, a_1 \neq 0 \text{ and } a_k = 0 \text{ otherwise.} \end{aligned} \quad (4.15)$$

Case (iii) is a spurious situation that gives place to non-normalizable eigenfunctions. For the first two cases [(i) and (ii)], the function αF becomes

$$\begin{aligned} \text{i) } & \alpha F = N a_0 u \\ \text{ii) } & \alpha F = N(N-1)/2 + N a_{-1} \end{aligned} \quad (4.16)$$

corresponding to the Morse oscillator potential as given by Eq. (4.1), and its mirror image $V_{MO}(-x)$, respectively.

b) Let $a_{-2} \neq 0$, $a_{-1} \neq 0$, $a_0 \neq 0$ and $a_k = 0$ otherwise; Then, from the system of Eqs. (4.14) we find

$$\alpha F = N a_0 u + N(N-1)/2 + N a_{-1} + N a_0 \langle\langle u \rangle\rangle, \quad (4.17)$$

and hence, according to Eq. (3.2), we have a QES potential.

c) Any other set of values for the a_k gives place to a PS potential.

Until now, we have applied the solubility analysis to identify the ES ($M = 0, L = 1$) and the QES ($M = 0, L = 2$) members of the family defined by Eq. (4.9). In the following subsections we pursue the analysis of such cases provided that the functional form of $\tilde{G}_{ES}$ and $\tilde{G}_{QES}$ is known.

A. The Morse oscillator ($M = 0, L = 1$)

First we consider the best known representative of this family, the Morse potential ($M = 0, L = 1$). Although its features are common place²², we find enlightening to analyze it from the present perspective. In this case we have $\tilde{G}(u) = a_0 + a_{-1}u^{-1}$, and thus,

$$F(u) = Na_0 u^{-1} \quad (4.18)$$

does not depend on the $\langle\langle u^k \rangle\rangle$ with $k \neq 0$. From Eqs. (4.9, 4.10) the resulting potential is

$$\begin{aligned} W(u) &= \sum_{k=-2}^0 w_k u^{k+2} = (a_0)^2 u^2 + 2a_0(N + a_{-1})u \\ &= (a_0)^2 e^{-2x} + 2a_0(N + a_{-1})e^{-x}, \end{aligned} \quad (4.19)$$

where we have fixed the energy origin such that $w_{-2} = 0$. We remark that this is the only potential within the family that has no constraining relations between the coefficients of $W(u)$.

The closed form eigenfunctions and corresponding energy levels [see Eqs. (4.11) and (4.12)] result

$$\psi(x) \propto C(u; N) u^{a_{-1}-1/2} e^{a_0 u}, \quad (4.20)$$

$$E = -(a_{-1} - 1/2)^2. \quad (4.21)$$

The wave function given in Eq. (4.20) can only represent a bound state because it has only a finite number of nodes, and therefore must vanish at the boundaries $x \rightarrow \pm\infty$. This requires

$$a_0 < 0, \quad N + a_{-1} > 1/2. \quad (4.22)$$

From Eq. (4.19) the potential has a minimum at $x_0 = \ln[-a_0/(N + a_{-1})]$. We can always fix the origin of coordinates at x_0 so

$$N + a_{-1} + a_0 = 0. \quad (4.23)$$

and therefore, the potential considered has *only one free-parameter*, e.g., $S > 0$ in terms of which $a_0 = -S$ and $a_{-1} = S - N$ are defined. The potential then acquires the usual expression for the Morse oscillator with a minimum at $x = 0$ and dissociation energy S^2 ,

$$V(x) = S^2 (e^{-2x} - 2e^{-x}), \quad (4.24)$$

with an energy level at

$$E_N = -(S - N - 1/2)^2. \quad (4.25)$$

For the potential (4.24), given a value of N we have obtained *only one* discrete level, which corresponding to a bound state it *must be* non degenerate. Thus, *only one* wave function in the form $\psi_N(x) \propto C(u; N) u^{S-N-1/2} \exp(-Su)$, can be obtained associated to that level.

Notice that the potential so written does not depend on the particular value chosen for N . Hence, we are free to choose the value of N in the range $0 \leq N \leq N_{\max} < S - 1/2$ in order to accomplish Eqs. (4.22,4.23). The Morse oscillator potential has then *all* its discrete energy levels given by Eq. (4.25).

To completely determine the wave functions, we have to look for the polynomial solution $C(u; N)$ of the differential Eq. (3.4)

$$u \frac{d^2 C}{du^2} + (2S - 2N - 2Su) \frac{dC}{du} + 2SNC = 0. \quad (4.26)$$

This is the well-known Kummer's equation²³ and its regular solution is proportional to the Laguerre polynomial $L_N^{2S-2N-1}(2Su)$ with *all* its N zeros in $\Re(u) > 0$. In summary, the N -th excited wave function of the Morse oscillator potential is

$$\psi_N(x) \propto L_N^{2S-2N-1}(2Se^{-x}) \exp[-(S - N - 1/2)x - Se^{-x}], \quad (4.27)$$

with energy given by Eq. (4.25) for $N = 0, 1, \dots, N_{\max} < S - 1/2$.

B. The QES member ($M = 0, L = 2$)

The other example that we will develop here corresponds to the case $M = 0, L = 2$, e.g.

$$\tilde{G}(u) = a_0 + a_{-1}u^{-1} + a_{-2}u^{-2}, \quad (4.28)$$

and thus from Eq. (4.17)

$$F(u) = Na_0u^{-1} + [N(N-1)/2 + Na_{-1} + Na_0\langle\langle u \rangle\rangle]u^{-2}, \quad (4.29)$$

which explicitly depends on the first moment $\langle\langle u \rangle\rangle$. According to Eqs. (4.9,4.10) the potential associated potential is

$$\begin{aligned} W(u) &= (a_0)^2 u^2 + 2a_0(N + a_{-1})u + 2a_{-2}(a_{-1} - 1)u^{-1} + (a_{-2})^2 u^{-2} \\ &= (a_0)^2 e^{-2x} + 2a_0(N + a_{-1})e^{-x} + 2a_{-2}(a_{-1} - 1)e^x + (a_{-2})^2 e^{2x}. \end{aligned} \quad (4.30)$$

In particular for $a_{-2} = 0$ or $a_0 = 0$ we recover the Morse oscillator potential. As before we have chosen the energy origin fixing $w_{-2} = 0$.

The closed form eigenfunctions and corresponding energy levels are

$$\psi(x) \propto C(u; N) u^{a_{-1}-1/2} \exp(a_0 u - a_{-2} u^{-1}). \quad (4.31)$$

$$E = -\frac{1}{4} - a_{-1}(a_{-1} - 1) - 2a_0 a_{-2} - N(N - 1) - 2Na_{-1} - 2Na_0 \langle u \rangle, \quad (4.32)$$

The potential (4.30) is a confining one as $V(|x| \rightarrow \infty) \rightarrow \infty$ and thus, all the eigenstates in such a potential are bounded. Therefore, the eigenfunctions (4.31) must vanish at $|x| \rightarrow \infty$. This imposes a restriction on the values that a_0 and a_{-2} can take, say $a_0 < 0$ and $a_{-2} > 0$. As was previously noted, for a given value of N the closed form solutions before mentioned can represent, at most, the first $N + 1$ states.

Unlike what happened with the Morse oscillator, there is no way to avoid the explicit dependence on N in the potential (4.30), and hence $W(u) = W(u; N)$. In other words, among all the potentials $\sum_{k=-4}^0 w_k u^{k+2}$, we can obtain closed form solution as given by (2.4-2.5), only when the constraint relation

$$w_{-3}/(w_{-4})^{1/2} + w_{-1}/(w_0)^{1/2} + 2(N + 1) = 0 \quad (4.33)$$

between the coupling constants is satisfied. Like for the QES potential considered in paper I, e.g., the symmetric sextic polynomial potential, the constraint (4.33) on the couplings depends on N but not on a particular solution set $\{u_j\}$. Hence, if there is more than one solution to Eqs. (2.13), all the states corresponds to the same potential.

Choosing the origin of coordinates such that the potential becomes asymptotically symmetric [$(a_0)^2 = (a_{-2})^2$] we introduce a new parameterization

$$a_{-2} = -a_0 = \eta > 0, \quad a_{-1} = (\Delta - N + 1)/2. \quad (4.34)$$

so the three-parameter potential (4.30) now reads

$$V(x; N) = 2\eta^2 \cosh 2x - 2\eta(N + 1) \cosh x + 2\eta\Delta \sinh x. \quad (4.35)$$

The parameter Δ clearly gives a direct measure of the asymmetry of the potential which results in a single-well or a double-well, depending on the values of the parameters. In particular, when $\Delta = 0$ (symmetric case) the potential develops two wells when $\eta < \eta_c \equiv (N + 1)/4$, and only one otherwise.²⁴ The wave functions (4.31) now read,

$$\begin{aligned} \psi(x) &\propto C(u; N) u^{-(N-\Delta)/2} \exp[-\eta(u^{-1} + u)] \\ &= C(u = e^{-x}; N) e^{(N-\Delta)x/2} \exp[-2\eta \cosh(x)]. \end{aligned} \quad (4.36)$$

As given in Eq. (4.32), the energy level of this wave function depends both on N and the first order moment $\langle u \rangle$ of the zeros of $C(u; N)$.

To completely specify the wave functions and energy levels, we need in addition the polynomial solutions $C(u; N)$ of the second order differential Eq. (3.4), which in this case reads

$$u^2 \frac{d^2 C}{du^2} + [2\eta - (N - 1 - \Delta)u - 2\eta u^2] \frac{dC}{du} + [E - 2\eta^2 + \frac{1}{4}(N - \Delta)^2 + 2\eta Nu] C = 0. \quad (4.37)$$

For the very particular symmetric case $\Delta = 0$, the Eq. (4.37) was first studied by Ince before the Schrödinger formulation of quantum mechanics, in relation with the vibrational modes of a membrane with variable density.²⁰ Then, our Eq. (4.37) gives place to a generalization of the Ince polynomials.

In terms of the coefficients c_k of the polynomial $C(u; N) = \sum_{k=0}^N c_k u^k$, the second order differential Eq. (4.37) can be read of as a finite eigenvalue problem of order $(N + 1)$

$$\sum_{l=0}^N H_{k,l} c_l = E c_k. \quad (4.38)$$

where the only non-null components the matrix $H_{k,l}$ are

$$\begin{aligned} H_{k,k-1} &= -2\eta(N + 1 - k); \\ H_{k,k} &= 2\eta^2 - [k - (N - \Delta)/2]^2; \\ H_{k,k+1} &= -2\eta(k + 1), \end{aligned} \quad (4.39)$$

for $k = 0, 1, \dots, N$. The coefficients of the tridiagonal matrix satisfy $H_{k,k+1} H_{k+1,k} > 0$. This property allows us to transform the matrix $H_{k,l}$ into a symmetric real one by a similarity transformation, and therefore, all the $(N + 1)$ eigenvalues are real. From each eigenvector we can reconstruct a polynomial $C(u; N)$ or equivalently a wave function $\psi(x)$ as given in (4.36). These $(N + 1)$ distinct eigenfunctions have at most N nodes and therefore, the associated eigenvalues are the $N + 1$ lowest energy levels of the spectrum.

In other words, the potential (4.35) is a QES one, with its ground and N first excited states belonging to the solvable Hilbert subspace. The complete specification of closed form eigenfunctions and corresponding energies only involves a finite matrix diagonalization.

In the general (non-symmetric) case the eigenvalue problem (4.38) can be solved explicitly up to $N = 3$. For larger values of N we must resort to a numerical diagonalization.

In the symmetric situation $V(x) = V(-x)$, or equivalently $W(u) = W(u^{-1})$, the parity of a wave function with m nodes (m zeros in $0 < \Re(u) < \infty$) is $(-1)^m$.

From Eq. (4.36) with $\Delta = 0$, we have $C(1/u) = (-1)^m u^N C_m(u)$ and hence $c_k^{(m)} = (-1)^m c_{N-k}^{(m)}$. This symmetry reduce the order of the eigenvalue problem, and explicit expressions can be obtained up to $N = 7$.

In the following we explicitly obtain the closed form eigensolutions for the potential (4.35) with a) $N = 1$, and b) $N = 3$. In the first case ($N = 1$) the potential reads

$$V(x; N = 1) = 2\eta^2 \cosh 2x - 4\eta \cosh x + 2\eta\Delta \sinh x. \quad (4.40)$$

The 2×2 H matrix to be diagonalised is

$$H = \begin{pmatrix} 2\eta^2 - (\Delta - 1)^2/4 & -2\eta \\ -2\eta & 2\eta^2 - (\Delta + 1)^2/4 \end{pmatrix}, \quad (4.41)$$

leading to the ground state and first excited energy levels,

$$E_{0,1} = 2\eta^2 - \frac{(\Delta^2 + 1)}{4} \pm (\Delta^2/4 + 4\eta^2)^{1/2}, \quad (4.42)$$

and the corresponding eigenfunctions are

$$\psi_0(x) \propto \left\{ 2\eta e^{x/2} + \left[\Delta/2 + (\Delta^2/4 + 4\eta^2)^{1/2} \right] e^{-x/2} \right\} \Phi(x; \eta, \Delta), \quad (4.43)$$

$$\psi_1(x) \propto \left\{ \left[\Delta/2 + (\Delta^2/4 + 4\eta^2)^{1/2} \right] e^{x/2} - 2\eta e^{-x/2} \right\} \Phi(x; \eta, \Delta),$$

where $\Phi(x) = \exp \left\{ -2\eta \cosh x - \frac{\Delta}{2} x \right\}$. The well defined even and odd parity eigenstates are recovered when $\Delta = 0$. Moreover, if we let $\eta \rightarrow 0$ (so the condition $\eta < \eta_c$ is fulfilled and the potential has a double-well structure) a pair of nearly degenerate levels results, as expected.²⁴

Let us now consider $N = 3$ for which the QES potential (4.35) becomes,

$$V(x; N = 3) = 2\eta \cosh 2x - 8\eta \cosh x + 2\eta\Delta \sinh x. \quad (4.44)$$

In this case, the matrix H is

$$H = \begin{pmatrix} 2\eta^2 - \frac{1}{4}(\Delta - 3)^2 & -2\eta & 0 & 0 \\ -6\eta & 2\eta^2 - \frac{1}{4}(\Delta + 1)^2 & -4\eta & 0 \\ 0 & -4\eta & 2\eta^2 - \frac{1}{4}(\Delta - 1)^2 & -6\eta \\ 0 & 0 & -2\eta & 2\eta^2 - \frac{1}{4}(\Delta + 3)^2 \end{pmatrix}, \quad (4.45)$$

giving place to a quartic secular equation for the solvable energy levels. The explicit expressions for the eigenvalues and eigenfunctions, although algebraic, are rather cumbersome, and we do not present them here. For the particular symmetric case ($\Delta = 0$) the problem decouples in the even and odd parity sectors, and the secular equation factorizes into two quadratic equations, whose solutions are

$$\begin{aligned} E_{0,2} &= - \left[\frac{9}{4} - (1 - 2\eta) \pm (1 - 4\eta + 16\eta^2)^{1/2} \right], \\ E_{1,3} &= - \left[\frac{9}{4} - (1 + 2\eta) \pm (1 + 4\eta + 16\eta^2)^{1/2} \right]. \end{aligned} \quad (4.46)$$

As before, when $\eta \rightarrow 0$ the energy levels degenerate in pairs. Finally, the corresponding eigenfunctions are

$$\begin{aligned} \psi_{0,2}(x) &\propto \left[\cosh \frac{3}{2}x + \frac{1}{2\eta} \left(E_{0,2} + \frac{9}{4} \right) \cosh \frac{x}{2} \right] e^{-2\eta \cosh x}, \\ \psi_{1,3}(x) &\propto \left[\sinh \frac{3}{2}x + \frac{1}{2\eta} \left(E_{1,3} + \frac{9}{4} \right) \sinh \frac{x}{2} \right] e^{-2\eta \cosh x}, \end{aligned} \quad (4.47)$$

with apparent well defined parity.

V. CONCLUDING REMARKS

In this paper we have extended the scope of the modified-Ricatti approach developed in paper I, to treat a more general class of potentials by introducing a coordinate transformation $u(x)$. The modified-Ricatti equation was shown to be particularly adequate to investigate, as well as to construct, solvable potentials which can be written as rational functions of the mapping $u(x)$, provided that $(du/dx)^2$ is a polynomial in u . We have refined the solubility analysis for this kind of potentials, by looking at the functional form that $2\alpha F$ acquires [See Eq. (3.2)].

Having noticed that all the exactly solvable potentials classified by Infeld and Hull³ and Natanzon,⁴ satisfy the above mentioned requirements, we have proposed¹³ to construct PS and QES potentials classified in families related to the already well-known ES ones.³ These families are generated by the addition of regular terms to the logarithmic derivative of the wave functions of the exactly solvable potentials, for which the identification of the mapping $u(x)$ is evident.¹³

In particular, we have explicitly constructed and analysed the Morse related family using the exponential mapping $u = e^{-x}$. Within this family we have identified a three-parameter QES potential which, depending on the value of its coupling constants,

leads to a symmetric ($\Delta = 0$) or asymmetric ($\Delta \neq 0$), and a single-well or a double well potential.

We want to remark that the existence of exact and closed form solutions to such a versatile model, with a symmetric or asymmetric bistability and a phase transition to monostability when the *control parameters* η , Δ and N are varied, is of great interest. Certainly, it allows us to check and compare different approximations schemes to tunneling rates in such a systems. This has motivated a much more detailed analysis of the QES potential found here. Our research on that line it is outside of the scope of this paper and will be given elsewhere.²⁴

To conclude we address some interesting points related to the same QES problem. First, the symmetric subset of the QES potential given in (4.35) suggests another possible mapping. Instead of the transformation $u = e^{-x}$ we could have use

$$u = \cosh x, \quad \alpha(u) = (u - 1)(u + 1). \quad (5.1)$$

The mapping (5.1), like $u = x^2$ treated in paper I, is only adequate to consider reflection symmetric potentials. With this new mapping the potential (4.35) in the symmetric case takes the form

$$V(x; N, \eta, \Delta = 0) = 4\eta^2 u^2 - 2\eta(N + 1)u - 2\eta^2, \quad (5.2)$$

which is also suitable to treat my means of the modified-Ricatti approach.

Second, A related interesting problem arises from the analytic continuation of the QES member (or any other symmetric member in the family). It is easy to show that, if $\psi_E(x)$ is an eigenfunction of the Schrödinger Eq. (2.1) with $V(x)$, then $\phi_e(x) \equiv \psi_E(ix)$ is an eigenfunction of the same equation with the potential $U(x) \equiv -V(ix)$ and the corresponding eigenvalue is $\epsilon = -E$. When this transformation is applied to the QES potential given in Eq. (4.35), we find that the periodic potential

$$U(x; N) = 2\eta(N + 1) \cos x - 2\eta^2 \cos 2x \quad (5.3)$$

is also QES. The energy levels obtained from the eigenvalue problem (4.38) correspond to some of the band edges of the spectrum.²⁴ The analytic continuation procedure is equivalently found when considering the mapping $u = \cos x$, and thus, the band edge eigenfunctions in the periodic potential (5.3) are also obtained from Eq. (4.38) with $\Delta = 0$. It can be shown that the two parameter (η, N) periodic potential (5.3) has a finite number of gaps and thus can be of interest in solid state physics as a, yet simple but much more realistic model, than the well known Kronig-Penney potential.²⁵

We postpone the use of the transformations $u = \cosh x$ and $u = \cos x$, for the analysis of the family related to the Pöschl-Teller potential in a future work.²⁴ The symmetric QES member here found, and the QES periodic potential (5.2) will be obtained there, as particular cases of the QES member in the new families.

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APPENDIX A:

In this appendix we derive some simple and useful relations satisfied by the coefficients

$$\Gamma_{Nj} \equiv \sum_{k \neq j} \frac{1}{u_k - u_j}, \quad j = 1, 2, \dots, N \geq 2, \quad (A1)$$

defined in Sec. II. From symmetry considerations, we can evaluate the successive *weighted moments* defined as

$$\langle\langle u^r \Gamma \rangle\rangle \equiv \frac{1}{N} \sum_{j=1}^N (u_j)^r \Gamma_{Nj} = -\frac{1}{2N} \sum_{j=1}^N \sum_{k \neq j} \frac{(u_k)^r - (u_j)^r}{u_k - u_j}. \quad (A2)$$

In particular, for $r = 0$ and $r = 1$, we have

$$\langle\langle \Gamma \rangle\rangle = 0, \quad (3)$$

$$\langle\langle u \Gamma \rangle\rangle = -(N-1)/2, \quad (4)$$

respectively. These relations have been used in Sec. IV to simplify the expressions in evaluating αF in order to determine PS, QES or ES solubility in the potentials there considered.

For the higher moments ($r \geq 2$),

$$\begin{aligned} \langle\langle u^r \Gamma \rangle\rangle &= -\frac{1}{2N} \sum_{j=1}^N \sum_{k \neq j} \left[(u_j)^{r-1} + (u_j)^{r-2} u_k + \dots + (u_k)^{r-1} \right] \\ &\quad - \frac{1}{2N} \left\{ r \left(\sum_{j=1}^N (u_j)^{r-1} \right) - \sum_{s=0}^{r-1} \left[\sum_{j=1}^N (u_j)^s \right] \left[\sum_{k=1}^N (u_k)^{r-s-1} \right] \right\} \\ &= \frac{r}{2} \langle\langle u^{r-1} \rangle\rangle - \frac{N}{2} \sum_{s=0}^{r-1} \langle\langle u^s \rangle\rangle \langle\langle u^{r-s-1} \rangle\rangle, \end{aligned} \quad (5)$$

where $\langle\langle u \rangle\rangle \equiv (1/N) \sum_{k=0}^N (u_k)^r$ are the *non-weighted moments* as defined in Eq. (4.8).

In summary, only the $r = 0$ and $r = 1$ weighted moments *do not* depend on the position of the zeros $\{u_j\}$ of $C(u; N)$.

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