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COLLECTIVE TREATMENT OF THE PAIRING HAMILTONIAN
(IV). The yrast approximation

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Abstract: The pairing collective excitations are analyzed within the framework of the yrast approximation. The states are supposed to have a large isospin and a large number of pairs outside the double magic core. The large centrifugal terms that are therefore present, induce a stable deformation in isospin and gauge spaces. A complete and physically meaningful classification of states is obtained in terms of rotations and small oscillations. This description reproduces the known harmonic results concerning energies and pair transfer matrix elements to leading order in the yrast quantum number Y, which is suitably defined. It is also shown how to introduce corrections to this zeroth order description. With this purpose a modification of the usual perturbation theory is developed. It is also shown how anharmonic terms may be introduced simultaneously with the above mentioned corrections.

1. Introduction

A collective treatment for the $T = 1$ pairing interaction has been presented in previous papers [1-3]. The Hamiltonian, which is analogous to the Bohr collective Hamiltonian [4], yields wave functions carrying the isospin T and the number of pairs of particles M as good quantum numbers.

These solutions are obtained in the two extreme situations corresponding to the harmonic potential and to rigid rotations of a system with an axially symmetric deformation. In principle, intermediate situations may be solved with numerical methods.

The collective treatment may be applied for instance, to the set of $J^{\pi} = 0^{+}$ states in the region around a doubly magic nucleus with equal number of protons and neutrons. The experimental situation around [5-9] ^{56}Ni suggests that we are in the presence of a phase transition between a normal and a superconducting system. Since the numerical methods become prohibitive for not very large values of T, we must resort to other methods to solve the corresponding Schrödinger equation.

The yrast approximation [10] is based on the assumption that the centrifugal terms in the kinetic energy may induce a permanent deformation in some states, even if the potential does not lead to a deformed situation. Thus, these states may be appropriately treated within the formalism of deformed systems. The system distributes the

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rotational quanta among the three axes, in such a way as to minimize the total energy. The states satisfying these conditions are the yrast states. To the extent that most of the kinetic energy is concentrated in the centrifugal term, the wave functions of the yrast states are stable under changes in the detailed structure of the potential energy surface and the vibrational degrees of freedom remain essentially frozen.

For a harmonic potential the yrast concept provides a new labelling of the states, which is equivalent to the one obtained by coupling phonons. It is expected that the consequences of the anharmonicities are smaller within the yrast approximation since in many cases they tend to induce deformations, and this effect is already accomplished in zero order by the centrifugal terms. In addition, the remaining anharmonicities should be easier to deal with, since the description of the deformed states may be simpler within the yrast scheme than within the vibrational picture.

In this paper we attempt the application of the yrast concept to the pairing degree of freedom. In sect. 2 we apply this approximation in the case of a harmonic motion. We define the yrast quantum number Y (Y = integer) which is related to the isospin and to the number of pairs M. This quantum number plays here the same role as the angular momentum in the quadrupole case. The yrast quantum number labels each member of a set of quasi-rotational states. In addition the different distribution of rotational quanta among the three axis, and the small oscillations that the system can perform around the equilibrium position, give rise to a subset of vibrational and rotational states, for each value of Y.

Up to this point we have applied to the pairing degree of freedom what is known about the yrast approximation to the quadrupole case. This yields the vibrational result to lowest order in an expansion in powers of $1/Y$. In addition, in sect. 3 (and appendices A and B), we develop the perturbation theory which is necessary to correct the approximation in successive orders of $1/Y$. In sect. 4 we show how to introduce anharmonic terms both in the kinetic and potential energies using similar techniques as those developed in sect. 3. This formalism is therefore able to be applied to the transitional region.

Together with the conclusions, we discuss possible applications of the present collective treatment.

2. Formulation of the yrast approximation

An important part of the two-body residual interaction between the nucleons is

$$H_{\mathrm{p}} = -G \sum_{\mu} P_{\mu} P_{\mu}^{+},$$

where P_{μ} creates two particles with angular momentum zero, isospin one, and isospin projection μ. Here G is the strength of the pairing force. The distorted field approximation corresponds to replacing the P_{μ} operators by their (complex) expectation values d_μ. These d_μ are the dynamical variables associated with the collective treatment. They transform according [1]) to

$$d_\mu = e^{2i\phi} \sum_\nu D^1_{\mu\nu}(\theta_i) d'_\nu, \tag{1}$$

where ϕ is the gauge angle (which is conjugate to the operator corresponding to the number of particles) and θ_i are Euler angles in isospace. Applying (1) we can transform to an intrinsic system in which only two (out of six) collective variables are different from zero. They may be chosen to represent the parameters† $\Delta_n = d'_1$ and $\Delta_p = d'_{-1}$ for the "intrinsic" neutrons and protons, respectively [11]). Alternatively, we may use as intrinsic variables Δ and Γ, which are defined by

$$\Delta = (\Delta_n^2 + \Delta_p^2)^{\frac{1}{2}}, \qquad \Gamma = \mathrm{tg}^{-1}\left(\frac{\Delta_p - \Delta_n}{\Delta_p + \Delta_n}\right).$$

Let us write down the collective Hamiltonian for the pairing degree of freedom†† [eq. (21) of ref. [1])]:

$$\begin{aligned} H &= T_{\mathrm{rot}} + T_{\mathrm{vib}} + V, \\ T_{\mathrm{rot}} &= \frac{1}{2B\Delta^2}\left\{\frac{(T_x+M)^2}{\cos^2 2\Gamma} - 2MT_x\frac{(1-\sin 2\Gamma)}{\cos^2 2\Gamma} + \frac{T_y^2}{\cos^2 \Gamma} + \frac{T_z^2}{\sin^2 \Gamma}\right\}, \\ T_{\mathrm{vib}} &= -\frac{1}{2B\Delta^5}\frac{\partial}{\partial \Delta}\Delta^5\frac{\partial}{\partial \Delta} - \frac{1}{2B\Delta^2 \sin 4\Gamma}\frac{\partial}{\partial \Gamma}\sin 4\Gamma\frac{\partial}{\partial \Gamma}, \\ V &= V(\Delta^2, \Delta^4\cos^2 2\Gamma). \end{aligned} \tag{2}$$

The domains of the variables Δ and Γ are

$$0 \leqq \Delta \leqq \infty, \qquad 0 \leqq \Gamma \leqq \tfrac{1}{4}\pi, \tag{3}$$

and the volume element is

$$d\tau = \Delta^5 |\sin 4\Gamma| d\Gamma \, d\Delta \, d\phi \, d^3\theta_i.$$

In eq. (2), M represents the operator corresponding to the number of pairs of particles outside the double closed shell, while T_x, T_y and T_z are the operators corresponding to the isospin components in the intrinsic system.

The harmonic solution can be constructed either from the Schrödinger equation corresponding to the Hamiltonian [1,2]) (2) with $V = \frac{1}{2}C\Delta^2$ or as a superposition of addition and removal phonons [12]), each of them carrying an isospin $t = 1$. The other simple solution of (2) corresponds to rigid rotations, in which case the low energy degrees of freedom are included in T_{rot}, with Δ and Γ being constant parameters.

† When the system stabilizes around definite values of Δ_n and Δ_p ($\neq 0$), the quantities $G\Delta_n$ and $G\Delta_p$ coincide with the BCS gap parameters for the system of neutrons and protons, respectively.

†† It has been assumed that the mass parameter B is a constant, which is equivalent to the hypothesis of irrotational flow for the quadrupole degree of freedom.

In order to find the yrast states, we minimize $T_{\rm rot}+V$ with respect to T_x, T_z, Γ and Δ, for fixed values of T and M:

$$\frac{\partial}{\partial T_x}(T_{\rm rot}+V) = \frac{2\sin\Gamma}{\cos^2\Gamma\cos^2 2\Gamma}(T_x\sin\Gamma+2M\cos^3\Gamma) = 0, \tag{4}$$

$$\frac{\partial}{\partial T_z}(T_{\rm rot}+V) = \frac{T_z 8\cos 2\Gamma}{\sin^2 2\Gamma} = 0, \tag{5}$$

$$\frac{\partial}{\partial \Gamma}(T_{\rm rot}+V) = \frac{1}{B\Delta^2}$$
$$\times\left\{\frac{2(T_x+M)^2\sin 2\Gamma}{\cos^3 2\Gamma}+\frac{2MT_x(1-\sin 2\Gamma)^2}{\cos^3 2\Gamma}+T_y^2\frac{\sin\Gamma}{\cos^3\Gamma}-T_z^2\frac{\cos\Gamma}{\sin^3\Gamma}\right\}+\frac{\partial V}{\partial\Gamma} = 0, \tag{6}$$

$$\frac{\partial}{\partial \Delta}(T_{\rm rot}+V) = -\frac{1}{B\Delta^3}$$
$$\times\left\{\frac{(T_x+M)^2}{\cos^2 2\Gamma}-2MT_x\frac{(1-\sin 2\Gamma)}{\cos^2 2\Gamma}+\frac{T_y^2}{\cos^2\Gamma}+\frac{T_z^2}{\sin^2\Gamma}\right\}+\frac{\partial V}{\partial \Delta} = 0. \tag{7}$$

The possible solutions to eq. (4) are

$$T_x = -2M\cos^3\Gamma/\sin\Gamma \tag{4a}$$

or

$$\Gamma = 0 \tag{4b}$$

or

$$T_x = M = 0, \tag{4c}$$

while the solutions of (5) imply

$$\Gamma = \tfrac{1}{4}\pi \tag{5a}$$

or

$$T_z = 0. \tag{5b}$$

We introduce in (6) different pairs of solutions of (4) and (5). If we assume V to be Γ-independent, we obtain using (4a)+ (5a)

$$\Gamma = \tfrac{1}{4}\pi, \qquad T_x = -M, \qquad T_z^2 = T_y^2 = \tfrac{1}{2}(T^2-M^2), \qquad T \geqq M, \tag{8a}$$

while (4b), (5b) and (6) yield

$$\Gamma = 0, \qquad T_x = T_z = 0; \qquad T = T_y. \tag{8b}$$

Other combinations of (4) and (5) do not yield independent solutions†.

† From eqs. (4a)+(5b) we obtain

$$\left\{\frac{M}{T}\right\}^2 = \frac{\sin^2\Gamma\cos^3 2\Gamma}{2\cos^4\Gamma\{2\cos^2\Gamma\cos^3 2\Gamma+\cos\Gamma\,(1-\sin 2\Gamma)^2-2(\sin\Gamma-2\cos^3\Gamma)^2\}}\,.$$

The right-hand side starts at the value 0 with negative slope and continues to be negative throughout the interval $0<\Gamma<\frac{1}{4}\pi$, being equal to zero again at $\Gamma=\frac{1}{4}\pi$. Thus, there is no value of Γ that satisfies the minimum condition, with the possible exception of $\Gamma=0$, if $M=0$. This is a particular case of (8b). Similarly, we can disregard the solution $M=0$, $\Gamma=\frac{1}{4}\pi$. Eqs. (4c) and (5a) yield $T_y{}^2 = T_z{}^2 = \frac{1}{2}T^2$, which is a particular case of (8a) for $M=0$. From (4c) and (5b), we obtain $\Gamma=0$, a result which is already included in (8b).

The eigenvalues of the rotational energy may be expressed as follows:

$$T_{\text{rot}}(\Gamma = \tfrac{1}{4}\pi) = (1/2B\Delta^2)(2T^2 - M^2), \tag{9a}$$

$$T_{\text{rot}}(\Gamma = 0) = (1/2B\Delta^2)(T^2 + M^2). \tag{9b}$$

The lowest energy is given by (9a) if $M^2 > \frac{1}{2}T^2$ and by (9b) if $M^2 < \frac{1}{2}T^2$. Both solutions correspond to an axially symmetric situation. The corresponding spectra are discussed in sect. 3 of ref. [2]).

Using the further assumption of a harmonic potential, eq. (7) gives the equilibrium values for Δ, namely

$$\Delta_{\text{eq}} = (2T^2 - M^2)^{\frac{1}{4}} b, \tag{10a}$$

$$\Delta_{\text{eq}} = (T^2 + M^2)^{\frac{1}{4}} b, \tag{10b}$$

with $b = \{BC\}^{-\frac{1}{4}}$ equal to the length parameter of the harmonic oscillator. When $\Gamma_{\text{eq}} = 0$ a solution including both rotational and vibrational degrees of freedom is obtained; it has been discussed in refs. [5, 11]).

We shall concentrate on the solution corresponding to $\Gamma_{\text{eq}} = \frac{1}{4}\pi$. For this value of Γ, the lowest state associated with a given M has [2]) $T = M$. This sequence of lowest states constitutes the lowest yrast band. It is convenient to introduce the quantum number Y (Y = positive integer) which for this yrast band coincides both with M and T.

The equilibrium value of Δ depends on the member of the band and can be obtained from eq. (10a):

$$\Delta_{\text{eq}} = bY^{\frac{1}{2}}. \tag{11}$$

The energy of an yrast state is

$$E = \frac{Y^2}{2B\Delta_{\text{eq}}^2} + \tfrac{1}{2}C\Delta_{\text{eq}}^2 = \omega Y, \tag{12}$$

where $\omega = (C/B)^{\frac{1}{2}}$ is the energy characteristic of the six-dimensional oscillator. For sufficiently large values of Y, the yrast band is a "quasi-rotational" band, in the sense that the intrinsic structures (as measured by Δ_{eq}) of two consecutive members of the band differ only by terms of the order of $1/Y$. An yrast excitation will be produced by going from the yrast state Y to the state with $Y+1$, and we see from (12) that the excitation energy is the harmonic frequency ω.

The yrast band corresponds to the set of the ground states of even single-closed-shell nuclei. Thus if A_0 is the number of particles in a double magic nucleus ($Y = 0$), the yrast states are the ground states of isotopes with $Z = \frac{1}{2}A_0$ for $A > A_0$, and the ground states of isotones with $N = \frac{1}{2}A_0$, for $A < A_0$. Here, and in the following physical interpretations of the elementary excitations of an yrast state, we assume always $T = \frac{1}{2}(N-Z)$. Obviously, the analogues of such states satisfy similar interpretations.

Presently, (small) departures from the equilibrium values $\Gamma = \frac{1}{4}\pi$, $T = M = -T_x = Y$ and $\Delta = bY^{\frac{1}{2}}$ are allowed. Thus, each member Y of the yrast band gives rise to a subset of vibrational or rotational states. Every member of each subset will be labelled by the quantum number Y characterizing the parent yrast state†.

The restoring force for the Γ-degree of freedom is obtained from T_{rot}:

$$C_\Gamma = \left.\frac{\partial^2 T_{\mathrm{rot}}}{\partial \Gamma^2}\right|_{\Gamma=\frac{1}{4}\pi} = \frac{Y^2}{B\Delta_{\mathrm{eq}}^2} = \omega Y. \tag{13}$$

Since the system may oscillate around $\Gamma = \frac{1}{4}\pi$, we define the (small) variable

$$\xi = (\Gamma - \tfrac{1}{4}\pi)Y^{\frac{1}{2}}. \tag{14}$$

The vibrational plus rotational kinetic terms in (2) yield to lowest order in ξ

$$H_\xi = \frac{\omega}{2}\left\{-\frac{1}{\xi}\frac{\partial}{\partial\xi}\xi\frac{\partial}{\partial\xi} + \frac{(T_x+M)^2}{4\xi^2} + \xi^2\right\}. \tag{15}$$

Thus, we see that the Γ-motion corresponds to a two-dimensional harmonic oscillator. Therefore, the first excited states have

$$\Lambda \equiv \tfrac{1}{2}(T_x+M) = \pm 1, \tag{16}$$

which we can produce by taking $M' = M+1$, $T'_x = T_x+1$, $T' = |T'_x| = M-1$ and $M' = M-1$, $T'_x = T_x-1$, $T' = |T'_x| = M+1$, respectively (fig. 1). The fact that these states violate the condition $M = -T_x$ (which is characteristic of the rigid rotor with $\Gamma = \frac{1}{4}\pi$) is consistent with the fact that they are associated with changes in Γ.

The magnitude Λ is the angular momentum in the two-dimensional rotation and has the eigenvalues

$$\Lambda = \pm n_\Gamma,\ \pm(n_\Gamma - 2), \ldots, \pm 1 \text{ or } 0, \tag{17}$$

where n_Γ is the number of Γ-phonons. If $A > A_0$, the one-phonon states with $\Lambda = 1$ correspond to the isotopes with $Z = \frac{1}{2}A_0+2$ (i.e., to the ground states of nuclei having two protons more than the closed shell) since they have one pair of particles more and one unit of isospin less than the original yrast states. On the contrary, states with $\Lambda = -1$ have one unit of isospin more, and thus two protons less than the closed shell. For $A < A_0$, the eigenvalues $n_\Gamma = 1$, $\Lambda = \pm 1$ correspond to the ground state of isotones with $N = \frac{1}{2}A_0 \pm 2$. In general, the $|\Lambda| = n_\Gamma$ states represent the ground states of even nuclei. If $|\Lambda| < n_\Gamma$, it is also possible to obtain excited states within the Γ-degree of freedom. For instance, the state with $n_\Gamma = 2$, $\Lambda = 0$ has the same values of T and M as the original yrast state. Therefore is an excited state with an excitation energy of 2ω. Since it is populated†† in the two-particle pick-up reaction from the $n_\Gamma = \Lambda = 1$ state (which has two protons more than the closed shell) and it has the same number of protons $Z_0 = \frac{1}{2}A_0$ as the original yrast state, the state $n_\Gamma = 2$, $\Lambda = 0$ may be interpreted as a two-particle, two-hole proton excitation of the closed

† In general, neither the number of pairs M nor the isospin T coincide with the yrast quantum number Y. Their relation is given by eq. (23) below.

†† See table 2 and corresponding discussion in the text.

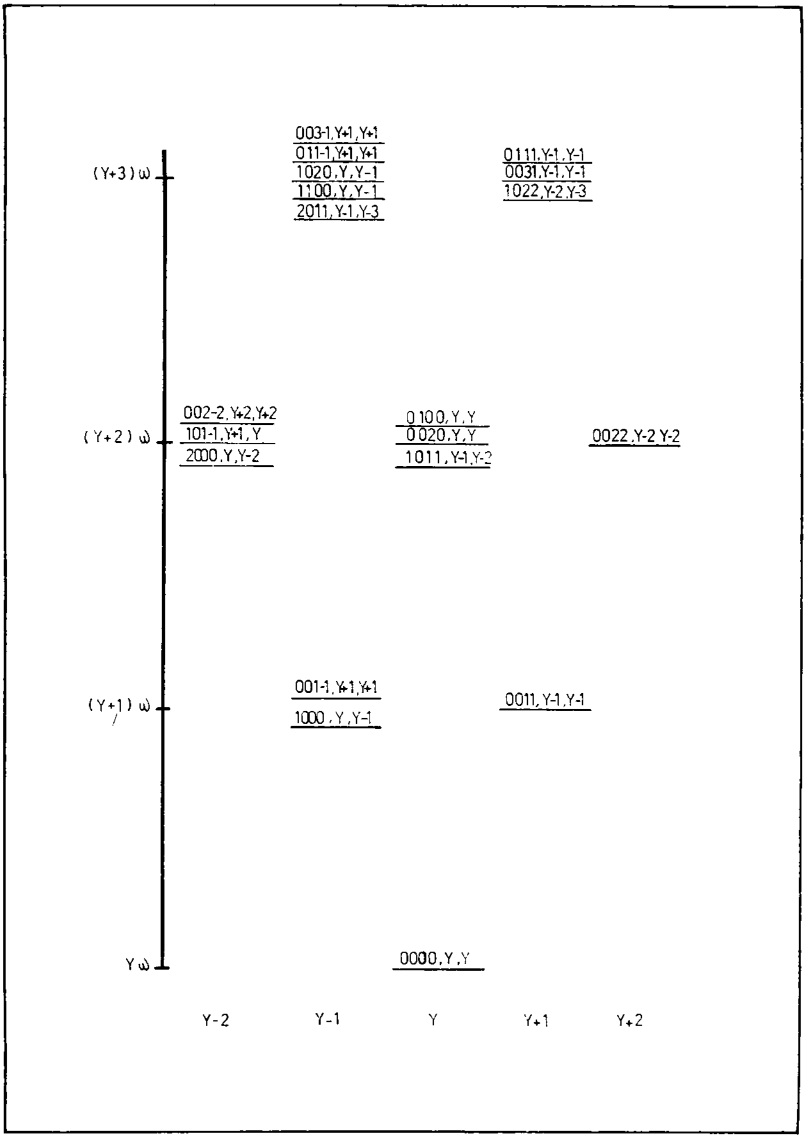

Fig. 1. Level spectrum associated with each yrast state. The states are labelled by the quantum numbers n_θ, n_Δ, n_Γ, A; T, $-T_x$. The numbers of pairs are indicated at the bottom.

shell, for $A > A_0$. The interpretation changes to a two-particle–two-hole neutron excitation of the yrast isotones, for $A < A_0$.

Another possibility for obtaining excited states is to produce Δ-vibrations. The restoring force C_Δ for these vibrations is computed as follows:

$$C_\Delta = \frac{\partial^2}{\partial \Delta^2}(T_{\mathrm{rot}} + \tfrac{1}{2}C\Delta^2)_{\Delta=\Delta_{\mathrm{eq}}} = 4C. \qquad (18)$$

Using the leading order term for the kinetic energy operators and the variable

$$\zeta = \frac{\sqrt{2}}{b}(\Delta - \Delta_{\text{eq}}), \tag{19}$$

we obtain

$$H_\Delta = \frac{\omega_\Delta}{2}\left\{-\frac{\partial^2}{\partial\zeta^2} + \zeta^2\right\}, \qquad \omega_\Delta = 2\omega. \tag{20}$$

Since Δ^2 is one of the fundamental invariants [1]), a Δ-vibration has the same $T = M = Y$ values as the yrast state from which it originates. Thus, the state with one Δ-phonon is degenerate and has the same T, M quantum numbers as the $n_\Gamma = 2$, $\Lambda = 0$ state. The state $(Y, n_\Delta = 1)$ is populated† in the two-neutron pick-up reaction from the next yrast state $(Y+1,\ n_\Delta = 0)$ and has the same number of neutrons as the original yrast state $(Y, n_\Delta = 0)$. Therefore, the state with $(Y, n_\Delta = 1)$ may be interpreted as two-neutron holes (below the $N_0 = \frac{1}{2}A_0$ magic core) in the state $(Y+1,\ n_\Delta = 0)$ if $A > A_0$. Similarly, it has two protons above the closed shell, if $A < A_0$.

So far, the rotational degree of freedom has only been included in the excitation of the yrast band. However, we may produce another type of rotation by transferring isospin quanta from the x-axis to the y- and z-axes. Thus, we keep constant the value of $T = Y$, but decrease the value of $|T_x| = Y$ to $|T'_x| = Y - n_\theta$. According to (9a),

$$T_{\text{rot}}(n_\theta) = \frac{1}{2B\Delta_{\text{eq}}^2}\{2Y^2 - (Y-n_\theta)^2\} \approx T_{\text{rot}}(n_\theta = 0) + \frac{n_\theta Y}{B\Delta_{\text{eq}}^2} = T_{\text{rot}}(0) + \omega n_\theta. \tag{21}$$

We have mentioned that we obtain the state $n_\Gamma = -\Lambda = 1$ by substracting a pair of protons from the closed shell (for $A > A_0$). This state has one unit more of isospin than the original yrast state. We may also substract a pair of particles from the closed shell, but now we couple the isospin $t = 1$ of this pair with the isospin $T = Y$ of the original yrast state to a total isospin Y (i.e., to the same isospin of the original yrast state). This is the physical interpretation†† of the state with $n_\theta = 1$. This state necessarily involves configurations in which a neutron-proton pair is substracted from the original yrast state corresponding to an even nucleus, and therefore occurs in doubly odd nuclei.

The two rotational excitations and the Δ- and Γ-vibrations exhaust the list of simple excitations of the yrast states. More complex excitations may be constructed using the previous ones as building blocks. Thus, the basic wave functions may be represented as

$$|Y, n_\theta, n_\Delta, n_\Gamma, \Lambda; M, T, T_x\rangle = \sqrt{(2T+1)/16\pi^3}\, G_{n_\Delta}(\zeta) f_{n_\Gamma\Lambda}(\xi) e^{i2M\phi} D^T_{\frac{1}{2}(N-Z),\, T_x}(\theta_i), \tag{22}$$

where $G_{n_\Delta}(\zeta)$ and $f_{n_\Gamma\Lambda}(\xi)$ are eigenfunctions of the one-dimensional harmonic oscillator and of the radial part of the two-dimensional harmonic oscillator, respectively.

† See table 2 and corresponding discussion in the text.

†† This state is sometimes called the antianalogue of the state $n_\Gamma = -\Lambda = 1$.

We also note that the isospin T, its projection over the intrinsic x-axis T_x, and the number of pairs of particles M, are good quantum numbers, with eigenvalues

$$T = Y - \Lambda,$$
$$T_x = -Y + \Lambda + n_\theta,$$
$$M = Y + \Lambda - n_\theta. \tag{23}$$

The solutions (22) become less reliable as we depart from the yrast hne, i.e., as the integers n_θ, n_Δ and n_Γ approach the value of Y. As we previously saw, the solution $\Gamma_{\text{eq}} = 0$ is expected to be lower than the present $\Gamma_{\text{eq}} = \frac{1}{4}\pi$, for values of $n_\theta \geqq \frac{1}{2}Y$.

Fig. 1 shows the spectrum associated with a single value of Y. Obviously, for a given number of pairs M, there exist more states which originate in other values of Y. Table 1 contains the quantum numbers for the spectrum associated with a given M. It is easy to verify that the number of states (and, in particular, the degeneracies for a given T) are thie same as in the coupling scheme [12]) in which the states are labelled

TABLE 1

The quantum numbers corresponding to states of the system with $M = Y$ pairs outside the closed shell

Number of phonons	Quantum numbers						
	Y	n_θ	n_Δ	n_Γ	Λ	T	$-T_x$
Y	Y	0	0	0	0	Y	Y
	$Y-1$	0	0	1	1	$Y-2$	$Y-2$
	$Y-2$	0	0	2	2	$Y-4$	$Y-4$
$Y+2$	$Y+1$	0	0	1	-1	$Y+2$	$Y+2$
	$Y+1$	1	0	0	0	$Y+1$	Y
	Y	0	1	0	0	Y	Y
	Y	0	0	2	0	Y	Y
	Y	1	0	1	1	$Y-1$	$Y-2$
	$Y-1$	0	1	1	1	$Y-2$	$Y-2$
	$Y-1$	0	0	3	1	$Y-2$	$Y-2$
	$Y-1$	1	0	2	2	$Y-3$	$Y-4$
$Y+4$	$Y+2$	0	0	2	-2	$Y+4$	$Y+4$
	$Y+2$	1	0	1	-1	$Y+3$	$Y+2$
	$Y+2$	2	0	0	0	$Y+2$	Y
	$Y+1$	0	0	3	-1	$Y+2$	$Y+2$
	$Y+1$	0	1	1	-1	$Y+2$	$Y+2$
	$Y+1$	1	0	2	0	$Y+1$	Y
	$Y+1$	1	1	0	0	$Y+1$	Y
	$Y+1$	2	0	1	1	Y	$Y-2$
	Y	0	1	2	0	Y	Y
	Y	0	0	4	0	Y	Y
	Y	0	2	0	0	Y	Y
	Y	1	1	1	1	$Y-1$	$Y-2$
	Y	1	0	3	1	$Y-1$	$Y-2$
	Y	2	0	2	2	$Y-2$	$Y-4$

TABLE 2

Correspondence between the phonon and the yrast coupling schemes

M	T	$\{n_r, t_r; n_a, t_a\}$	$(Y, n_\theta, n_\Delta, n_\Gamma, \Lambda)$
Y	Y	$\{0, 0; Y, Y\}$	$(Y, 0, 0, 0, 0)$
Y	$Y-2$	$\{0, 0; Y, Y-2\}$	$(Y-1, 0, 0, 1, 1)$
Y	$Y+2$	$\{1, 1; Y+1, Y+1\}$	$(Y+1, 0, 0, 1, -1)$
Y	$Y+1$	$\{1, 1; Y+1, Y+1\}$	$(Y+1, 1, 0, 0, 0)$
Y	Y	$\{1, 1; Y+1, Y+1\}$	$(Y, 0, 1, 0, 0)$
Y	Y	$\{1, 1; Y+1, Y-1\}$	$(Y, 0, 0, 2, 0)$

by $\{n_r, t_r; n_a, t_a\}^T_{\frac{1}{2}(N-Z)}$. This notation represents n_r removal phonons coupled to isospin t_r and n_a addition phonons coupled to t_a. The parities of t_r and t_a are the parities of n_r and n_a, respectively. The two partial isospins t_r, t_a are coupled to the total isospin T. From the expressions for the total number of phonons and the number of pairs outside the closed shell, we obtain the number of addition and removal phonons in terms of the yrast quantum numbers:

$$n_a = Y + n_\Delta + \tfrac{1}{2}(n_\Gamma + \Lambda), \qquad n_r = n_\theta + n_\Delta + \tfrac{1}{2}(n_\Gamma - \Lambda). \tag{24}$$

For instance, the yrast states have $n_a = Y$ addition quanta coupled to the maximum isospin $t_a = T = Y$ and zero removal quanta.

If we consider the states with two phonons more, the correspondence is easily established for those states in which the quantum numbers (M, T) specify the state completely (table 2). However the two last states in table 2 have the same value of $M = T = Y$. The one-to-one correspondence is obtained by noting that the matrix element between the states $\{1,1; Y+1, Y-1\}^Y$ and $\{0,0; Y+1, Y+1\}^{Y+1}$ vanishes in the phonon scheme†.

Similarly, in the yrast scheme, the transition between the $(Y, 0,0, 2,0)$ and $(Y+1, 0,0, 0,0)$ states is forbidden, since it implies†† $\Delta n_\Gamma = 2$. Thus, the correspondence is established. However, since there exists a small matrix element violating this selection rule (sect. 3), this correspondence is only established up to the order $1/Y$.

The transfers of pairs of particles coupled to $J^\pi = 0^+$, isospin one, are the specific processes for the present collective motion. The corresponding operators $\mathscr{P}^+_\mu (\mathscr{P}^-_\mu)$ are the variables d_μ (d^+_μ) representing the collective coordinates in the laboratory frame. Using (1) they may be expressed as a function of the intrinsic parameters Δ and Γ. The result is given by eq. (16) of ref. [2]):

$$\mathscr{P}^\pm_\mu = \Delta\, e^{\pm 2i\phi}\{\sin(\Gamma \pm \tfrac{1}{4}\pi) D^1_{\mu,-1}(\theta_i) - \sin(\Gamma \mp \tfrac{1}{4}\pi) D^1_{\mu,1}(\theta_i)\}. \tag{25}$$

The condition $|\Delta M| = 1$ for the two-particle transfer, plus the selection rules $|\Delta T| \leqq 1$ and $|\Delta T_x| \leqq 1$, limit to five (and their h.c.) the number of possible matrix elements within our basic set (22). See table 3.

† This transition implies a simultaneous change in n_r, t_r and t_a, which is forbidden since the transfer operators depend linearly on the phonon creation and annihilation operators [11,6]).

†† See the discussion on transition probabilities at the end of this section.

TABLE 3

The reduced matrix elements of the operator (25) within the basic set (22) corresponding to the possible transitions from M to $M+1$

Y'	n'_θ	Λ'	T'	T'_x	$\langle n'\|\|P^+\|\|n\rangle/\langle n'_\Delta\|\Delta\|n_\Delta\rangle\ \langle n'_\Gamma\Lambda'\|(\cos\Gamma-(\Delta T_x)\sin\Gamma)\|n_\Gamma\Lambda\rangle$
Y	n_θ	$\Lambda+1$	$T-1$	T_x+1	$[(Y-\Lambda-\frac{1}{2}n_\theta-\frac{1}{2})(Y-\Lambda-\frac{1}{2}n_\theta)/(Y-\Lambda)]^{\frac{1}{2}}$
Y	$n_\theta-1$	Λ	T	T_x-1	$[(Y-\Lambda+\frac{1}{2})(Y-\Lambda-\frac{1}{2}n_\theta+\frac{1}{2})n_\theta/(Y-\Lambda)(Y-\Lambda+1)]^{\frac{1}{2}}$
$Y+1$	n_θ	Λ	$T+1$	T_x-1	$[(Y-\Lambda-\frac{1}{2}n_\theta+\frac{1}{2})(Y-\Lambda-\frac{1}{2}n_\theta+1)/(Y-\Lambda+1)]^{\frac{1}{2}}$
$Y+1$	$n_\theta+1$	$\Lambda+1$	T	T_x+1	$-[(Y-\Lambda+\frac{1}{2})(Y-\Lambda-\frac{1}{2}n_\theta)(n_\theta+1)/(Y-\Lambda)(Y-\Lambda+1)]^{\frac{1}{2}}$
$Y+2$	$n_\theta+2$	$\Lambda+1$	$T+1$	T_x+1	$[(n_\theta+1)(n_\theta+2)/4(Y-\Lambda+1)]^{\frac{1}{2}}$

The primed quantities are quantum numbers of the final state.

In order to evaluate these matrix elements to leading orders we use (19) and (11) to expand

$$\Delta = b\sqrt{Y}+\frac{b}{\sqrt{2}}\zeta, \tag{26}$$

and with (14) we get

$$\cos\Gamma-(\Delta T_x)\sin\Gamma = \begin{cases}\sqrt{2} & \text{for } \Delta T_x = -1\\ -\sqrt{2}\xi/\sqrt{Y} & \text{for } \Delta T_x = 1,\end{cases} \tag{27}$$

with $\Delta T_x = T'_x-T_x$. Using the values for $\Delta T_x = -1$, the third line of table 3 yields the (strong) transitions between two states differing only in the Y quantum number and thus belonging to the same yrast band. The corresponding reduced matrix elements are, to leading order in $1/Y$,

$$\langle Y+1, n_\theta, n_\Delta, n_\Gamma, \Lambda\|\mathscr{P}^+\|Y, n_\theta, n_\Delta, n_\Gamma, \Lambda\rangle = \sqrt{2}bY. \tag{28}$$

For the leading values of (26) and (27), we obtain the transitions to the other rotational excitations, from the second line of table 3. The corresponding reduced matrix elements are a factor of $\sqrt{Y}$ smaller than the previous transitions

$$\langle Y, n_\theta+1, n_\Delta, n_\Gamma, \Lambda\|\mathscr{P}^-\|Y, n_\theta, n_\Delta, n_\Gamma, \Lambda\rangle = b\{2Y(n_\theta+1)\}^{\frac{1}{2}}. \tag{29}$$

For $\Delta T_x = 1$ transitions, the first line of table 3 yields the transitions corresponding to the selection rule $|\Delta n_\Gamma| = |\Delta\Lambda| = 1$, all other quantum numbers being kept constant. These transitions are

$$\langle Y, n_\theta, n_\Delta, n_\Gamma\pm1, \Lambda+1\|\mathscr{P}^+\|Y, n_\theta, n_\Delta, n_\Gamma, \Lambda\rangle = \mp b\{Y(n_\Gamma\pm\Lambda\pm1+1)\}^{\frac{1}{2}}, \tag{30}$$

again a factor of $\sqrt{Y}$ smaller than the transitions within an yrast band and of the same order of magnitude as the transitions corresponding to the selection rule $|\Delta n_\theta| = 1$.

Finally, the reduced matrix elements corresponding to excitations of the Δ-degree of freedom are obtained using the second term in the expansion of Δ and the third line of table 3,

$$\langle Y-1, n_\theta, n_\Delta+1, n_\Gamma, \Lambda\|\mathscr{P}^-\|Y, n_\theta, n_\Delta, n_\Gamma, \Lambda\rangle = b\{\tfrac{1}{2}Y(n_\Delta+1)\}^{\frac{1}{2}}, \tag{31}$$

which are again smaller than the yrast transitions.

It is also easy to verify that the (so far unused) fourth and fifth lines of table 3 yield reduced matrix elements which are at least of order 1 or $Y^{-\frac{1}{2}}$, respectively.

Therefore, as in the quadrupole case, we conclude that the strongest transitions occur within an yrast band. A second group of weaker (but still strong) transitions involve only one of the elementary excitations that have been described in this section.

It is possible to verify that the reduced matrix elements (28)–(30) are reproduced in the phonon scheme, to leading order in $1/Y$. However, expression (31) is only half of the corresponding value. This discrepancy is explained in the next section.

3. Perturbative treatment of the corrections to the yrast approximation

We want to maintain the wave functions (22) as our basic set. In these functions, the variable ζ and ξ represent the coordinate in a one-dimensional harmonic oscillator and the radius in a two-dimensional harmonic oscillator, respectively. Thus, their domain is

$$-\infty < \zeta < \infty,$$
$$0 \leqq \xi < \infty, \tag{32}$$

which is different from the one that can be obtained using the definitions (14) and (19) plus equation (3). Therefore, we must replace (14) and (19) by a more suitable transformation which agrees with them to lowest order in $1/Y$ and such that the new variables ζ and ξ have the domains (32). A convenient transformation is

$$\zeta = \sqrt{2Y}\ln(\Delta/bY^{\frac{1}{4}}),$$
$$\xi = -\tfrac{1}{2}\sqrt{Y}\cot 2\Gamma. \tag{33}$$

The sum of (12), (15), (20) and (21) take into account only part of the original kinetic energy (2). In order to obtain the perturbation Hamiltonian, we write (2) in terms of the new variables (33),

$$\begin{aligned} H = \frac{\omega}{2} e^{-\zeta\sqrt{2/Y}} \Bigg\{ & -2\frac{\partial^2}{\partial\zeta^2} - \frac{\sqrt{32}}{\sqrt{Y}}\frac{\partial}{\partial\zeta} + Y e^{\zeta\sqrt{8/Y}} - (1+4\xi^2/Y)^2 \frac{1}{\xi}\frac{\partial}{\partial\xi}\xi\frac{\partial}{\partial\xi} \\ & + \frac{(T_x+M)^2(1+4\xi^2/Y)}{4\xi^2} - \frac{MT_x\{1+4\xi^2/Y-(1+4\xi^2/Y)^{\frac{1}{2}}\}}{2\xi^2} \\ & + \frac{2}{Y}(T^2-T_x^2)(1+4\xi^2/Y) + 4Y^{-\frac{3}{2}}(T_z^2-T_y^2)\xi(1+4\xi^2/Y)^{\frac{1}{2}} \Bigg\}. \end{aligned} \tag{34}$$

Similarly, the volume element (3) has to be expressed in terms of the new variables

$$d\tau = \frac{|\xi| e^{3\zeta\sqrt{2/Y}}}{(1+4\xi^2/Y)^2} d\xi\, d\zeta\, d\phi\, d\theta_i. \tag{35}$$

We have dropped from this volume element a numerical factor which would only lead to a different normalization in (22).

In order to solve (34) with (35) we perform an expansion in powers of $1/Y^{\frac{1}{2}}$. After some algebra, the lower orders of (34) yield

$$H = \sum_{\nu} h_{\nu},$$

$$h_0 = \omega(Y+2n_\Delta+n_\theta+n_\Gamma+3),$$

$$h_1 = \omega\sqrt{2/Y}\left\{-\zeta(2n_\Delta+n_\theta+n_\Gamma+3)+\zeta^3-2\frac{\partial}{\partial\zeta}+\frac{\sqrt{2}}{Y}\xi(T_y^2-T_z^2)\right\},$$

$$h_2 = \frac{\omega}{Y}\left\{\zeta^2(2n_\Delta+n_\theta+n_\Gamma+3)-\tfrac{5}{6}\zeta^4+4\zeta\frac{\partial}{\partial\zeta}+\xi^2(12+8n_\Gamma+7n_\theta)-5\xi^4\right.$$

$$\left.-\Lambda-\tfrac{5}{2}\Lambda^2-2\Lambda n_\theta-\tfrac{1}{2}n_\theta^2-2\sqrt{2}\frac{\zeta\xi}{Y}(T_y^2-T_z^2)\right\}, \qquad (36)$$

while the volume element gives

$$\mathrm{d}\tau = \sum_{\nu} t_\nu \mathrm{d}t,$$

$$\mathrm{d}t = |\xi|\mathrm{d}\xi\mathrm{d}\zeta\mathrm{d}\phi\mathrm{d}\theta_i,$$

$$t_0 = 1, \qquad t_1 = 3\sqrt{2/Y}\zeta, \qquad t_2 = \frac{1}{Y}(9\zeta^2-8\xi^2). \qquad (37)$$

Here, Y, n_Δ, n_θ, n_Γ and Λ should be considered as operators.

The consistency with the order of magnitude of the terms in (36) implies that the matrix elements of $T_y^2-T_z^2$ in h_1 and h_2 can be taken to leading order in $1/Y$, namely

$$\langle Y+1, n_\theta+2, \Lambda+1||(T_y^2-T_z^2)||Y, n_\theta, \Lambda\rangle = -Y\sqrt{(n_\theta+1)(n_\theta+2)}. \qquad (38)$$

The yrast approximation which was developed in the previous section only takes into account h_0 and t_0. In order to use the remaining terms of H and of $\mathrm{d}\tau$, we must apply a perturbation theory in which both the Hamiltonian and the volumes element are taken into account up to the same order. This theory is developed in appendix A.

Note that some of the terms in h_1 and h_2 are non-hermitian. This situation is connected with the change of the volume element as is shown for instance in eq. (A.7) of appendix A.

The correct energies are already given by the zero-order Hamiltonian h_0. Thus, a convenient check of the h and t of perturbation theory, is to verify that the correction to the energy does vanish. This is done up to second order in appendix B.

In order to calculate corrections to the transition matrix elements of table 3, we expand the transfer operator (25) in powers of $1/Y^{\frac{1}{2}}$:

$$\mathscr{P}_\mu^{\pm} = \pm b\sqrt{Y}\,\mathrm{e}^{\pm 2i\phi}\left[D^1_{\mu,\mp 1}+Y^{-\frac{1}{2}}\left\{\frac{\zeta}{\sqrt{2}}D^1_{\mu,\mp 1}-\xi D^1_{\mu,\pm 1}\right\}\right.$$

$$\left.+Y^{-1}\left\{\frac{\zeta^2}{4}D^1_{\mu,\mp 1}-\frac{\zeta\xi}{\sqrt{2}}D^1_{\mu,\pm 1}-\frac{\xi^2}{2}D^1_{\mu,\mp 1}\right\}\right]. \qquad (39)$$

Firstly, we calculate the value of a "strong" matrix element (i.e., the matrix elements involving only one elementary excitation). This is done in appendix B for the transitions $|\Delta n_\theta| = 1$. The result (eq. (B.9)) agrees with the corresponding matrix elements in the boson scheme [6]) to the required order of $Y^{-\frac{1}{2}}$.

Secondly, we calculate the transition $\Delta Y = \Delta n_\theta = \Delta n_\Gamma = \Delta \Lambda = 1$, $\Delta n_\Delta = 0$. It corresponds to an increase of three phonons, and thus should vanish in the harmonic limit. However, table 3 yields a non-vanishing matrix element of order unity between the unperturbed states. We verify in eq. (B.10) that there exists a contribution cancelling the direct matrix element through the states which are admixed by the perturbation theory.

In the evaluation of the two previous matrix elements, we have assumed that the initial and final states have the same equilibrium deformation. This is not exactly true for the second transition, where the selection rule $\Delta Y = 1$ implies that there is a small difference in Δ_{eq} for the initial and final states. However, this effect only introduces a correction of order Y^{-1} in both the direct and indirect contributions, which is of the same order as the neglected terms in (B.12).

However, in some cases it is essential to take into account the difference in Δ_{eq} between initial and final states. This is the case, for instance, for the transitions $\Delta n_\Delta = 1$. We have already mentioned that expression (31) yields only half of the correct value. The matrix element corresponding to the transition $|Y, n_\theta, n_\Delta, n_\Gamma, \Lambda\rangle \to |Y+1, n_\theta, n_\Delta+1, n_\Gamma, \Lambda\rangle$, which is also given by (31), also constitutes a problem since it does not vanish in spite of the fact that it is associated with a change of three in the number of phonons.

If $Y' = Y + \Delta Y$ and one uses for the set of states carrying the quantum number Y' the same equilibrium deformation Δ_{eq} as for the set of states that are characterized by the quantum number Y, one must introduce some further correction terms to the Hamiltonian. For instance, the term which is proportional to $Y\zeta$ is no longer completely cancelled by the minimization procedure, but yields a perturbation

$$h_1' = -\omega\sqrt{2/Y}\,\zeta\Delta Y. \tag{40}$$

Including this term, the matrix element (31) should be replaced, to lowest order in Y^{-1}, by

$$\begin{aligned}
&\langle Y', n_\theta, n_\Delta+1, n_\Gamma, \Lambda||\mathscr{P}^{\pm}||Y, n_\theta, n_\Delta, n_\Gamma, \Lambda\rangle \\
&\qquad = \langle Y', n_\theta, n_\Delta+1, n_\Gamma, \Lambda; \Delta_{\text{eq}} = \Delta_{\text{eq}}(Y)||\mathscr{P}^{\pm}||Y, n_\theta, n_\Delta, n_\Gamma, \Lambda\rangle \\
&\qquad + \frac{1}{2\omega}\langle Y', n_\theta, n_\Delta, n_\Gamma, \Lambda|h_1'|Y', n_\theta, n_\Delta+1, n_\Gamma, \Lambda\rangle \\
&\qquad \times \langle Y', n_\theta, n_\Delta, n_\Gamma, \Lambda; \Delta_{\text{eq}} = \Delta_{\text{eq}}(Y)||\mathscr{P}^{\pm}||Y, n_\theta, n_\Delta, n_\Gamma, \Lambda\rangle \\
&\qquad = b\{\tfrac{1}{2}Y(n_\Delta+1)\}^{\frac{1}{2}}(1-\Delta Y).
\end{aligned} \tag{41}$$

For the pick-up reaction ($\Delta Y = -1$) we obtain twice the value (31) and the stripping cross section ($\Delta Y = 1$) vanishes, as required by the exact calculation. Here again, the admixed state has a strong matrix element with the initial state (order of yrast transitions). Thus, the indirect contribution is of the same order of magnitude as the direct one, in spite of the smallness of the admixed amplitude in the final state ($O(Y^{-\frac{1}{2}})$).

4. Anharmonic terms

The yrast concept should also simplify the treatment of situations departing from the harmonic limit. We show now how anharmonic terms may be introduced in the formalism. We assume that the solutions remain of the form (22), i.e., that the system presents a stable deformation for $\Delta = \Delta_{eq}$ and $\Gamma = \Gamma_{eq} = \frac{1}{4}\pi$, and that the (small) fluctuations around the equilibrium deformation and the two kind of rotations constitute the building blocks of the spectrum. Even if the anharmonicities are such that the potential has a minimum for $\Gamma = 0$, for those states which have a large excitation energy as compared to the depth of the minimum, the yrast description will still be valid.

Since the most general potential energy surface is a function of the two invariants Δ^2 and $\Delta^4 \cos^2 2\Gamma$, we confine our attention to those cases in which the dependence on the second invariant does not change the value of Γ_{eq}.

There may also be anharmonic terms arising from the kinetic energy. Here, there are eight arbitrary functions of the two invariants [1]). The most general quantum mechanical expression for the kinetic energy and volume element has not been given yet. We obtain some simple anharmonic terms by noting that, after the transformation to intrinsic axes, an irrotational classical kinetic energy has the form

$$T_{rot} = \tfrac{1}{2}B\Delta^2\{\dot{\theta}_x^2 + \dot{\theta}_y^2 \sin^2 \Gamma + \dot{\theta}_z^2 \cos^2 \Gamma + 4\dot{\phi}\dot{\theta}_x \sin 2\Gamma + 4\dot{\phi}^2\}, \tag{42}$$

$$T_{vib} = \tfrac{1}{2}B\{\dot{\Delta}^2 + \Delta^2\dot{\Gamma}^2\}, \tag{43}$$

with a common constant B. We generalize (1) by multiplying each term in (42) by different factors. However, this freedom is limited if we require the volume element to remain of the same form (apart from a constant). Since the volume element is proportional to the determinant of the coefficients of the products of derivatives [13]), the terms proportional to $\dot{\theta}_x^2$, $\dot{\phi}\dot{\theta}_x$ and $\dot{\phi}^2$ should be multiplied by the same constant. Thus, the number of arbitrary constants is reduced to 4. This number if further reduced by one if the Γ-motion is to be described by a two-dimensional harmonic oscillator†, because the centrifugal term Λ^2/ξ^2 comes from T_{rot} while the radial derivative $(-1/\xi)(\partial/\partial\xi)\xi(\partial/\partial\xi)$ arises from the term proportional to $\dot{\Gamma}^2$ in T_{vib}.

† These requirements also exist for the quadrupole case and for the pairing between identical particles.

Therefore, we may add to (1) three terms, namely

$$T'_{\rm vib} = \frac{-\alpha_\Delta}{2B\Delta^5}\frac{\partial}{\partial\Delta}\Delta^5\frac{\partial}{\partial\Delta}, \tag{44}$$

$$T'_{\rm rot} = \frac{-\alpha_\theta}{B\Delta^2}\frac{(T^2-T_x^2)}{\sin^2 2\Gamma}, \tag{45}$$

$$T_{\rm coup} = \frac{-\alpha_c}{B\Delta^2}(T_z^2-T_y^2)\frac{\cos 2\Gamma}{2\sin^2 2\Gamma}. \tag{46}$$

The contributions (44) and (45) merely change the frequency and length of the Δ-vibrations and the frequency of the small rotations, respectively. The third term represents the coupling between the rotational and vibrational degrees of freedom.

Following the same procedure as in the previous two sections, we obtain the equilibrium value of Δ from the equation

$$\Delta^4 = \frac{Y^2}{2B\left.\dfrac{\partial V}{\partial(\Delta^2)}\right|_{\Gamma=\frac{1}{4}\pi}}, \tag{47}$$

and the excitational spectrum is given in lowest order by

$$h_0 = \frac{Y^2}{2B\Delta_{\rm eq}^2}+V(\Delta_{\rm eq}^2, \Gamma = \tfrac{1}{4}\pi)+\omega_\theta(n_\theta+1)+\omega_\Delta(n_\Delta+\tfrac{1}{2})+\omega_\Gamma(n_\Gamma+1)-\frac{\alpha_\theta Y}{B\Delta_{\rm eq}^2}, \tag{48}$$

where the frequencies are defined as

$$\begin{aligned}
\omega_\theta &= (1+2\alpha_\theta)Y/B\Delta_{\rm eq}^2,\\
\omega_\Delta &= \{C_\Delta(1+\alpha_\Delta)/B\Delta_{\rm eq}^2\}^{\frac{1}{2}},\\
C_\Delta &= \frac{2Y^2}{B\Delta_{\rm eq}^2}+4\Delta_{\rm eq}^4\frac{\partial^2 V}{\partial(\Delta^2)^2}+4\Delta_{\rm eq}^2\frac{\partial V}{\partial(\Delta^2)},\\
\omega_\Gamma &= \{C_\Gamma/B\Delta_{\rm eq}^2\}^{\frac{1}{2}},\\
C_\Gamma &= \frac{Y^2}{B\Delta_{\rm eq}^2}+8\frac{\partial V}{\partial\cos^2 2\Gamma}.
\end{aligned} \tag{49}$$

As before, it is convenient to use oscillators with unit length parameters. Therefore, the variables measuring the departure from the equilibrium position are given by

$$\zeta = \{B\Delta_{\rm eq}^2 C_\Delta/(1+\alpha_\Delta)\}^{\frac{1}{4}}\ln(\Delta/\Delta_{\rm eq}), \tag{50}$$

$$\xi = -\tfrac{1}{2}\{B\Delta_{\rm eq}^2 C_\Gamma\}^{\frac{1}{4}}\cot 2\Gamma. \tag{51}$$

The first and second order corrections to the Hamiltonian now read

$$
\begin{aligned}
h_1 = & -2\left\{\frac{1+\alpha_\Delta}{BC_\Delta \Delta_{\mathrm{eq}}^2}\right\}^{\frac{1}{4}}\zeta\left\{\omega_\theta(n_\theta+1)+\omega_\Delta(n_\Delta+\tfrac{1}{2})+\omega_\Gamma(n_\Gamma+1)-\frac{\alpha_\theta Y}{B\Delta_{\mathrm{eq}}^2}\right\} \\
& -\frac{2(1+\alpha_\theta)}{B\Delta_{\mathrm{eq}}^2}\left\{\frac{1+\alpha_\Delta}{BC_\Delta \Delta_{\mathrm{eq}}^2}\right\}^{-\frac{1}{4}}\frac{\partial}{\partial\zeta} \\
& +\frac{4}{3}\left\{\frac{1+\alpha_\Delta}{BC_\Delta \Delta_{\mathrm{eq}}^2}\right\}^{\frac{3}{4}}\zeta^3\left\{\Delta_{\mathrm{eq}}^6\frac{\partial^3 V}{\partial(\Delta^2)^3}+6\Delta_{\mathrm{eq}}^4\frac{\partial^2 V}{\partial(\Delta^2)^2}+4\Delta_{\mathrm{eq}}^2\frac{\partial V}{\partial(\Delta^2)}+\frac{Y^2}{B\Delta_{\mathrm{eq}}^2}\right\} \\
& +8\left\{\frac{(1+\alpha_\Delta)^2}{B^3\Delta_{\mathrm{eq}}^6 C_\Delta C_\Gamma^2}\right\}^{\frac{1}{4}}\zeta\xi^2\left\{\Delta_{\mathrm{eq}}^2\frac{\partial^2 V}{\partial(\Delta^2)\partial(\cos^2 2\Gamma)}+\frac{\partial V}{\partial(\cos^2 2\Gamma)}\right\} \\
& +\frac{2(1+\alpha_c)}{(B^5\Delta_{\mathrm{eq}}^{10}C_\Gamma)^{\frac{1}{4}}}\xi(T_z^2-T_y^2),
\end{aligned}
\tag{52}
$$

$$
\begin{aligned}
h_2 = & \frac{-1}{2B\Delta_{\mathrm{eq}}^2}\{n_\theta^2(1+2\alpha_\theta)+2\Lambda(1+\alpha_\theta)(1+2n_\theta)+5\Lambda^2\}+\frac{4(1+\alpha_\Delta)}{B\Delta_{\mathrm{eq}}^2}\zeta\frac{\partial}{\partial\zeta} \\
& +2\left\{\frac{1+\alpha_\Delta}{BC_\Delta \Delta_{\mathrm{eq}}^2}\right\}^{\frac{1}{2}}\zeta^2\left\{\omega_\theta(n_\theta+1)+\omega_\Delta(n_\Delta+\tfrac{1}{2})+\omega_\Gamma(n_\Gamma+1)-\frac{\alpha_\theta Y}{B\Delta_{\mathrm{eq}}^2}\right\} \\
& +\frac{1}{3}\frac{1+\alpha_\Delta}{BC_\Delta \Delta_{\mathrm{eq}}^2}\zeta^4\left\{\frac{-5Y^2}{B\Delta_{\mathrm{eq}}^2}-10\Delta_{\mathrm{eq}}^2\frac{\partial V}{\partial(\Delta^2)}+2\Delta_{\mathrm{eq}}^4\frac{\partial^2 V}{\partial(\Delta^2)^2}+4\Delta_{\mathrm{eq}}^6\frac{\partial^3 V}{\partial(\Delta^2)^3}+\tfrac{2}{3}\Delta_{\mathrm{eq}}^8\frac{\partial^4 V}{\partial(\Delta^2)^4}\right\} \\
& -\frac{8(1+\alpha_\Delta)^2}{B\Delta_{\mathrm{eq}}^2\sqrt{C_\Delta C_\Gamma}}\xi^2\zeta^2\left\{\frac{\partial V}{\partial(\cos^2 2\Gamma)}-4\Delta_{\mathrm{eq}}^2\frac{\partial^2 V}{\partial(\Delta^2)\partial(\cos^2 2\Gamma)}+\Delta_{\mathrm{eq}}^4\frac{\partial^3 V}{\partial(\Delta^2)^2\partial(\cos^2 2\Gamma)}\right\} \\
& +\frac{1}{\sqrt{C_\Gamma B\Delta_{\mathrm{eq}}^2}}\xi^2\left\{4\omega_\theta(n_\theta+1)+8\omega_\Gamma(n_\Gamma+1)+\frac{(3n_\theta-4\alpha_\theta)Y}{B\Delta_{\mathrm{eq}}^2}\right\} \\
& +\frac{1}{BC_\Gamma\Delta_{\mathrm{eq}}^2}\xi^4\left\{\frac{-5Y^2}{B\Delta_{\mathrm{eq}}^2}+8\frac{\partial^2 V}{\partial(\cos^2 2\Gamma)}-48\frac{\partial V}{\partial(\cos^2 2\Gamma)}\right\} \\
& +2(1+\alpha_c)\left\{\frac{1+\alpha_\Delta}{B^6\Delta_{\mathrm{eq}}^{12}C_\Gamma C_\Delta}\right\}^{\frac{1}{4}}(T_z^2-T_y^2)\xi\zeta.
\end{aligned}
\tag{53}
$$

In C_Δ, C_Γ, h_1 and h_2 the derivatives of V have obviously to be evaluated at $\Delta = \Delta_{\mathrm{eq}}$ and $\Gamma = \frac{1}{4}\pi$. In the present case there is no single parameter (like $Y^{-\frac{1}{2}}$) which allows us to group together terms of the same power.

Therefore, the distribution of the terms among the different h_ν is now somewhat arbitrary. However, it has been assumed that the fluctuations around the equilibrium positions are small, more precisely that the mass parameters $(B\Delta_{\mathrm{eq}}^2 C_\Delta)^{-\frac{1}{2}}$ and $(B\Delta_{\mathrm{eq}}^2 C_\Gamma)^{-\frac{1}{2}}$ are of order $Y^{-\frac{1}{2}}$ or smaller.

5. Conclusions

In the present paper we have seen how the yrast classification scheme provides a new interpretation of the phonon description of the pairing collective states. This is similar to the new insight that the yrast treatment yields in the case of quadrupole vibrations [10]. This interpretation already appears in lowest order, and is a consequence of the applicability of the well known formalism of deformed systems.

The parameter measuring the accuracy of the lowest order yrast approximation is $1/Y$, where Y is the yrast quantum number which plays a similar role to the angular momentum for the quadrupole vibrations. It corresponds to the number of pairs of particles M and to the isospin T for the lowest yrast band ($n_\Delta = n_\theta = n_\Gamma = 0$).

In this paper we have also shown how to correct the lowest order results via a perturbation procedure, which differs from the usual one by the fact that both the Hamiltonian and the volume element are to be expanded in powers of a small parameter. It will be necessary to use these corrections in the region around ^{56}Ni, since the values of Y are not too large there.

We have also shown how anharmonic terms may be introduced, while simultaneously keeping the $1/Y$ corrections. Therefore, we have developed a formalism having a range of applicability which is complementary to the one constructed [2] for the collective treatment of the $T = 1$ pairing interaction. For practical reasons, the possibility of solving Bohr's collective Hamiltonian using the methods of ref. [2] is limited to $T \leqq 2$. By contrast, the present collective treatment can only be applied for larger values of T and relatively large values of M.

Throughout the periodic table, there are several possible magic nuclei with $T = 0$. In principle, each one may give rise to two yrast lines, corresponding to the addition and substraction of particles, respectively. Experimentally, the best known case corresponds to the yrast line which is associated with the $N = 28$ isotones (^{54}Fe, ^{52}Cr, ^{50}Ti and ^{48}Ca). For this region, the experimental information concerning (t, p), (h, p) and (h, n) reactions is fairly complete [7, 8, 14]. These reactions excite the Γ; θ and Γ; and Y, Δ, θ and Γ degrees of freedom, respectively†. Preliminary calculations using the present collective formalism have been reported in reference [15].

Much less information is available on other possible yrast bands, such as the ones corresponding, on the addition side, to the Ni isotopes ($A_0 = 56$) and the Sn isotopes ($A_0 = 100$) and, on the removal side, to the $N = 50$ isotones (^{92}Mo, ^{70}Zr, ^{88}Sr, ^{86}Kr) and to the $N = 82$ isotones (^{144}Sm, ^{142}Nd, ^{140}Ce, ^{138}Ba, ^{136}Xe). Since our scheme relates energies and transition rates between states involving the excitation of two core-particles (sect. 2), the relevant experiments to be made are $J^\pi = 0^+$ transfer processes in the region between ≈ 2 and ≈ 7 MeV. The corresponding states will not be found, for instance if the seniority conserving forces (as the quadrupole force) become

† The fact that the target nuclei have $T_z = T$ implies that our different degrees of freedom are selectively excited by the different two-body transfer processes. This correspondence can be found in table 1 of ref. [15].

dominant. However, this is unlikely to happen for the single-closed-shell nuclei, which constitute our region of interest.

Discussions with Professors A. Bohr, R. Broglia and O. Nathan have greatly stimulated this work.

Appendix A

PERTURBATION THEORY

Here we write expressions for a perturbation expansion corresponding to a situation in which the volume element (as well as the Hamiltonian) may be developed in power series of a small parameter:

$$\begin{aligned} H &= \sum_\nu h_\nu, \\ \mathrm{d}\tau &= \sum_\nu t_\nu \mathrm{d}t \qquad (t_0 = 1). \end{aligned} \tag{A.1}$$

As in the usual case, the energies E_n and wave functions $|\psi(n)\rangle$ are determined through the two equations corresponding to the energy eigenvalues and the orthonormality condition

$$\begin{aligned} E_n\delta(n, m) &= \langle\psi(m)|H|\psi(n)\rangle, \\ \delta(n, m) &= \langle\psi(m)|\psi(n)\rangle. \end{aligned} \tag{A.2}$$

Let us denote by $|n\rangle$ the ket representing our basic set. Then,

$$|\psi(n)\rangle = \sum_\nu \sum_m a^{(\nu)}(n, m)|m\rangle. \tag{A.3}$$

Using (A.2) we obtain, after some algebra,

Zeroth order:

$$a^{(0)}(n, m) = \delta(n, m), \tag{A.4}$$

First order:

$$\begin{aligned} E_n^{(1)} &= \langle n|h_1|n\rangle, \\ a^{(1)}(n, m) &= \langle m|h_1|n\rangle/(E_n^{(0)} - E_m^{(0)}), \qquad (n \neq m) \\ a^{(1)}(n, n) &= \frac{-1}{2}\langle n|t_1|n\rangle, \end{aligned} \tag{A.5}$$

Second order:

$$E_n^{(2)} = \langle n|t_1 h_1|n\rangle + \langle n|h_2|n\rangle + \sum_m |\langle m|h_1|n\rangle|^2/(E_n^{(0)} - E_m^{(0)}),$$

$$\begin{aligned} a^{(2)}(n, m) &= \frac{\langle m|t_1 h_1|n\rangle + \langle m|h_2|n\rangle}{E_m^{(0)} - E_n^{(0)}} \\ &\quad + \sum_k \frac{\langle k|h_1|n\rangle}{E_k^{(0)} - E_m^{(0)}}\left\{\frac{\langle m|h_1|k\rangle}{E_k^{(0)} - E_n^{(0)}} + \frac{\langle k|h_1|m\rangle}{E_n^{(0)} - E_m^{(0)}}\right\} \qquad (n \neq m), \\ a^{(2)}(n, n) &= \frac{-1}{2}\langle n|t_2|n\rangle + \sum_m \frac{\langle m|h_1|n\rangle}{(E_n^{(0)} - E_m^{(0)})^2}\left(\tfrac{1}{2}\langle m|h_1|n\rangle - \langle n|h_1|m\rangle\right). \end{aligned} \tag{A.6}$$

Note that, in general, h_n is non-hermitian. Thus, it is important to keep track of the quantum numbers labelling the bras and kets in (A.5) and (A.6). The non-hermiticity of h_1 is intimately connected with the variation of the volume element. This can be seen, for instance, if we write the orthogonality condition in first order, namely

$$a^{(1)}(n, m)+a^{(1)}(m, n)+\langle n|t_1|m\rangle = \frac{\langle m|h_1|n\rangle-\langle n|h_1|m\rangle}{E_n^{(0)}-E_m^{(0)}}+\langle n|t_1|m\rangle = 0. \quad \text{(A.7)}$$

Appendix B

CALCULATION OF THE CORRECTION TO THE EIGENVALUES AND TRANSITION MATRIX ELEMENTS

We apply the expressions for the perturbation energies and amplitudes which are give in appendix A to the Hamiltonian (34) and volume element (35). Thus, the wave functions corresponding to the basic set are written in (22).

(*i*) *Calculation of the energy eigenvalues.* Since h_1 contains only non-diagonal terms, the first order correction $E_n^{(1)}$ vanishes (see (A.5)). For the second order correction, we obtain from (A.6)

$$\langle Y, n_\theta, n_\Delta, n_\Gamma, \Lambda|t_1 h_1|Y, n_\theta, n_\Delta, n_\Gamma, \Lambda\rangle$$
$$= \frac{-3\omega}{2Y}(2n_\Delta^2+4n_\Delta n_\theta+4n_\Delta n_\Gamma+10n_\Delta+2n_\theta+2n_\Gamma+1), \quad \text{(B.1)}$$

$$\langle Y, n_\theta, n_\Delta, n_\Gamma, \Lambda|h_2|Y, n_\theta, n_\Delta, n_\Gamma, \Lambda\rangle$$
$$= \frac{\omega}{8Y}(6n_\Delta^2+4n_\Gamma^2-4n_\theta^2+8n_\Delta n_\theta+8n_\Delta n_\Gamma+56n_\theta n_\Gamma-16\Lambda n_\theta$$
$$+22n_\Delta+44n_\Gamma+60n_\theta-8\Lambda+7), \quad \text{(B.2)}$$

$$\sum_{n'_\Delta}\frac{\langle Y, n_\theta, n'_\Delta, n_\Gamma, \Lambda|h_1|Y, n_\theta, n_\Delta, n_\Gamma, \Lambda\rangle^2}{E(n'_\Delta)-E(n_\Delta)}$$
$$= \frac{\omega}{8Y}(18n_\Delta^2-4n_\Gamma^2-4n_\theta^2+40n_\Delta n_\theta+40n_\Delta n_\Gamma-8n_\theta n_\Gamma+98n_\Delta+4n_\theta+4n_\Gamma-3), \quad \text{(B.3)}$$

$$\sum_{n'_\theta, n'_\Gamma, \Lambda'}\frac{\langle Y, n'_\theta, n_\Delta, n'_\Gamma, \Lambda'|h_1|Y, n_\theta, n_\Delta, n_\Gamma, \Lambda\rangle^2}{E(n'_\theta, n'_\Gamma, \Lambda')-E(n_\theta, n_\Gamma, \Lambda)}$$
$$= \frac{\omega}{Y}(n_\theta^2-6n_\theta n_\Gamma+2\Lambda n_\theta-5n_\theta-3n_\Gamma+\Lambda-2). \quad \text{(B.4)}$$

We verify that $E_n^{(2)}$ vanishes, if we add the contributions (B.1) to (B.4).

(*ii*) *Calculation of transfer matrix elements.* First we obtain the corrections corresponding to a large cross section, like the ones involving an elementary excitation (see the end of sect. 2).

Let us consider, for instance, transitions with $\Delta T = 0$. The strong matrix elements ($\mathrm{O}(Y^{\frac{1}{2}})$) satisfy the selection rule $|\Delta n_\theta| = 1$, $\Delta Y = \Delta n_\Delta = \Delta n_\Gamma = \Delta\Lambda = 0$ (table 3). The terms proportional to $Y^{\frac{1}{2}}$ in the matrix elements are given by (27).

Using table 3, we calculate the lowest order correction in the matrix element between unperturbed states:

$$\langle Y, n_\theta+1, n_\Delta, n_\Gamma, \Lambda||\mathscr{P}^-||Y, n_\theta, n_\Delta, n_\Gamma, \Lambda\rangle$$
$$= b\sqrt{2Y(n_\theta+1)}\left\{1+\frac{1}{8Y}(98n_\Delta-68n_\Gamma-2n_\theta-21)+\mathrm{O}(Y^{-2})\right\}. \quad \text{(B.5)}$$

There are unperturbed matrix elements of order unity between the initial (final) state and an admixture in the final (initial) state. For such cases in which the amplitude of the admixture is of order $Y^{-\frac{1}{2}}$, we obtain a contribution which is of the same order as the correction in (B.5):

$$\sum_{n'_\Delta}\{a^{(1)}(Y, n_\theta, n_\Delta, n_\Gamma, \Lambda;\, Y, n_\theta, n'_\Delta, n_\Gamma, \Lambda)\langle Y, n_\theta+1, n_\Delta, n_\Gamma, \Lambda||\mathscr{P}^-||Y, n_\theta, n'_\Delta, n_\Gamma, \Lambda\rangle$$
$$+a^{(1)}(Y, n_\theta+1, n_\Delta, n_\Gamma, \Lambda;\, Y, n_\theta+1, n'_\Delta, n_\Gamma, \Lambda)$$
$$\times\langle Y, n_\theta+1, n'_\Delta, n_\Gamma, \Lambda||\mathscr{P}^-||Y, n_\theta, n_\Delta, n_\Gamma, \Lambda\rangle\}$$
$$= 7b\sqrt{(n_\theta+1)/2Y}(n_\theta-5n_\Delta+n_\Gamma)+\mathrm{O}(Y^{-\frac{3}{2}}), \quad \text{(B.6)}$$

$$\sum_{n'_\Gamma}\{a^{(1)}(Y, n_\theta, n_\Delta, n_\Gamma, \Lambda;\, Y+1, n_\theta+2, n_\Delta, n'_\Gamma, \Lambda+1)$$
$$\times\langle Y, n_\theta+1, n_\Delta, n_\Gamma, \Lambda||\mathscr{P}^-||Y+1, n_\theta+2, n_\Delta, n'_\Gamma, \Lambda+1\rangle$$
$$+a^{(1)}(Y, n_\theta+1, n_\Delta, n_\Gamma, \Lambda;\, Y-1, n_\theta-1, n_\Delta, n'_\Gamma, \Lambda-1)$$
$$\times\langle Y-1, n_\theta-1, n_\Delta, n'_\Gamma, \Lambda-1||\mathscr{P}^-||Y, n_\theta, n_\Delta, n_\Gamma, \Lambda\rangle\}$$
$$= b\sqrt{(n_\theta+1)/2Y}(n_\theta-3n_\Gamma+\Lambda-2)+\mathrm{O}(Y^{-\frac{3}{2}}). \quad \text{(B.7)}$$

Another correction is obtained through the second order renormalization amplitudes $a^{(2)}(n, n)$. Finally†, there are admixtures of order $Y^{-\frac{1}{2}}$ in both the initial and final states, such that there is a strong matrix element between the two admixed

† Since the second order admixtures in the initial (final) state are connected with the final (initial) unperturbed state via a transfer matrix element which is always smaller than $Y^{\frac{1}{2}}$, we do not need in the present case to consider the amplitudes $a^{(2)}(n, m)$ for $n \neq m$.

states. These last two corrections strongly cancel each other, and the net result is

$$\begin{aligned}
&\{a^{(2)}(Y, n_\theta, n_\Delta, n_\Gamma, \Lambda;\ Y, n_\theta, n_\Delta, n_\Gamma, \Lambda) \\
&\quad + a^{(2)}(Y, n_\theta+1, n_\Delta, n_\Gamma, \Lambda;\ Y, n_\theta+1, n_\Delta, n_\Gamma, \Lambda)\} \\
&\quad \times \langle Y, n_\theta+1, n_\Delta, n_\Gamma, \Lambda||\mathscr{P}^-||Y, n_\theta, n_\Delta, n_\Gamma, \Lambda\rangle \\
&\quad + \sum_{Y', n'_\theta, n'_\Delta, n'_\Gamma, \Lambda'} a^{(1)}(Y, n_\theta, n_\Delta, n_\Gamma, \Lambda;\ Y', n'_\theta, n'_\Delta, n'_\Gamma, \Lambda') \\
&\quad \times a^{(1)}(Y, n_\theta+1, n_\Delta, n_\Gamma, \Lambda;\ Y', n'_\theta+1, n'_\Delta, n'_\Gamma, \Lambda') \\
&\quad \times \langle Y', n'_\theta+1, n'_\Delta, n'_\Gamma, \Lambda'||\mathscr{P}^-||Y', n'_\theta, n'_\Delta, n'_\Gamma, \Lambda'\rangle \\
&\quad = \tfrac{1}{4} b\sqrt{(n_\theta+1)/2Y}(46n_\Delta+50n_\Gamma-30n_\theta-6\Lambda+31)+\mathrm{O}(Y^{-\frac{3}{2}}).
\end{aligned} \tag{B.8}$$

Adding (B.5) to (B.8) we obtain

$$\begin{aligned}
&\langle\psi(Y, n_\theta+1, n_\Delta, n_\Gamma, \Lambda)||\mathscr{P}^-||\psi(Y, n_\theta, n_\Delta, n_\Gamma, \Lambda)\rangle \\
&\qquad = b\sqrt{2Y(n_\theta+1)}\{1+(1/4Y)(2n_\Delta-n_\Gamma-\Lambda+1)\}+\mathrm{O}(Y^{-\frac{3}{2}}).
\end{aligned} \tag{B.9}$$

In order to see how this result agrees with the corresponding matrix elements in the boson scheme to the required order of $Y^{-\frac{1}{2}}$ we apply eq. (B.9) to the transitions starting from the yrast and the one-Γ-phonon states,

$$\langle Y, 1, 0, 0, 0||\mathscr{P}^-||Y, 0, 0, 0, 0\rangle = b\sqrt{2Y}\left(1+\frac{1}{4Y}+\ldots\right), \tag{B.10a}$$

$$\begin{aligned}
\langle Y-1, 1, 0, 1, 1||\mathscr{P}^-||Y-1, 0, 0, 1, 1\rangle &= b\sqrt{2(Y-1)}\left(1-\frac{1}{4Y}+\ldots\right) \\
&= b\sqrt{2Y}\left(1-\frac{3}{4Y}+\ldots\right),
\end{aligned} \tag{B.10b}$$

while the corresponding transitions in the boson description [see sect. 2 and ref. [6])] give

$$\langle 1, 1;\ Y, Y;\ Y||\mathscr{P}^-||0, 0;\ Y, Y;\ Y\rangle = b\sqrt{2Y+1} = b\sqrt{2Y}\left(1+\frac{1}{4Y}+\ldots\right), \tag{B.11a}$$

$$\begin{aligned}
\langle 1, 1;\ Y, Y-2;\ Y-2||\mathscr{P}^-||0, 0;\ Y, Y-2;\ Y-2\rangle &= b\sqrt{2Y-3} \\
&= b\sqrt{2Y}\left(1-\frac{3}{4Y}+\ldots\right).
\end{aligned} \tag{B.11b}$$

Secondly, we calculate the transfer matrix element between the state $|\psi(Y, n_\theta, n_\Delta, n_\Gamma, \Lambda)\rangle$ and the state $|\psi(Y+1, n_\theta+1, n_\Delta, n_\Gamma+1, \Lambda+1)\rangle$ having a pair of particles more. This matrix element is small (O(1)) according to table 3. In this case, there are states which are mixed in first order with the initial (final) state and these admixtures have large transfer matrix elements ($O(Y^{\frac{1}{2}})$) with the final (initial) state. Thus we may obtain significant corrections to the unperturbed matrix element:

$$
\begin{aligned}
&\langle\psi(Y+1, n_\theta+1, n_\Delta, n_\Gamma+1, \Lambda+1)||\mathscr{P}^+||\psi(Y, n_\theta, n_\Delta, n_\Gamma, \Lambda)\rangle \\
&\quad = \langle Y+1, n_\theta+1, n_\Delta, n_\Gamma+1, \Lambda+1||\mathscr{P}^+||Y, n_\theta, n_\Delta, n_\Gamma, \Lambda\rangle \\
&\quad + a^{(1)}(Y, n_\theta, n_\Delta, n_\Gamma, \Lambda;\ Y+1, n_\theta+2, n_\Delta, n_\Gamma+1, \Lambda+1) \\
&\quad \times\langle Y+1, n_\theta+1, n_\Delta, n_\Gamma+1, \Lambda+1||\mathscr{P}^+||Y+1, n_\theta+2, n_\Delta, n_\Gamma+1, \Lambda+1\rangle \\
&\quad + a^{(1)}(Y+1, n_\theta+1, n_\Delta, n_\Gamma+1, \Lambda+1;\ Y, n_\theta-1, n_\Delta, n_\Gamma, \Lambda) \\
&\quad \times\langle Y, n_\theta-1, n_\Delta, n_\Gamma, \Lambda||\mathscr{P}^+||Y, n_\theta, n_\Delta, n_\Gamma, \Lambda\rangle + O(1/Y) \\
&\quad = b\{((n_\theta+1)(n_\Gamma+\Lambda+2))^{\frac{1}{2}} - ((n_\theta+1)(n_\theta+2)(n_\Gamma+\Lambda+2)/8Y)^{\frac{1}{2}}(2Y(n_\theta+2))^{\frac{1}{2}} \\
&\quad + (n_\theta(n_\theta+1)(n_\Gamma+\Lambda+2)/8Y)^{\frac{1}{2}}(2Yn_\theta)^{\frac{1}{2}} + O(1/Y)\}. \qquad \text{(B.12)}
\end{aligned}
$$

This matrix element vanishes in the harmonic limit, since it corresponds to a transition between states differing by three phonons. Therefore, the perturbative treatment reproduces the correct result to the required order.

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