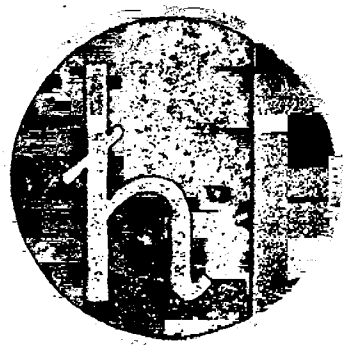

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Fermions of integer spin

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FERMIONS OF INTEGER SPIN

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ABSTRACT: It is shown that the problem of a two-level laser leads to processes in which bosons transform into fermions and vice versa. Such processes lie beyond the frame work of present field theories.

1. The main features of the passage of a light beam trough a transparent medium are well described by Lorentz' classical model of a system of continuously distributed, independent harmonic atomic oscillators, (oscillator plasma), which are coupled to the radiation field (electromagnetic oscillator). Lorentz' dispersion theory can be easily transcribed into quantum theory and leads to quantised collective vibrations of both atomic andelectromagnetic oscillators, which, in second quantization, appear as the result os processes in which bosonic photons become transformed into bosonic excitations of the atomic oscillator system and vice versa (x)

The phenomena, however, which are actually of interest in dispersion theory require generalisations of Lorentz' classical model in several respects. The problem of optical pumping requires the atoms or oscillators to be coupled among eachother, either by forces acting between them directly, or indirectly

(x) See: L.Masperi, Supl. Nuovo Cimento 4, 1, (1969).-

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by coupling with a new field able to carry the excitatation energy which feeds a laser. (x)

Another modification of Lorentz' model is necessary in order to account for the fact, that dispersion is not caused by harmonic oscillators, but by more realistic atomic models, which make the dispersion phenomenon model dependent in the resonance region. In the present paper we shall deal with a model in which the oscillators are replaced by two-level atoms for which the ground state and the excited state of energy $\hbar.\omega_o$ of the i-th atom are respectively given by $\psi_i(x_{1)}$ and $\varphi_i(x_i)$.

2. We shall assume that N two-level atoms are densely distributed over a volume V, and we shall, at first, omit the interaction of these atoms with the radiation field. The system posesses, then, 2^N states with a high degree of degeneracy. The complete set of eigenfuctions, their energy values and their multiplicity are given in I.-

Table I

Energy	Eigenfunctions	Multiplicity
$N.\hbar.\omega_o$	$\Phi_{1,\ 2,\ldots,N}$	1
	.....	
$n.\hbar.\omega_o$	$\Phi_{1,2,\ldots\ n,}$	$\binom{N}{n}$
	.....	
$2.\hbar.\omega_o$	$\Phi_{1,2}$, $\Phi_{1,3}$ ..... $\Phi_{N-1,N}$	$\frac{N.(N-1)}{2}$
$\hbar.\omega_o$	Φ_1, Φ_2 ..... Φ_N	N
0	Φ	1

(x) One aspect of the new features which are introduced by the feeding process shall be dealt with in a forthcoming paper by I.C. da Cunha Lima.-

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The eigenfunctions $\phi_{r,s,\ldots}$ of table I are products of N individual eigenfunctions and tend, in the limiting case, to functionals of field theory. The ground state eigenfunctions is given by.

$$\phi = \prod_{i=1}^{N} \psi_i (x_i)$$

ϕ_r means, that in (1) $\psi_r(x_r)$ has been replaced by $\varphi_r(x_r)$ etc. The complete set of eigenfunctions in table I has been ordered so that in $\phi_{r,s,t,\ldots..}$ $r < s < t < \ldots..$ The first two lines from below of table I do not present any difficulties. The relevant properties of the eigenfunctions from the other lines follow by straighforward generalisation from the behaviour of the ones of the third line, indicated by an arrow. We shall write the eigenfunctions of this line more explicitly in table II

$$\begin{vmatrix} 0, & \phi_{1,2}, & \phi_{1,3}, \ldots\ldots\ldots & \phi_{1,N} \\ & 0 & \phi_{2,3}, \ldots\ldots\ldots & \phi_{2,N} \\ & & 0 \quad \ldots\ldots\ldots & \phi_{3,N} \\ & & \ldots\ldots\ldots\ldots\ldots & \\ & & 0 \quad \phi_{N-1,N} & \\ & & 0 & \end{vmatrix}$$

Table II

They fill a half square of diagonal zero.

Formally, however, it is more convenient to work with a quadratic scheme of eigenfunctions. We may complete, therefore, our half square to a complete square, e.g. by making it symmetric or antisymmetric. We have to be aware, however, that such an extension is purely formal, without any physical meaning, and makes the system of eigenfunctions we use overcomplete. This implies that series developments in this overcomplete sets of eigen-

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functions are no longer unique. Further, the extension we want to make is arbitrary and requires a new convention.

We shall, now, introduce creation operators by

$$\phi_r = b_r^+ . \phi \quad , \quad \phi_{r,s} = b_s^+ . \phi_r = b_s^+ . b_r^+ . \phi \quad , \text{ etc.} \qquad (2)$$

The b_r^+ and the corresponding destruction operators b_r have to satisfy anticommutation relations

$$b_r . b_r^+ + b_r^+ . b_r = 1 \qquad (3)$$

in order to assure, that the r-th two level system cannot be excited twice. The commutation relations between b_r^+ and b_s^+ remain, however, indetermined for $r \neq s$ and can be determined by convention.

We shall examine two particularly simple conventions:

a) $$b_r^+ . b_s^+ - b_s^+ . b_r^+ = 0 \qquad (4)$$

b) $$b_r^+ . b_s^+ + b_s^+ . b_r^+ = 0 \qquad (5)$$

Both possible conventions extend the half square of table II to a complete square. Convention (4) makes the eigenfuctions, according to (2), symmetrically over-complete

$$\phi_{r,s} = \phi_{s,r} \quad , \quad \phi_{r,r} = 0 \qquad (6)$$

while convention (5) makes them antisymmetrically over-complete

$$\phi_{r,s} = - \phi_{s,r} \qquad (7)$$

As long as we deal with the basic set of eigenfunctions given in Tables I and II their properties can be easily overlooked and we may, without difficulties use either

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convention (4) or convention (5). Applications of the formalism require, however, that we shall be able to pass from one basis system to another, e.g. by a unitary transformation

$$b_k^+ = \sum_r a_k^r \cdot b_r^+ \qquad (8)$$

Under the transformation (8) the extended overcomplete eigenfunctions of Table I transform like tensors. According to convention (5) these tensors are antisymmetric and their linear dependency remains easily overlookable. According to convention (4), however, the tensors become symmetric and contain N additional linear dependencies, due to the fact, that the diagonal elements in the original basis system have to vanish. Convention (5) is, therefore, easier to be handled than convention (4).

3. Convention (4) has been adopted in a paper by SCHWABL and THIRRING (x) and they have shown, that this convention is convenient in the case of low energy excitations of the two-level atoms. The excitations, according to relations (3) and (4) are certainly no bosons, but they behave, in the considered limiting case in good approximation as if they were bosons. The results of Schwabl and Thirring show, that in the considered limit Lorentz' dispersion formulae remain valid even in the case of two-level atoms. In the general case of arbitrarily high excitations, however, convention (4) is no longer useful.

(x) F. SCHWABL u. W. THIRRING: Ergeb. exakt. Naturw. 36, 219 (1964).

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If we adopt, however, convention (5), relations (3) and (5) show, that excitations of a system of two-level atoms can be treated rigolously as fermions. Since they are excited by dipole transitions they have necessarily spin one. We are, therefore, lead to conclude that fermions of integer spin exist and play an important role in phenomena implying two-level atoms.

On the other hand, we know that the axioms of field theory are chosen in such a way, that they exclude the possibility to couple fields of fermions of integer spin (spin - statistics theorem). We can, therefore, hardly assume, that the problem of dispersion by two-level atoms, required for a theory of the laser, can be solved within the framework of present field theory and we have to conjecture, that the essentially non-linear interactions which take place between the involved bosons and fermions of integer spin require a broader axiomatic basis then the one we know at present.-