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Irreducibility Constraints and Field Equations for the Elementary Particles.

I. - Bosons.

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Summary. — The conditions needed to describe an elementary particle by an irreducible quantity in the Lorentz group, are here taken as constraint equations to be imposed on the field entity. The independent components are extracted in a covariant way. The field equations are deduced from a Lagrangian in which only the free components are permitted to be varied. The commutation relations, compatible with the constraints, are also given. In Sect. **1** the field entity is a tensor, in a forthcoming paper we will consider spin-tensors and simple Dirac spinors for which second order wave equations will be proposed.

1. – Introduction.

The field of an elementary particle with integer spin, is represented by means of a tensor A. Irreducibility under the Lorentz group and uniqueness of spin is assured if the following conditions are imposed [1]. A must be *a*) completely symmetric, *b*) traceless, *c*) divergenceless. These three conditions reduce the number of independent components to $2s+1$ which is the right number of possible orientations of a particle with spin s.

Condition *c* for example, implies the spacial character of the spin in the rest system. Neverteheless this condition is usually deduced, whenever possible ($m \neq 0$), from a suitable Lagrangian, being then dependent on the equations of motion.

(*) Now on leave of absence.

(1) H. Umezawa: *Quantum field theory* (1956), pp. 64-66.

Our viewpoint is different. We are going to take *a*), *b*) and *c*) as constraint conditions to be imposed on the field tensor independently of the equation of motion. It will be shown that no difference with the usual procedure will result in the free field case. But new possibilities arise in the presence of interactions.

Something similar happens when the particle is a fermion. The representing field may be chosen to be a spin-tensor [2] which must be: *a'*) completly symmetric, *b'*) «perpendicular» to γ_μ, and *c'*) divergenceless. The traceless character can be deduced from *a'*) and *b'*). Here too, complicated Lagrangians must be written down in order to be able to deduce *b'*) and *c'*) as equations of motion [2,3]. We have now two ways opened. We can take either a first or a second order wave equation [4]. With our point of view the second possibility seems more logical because a Dirac spinor in itself has two redundant components [4]. Also, the connection between spin and statistics is indeed independent of the equation of motion [5]. In order to eliminate the redundant components we are going to change *c'*) in favour of a natural and more stringent condition which will be mentioned in Part II.

Summarizing. – Our aim is: 1) The statement of the constraint equations. 2) The extraction, in a covariant way, of the independent components of the field entity representing any particle with spin greater then zero (the $S=0$ case being trivial). 3) The deduction of the field equations for the independent components and the field entity. 4) The quantization of the field in a way compatible with the constraints.

For the sake of clearness we only treat in this paper the integer spin case. In a forthcoming second part we are going to treat the fermion fields.

2. – Basic tensors.

We first arbitrarily choose three space-like unit vector operators which are perpendicular to the impulse vector $p_\mu = -i\partial_\mu$

$$\left\{ \begin{aligned} & a_\mu a_\mu = b_\mu b_\mu = c_\mu c_\mu = 1\,, \\ & a_\mu b_\mu = b_\mu c_\mu = c_\mu a_\mu = 0\,, \\ & a_\mu\, p_\mu = b_\mu\, p_\mu = c_\mu\, p_\mu = 0\,. \end{aligned} \right. \tag{2.1}$$

(2) W. RARITA and J. SCHWINGER: *Phys. Rev.*, **60**, 61 (1941).
(3) H. UMEZAWA: *Quantum field theory* (1956), p. 120.
(4) R. P. FEYNMAN and M. GELL-MANN: *Phys. Rev.*, **409**, 193 (1958).
(5) N. BURGOYNE: *Nuovo Cimento*, **8**, 607 (1958).

These three operators, together with the properly normalized impulse vector [6] d_μ

(2.2) $$d_\mu = p^{-1} p_\mu \,, \qquad d_\mu d_\mu = -1$$

form a basic system of vectors. We have

(2.3) $$\delta_{\mu\nu} = a_\mu a_\nu + b_\mu b_\nu + c_\mu c_\nu - d_\mu d_\nu \,.$$

Any field tensor $A_{\mu_1 \dots \mu_s}$ satisfying conditions a, b and c, can be expressed in terms of a basic system of tensors satisfying also the same constraints. Such a basis may be formed with the following tensor operators

(2.4) $$\left\{ \begin{aligned} N^{(r)} p^{(2r-1)}_{\mu_1 \dots \mu_s} &= p^{(2r-2)}_{\mu_1 \dots \mu_{s+1}} a_{\mu_{s+1}} + p^{(2r-3)}_{\mu_1 \dots \mu_{s+1}} b_{\mu_{s+1}} \\ N^{(r)} p^{(2r)}_{\mu_1 \dots \mu_s} &= p^{(2r-2)}_{\mu_1 \dots \mu_{s+1}} b_{\mu_{s+1}} - p^{(2r-3)}_{\mu_1 \dots \mu_{s+1}} a_{\mu_{s+1}} \end{aligned} \right. \qquad r = 1, 2, \dots, s.$$

This is a recurrence chain which allows any tensor to be written in terms of $p^{(0)}_{\mu_1 \dots \mu_{s+1}}$ [7]. A formula for the latter is given in the appendix. $N^{(r)}$ are normalization constants. We have

(2.5) $$p^{(\varrho)}_{\mu_1 \dots \mu_s} p^{(\sigma)}_{\mu_1 \dots \mu_s} = \delta^{\varrho\sigma} \qquad \varrho, \sigma = 0, 1, \dots, 2s.$$

Any one of the tensors $p^{(\varrho)}_{\mu_1 \dots \mu_s}$ is symmetric, traceless and perpendicular to d_μ (*i.e.*, ∂_μ). They may be called « polarization tensors » for the fields $A_{\mu_1 \dots \mu_s}$ satisfying the constraints a, b and c.

3. – Projection operators.

As is always the case when we have a base for a given subspace, the corresponding projection operator may be formed by means of

(3.1) $$P_{\mu_1 \dots \mu_s;\, \nu_1 \dots \nu_s} = \sum_{\varrho=0}^{2s} p^{(\varrho)}_{\mu_1 \dots \mu_s} p^{(\varrho)}_{\nu_1 \dots \nu_s} \,.$$

Recalling the properties of the polarization tensors we come to the conclusion that the projection operator coincides with that investigated by C.

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(6) Although some caution is needed, we are going to use the operator p^{-1} defined by

$$p^{-1} f(x) = \int \mathrm{d}^4 p (-p_\mu p_\mu)^{-\frac{1}{2}} f(p) \exp [i p_\mu x_\mu] \,.$$

(7) When $r = 1$, in (2.4) we put $p^{(-1)}_{\mu_1 \dots \mu_s} = 0$.

FRONSDAL [8]. $\boldsymbol{P}$ is independent of the chosen base because it is unique [9]. Also

$$P_{\mu_1\dots\mu_s;\,\alpha_1\dots\alpha_s}P_{\alpha_1\dots\alpha_s;\,\nu_1\dots\nu_s}=P_{\mu_1\dots\mu_s;\,\nu_1\dots\nu_s} \tag{3.2}$$

and

$$P_{\mu_1\dots\mu_s;\,\mu_1\dots\mu_s}=2s+1\,. \tag{3.3}$$

For the vector field we have

$$P_{\mu;\nu}=a_\mu a_\nu+b_\mu b_\nu+c_\mu c_\nu\;(=p^{(0)}_\mu p^{(0)}_\nu+p^{(1)}_\mu p^{(1)}_\nu+p^{(2)}_\mu p^{(2)}_\nu)\,.$$

It follows from (2.2) and (2.3) that

$$P_{\mu;\nu}=\delta_{\mu\nu}+d_\mu d_\nu=\delta_{\mu\nu}+p^{-2}p_\mu p_\nu\,. \tag{3.4}$$

4. – Constraint equations and free components.

The fact that a given tensor field A satisfies conditions a, b and c, is expressed by the constraint equation

$$A_{\mu_1\dots\mu_s}=P_{\mu_1\dots\mu_s;\,\nu_1\dots\nu_s}A_{\nu_1\dots\nu_s}\,. \tag{4.1}$$

By considering (3.1), we may draw from (4.1) the consequence that

$$\left\{\begin{aligned}A_{\mu_1\dots\mu_s}&=\sum_\varrho p^{(\varrho)}_{\mu_1\dots\mu_s}p^{(\varrho)}_{\nu_1\dots\nu_s}A_{\nu_1\dots\nu_s}\,,\\ A_{\mu_1\dots\mu_s}&=\sum_{\varrho=0}^{2s}p^{(\varrho)}_{\mu_1\dots\mu_s}A^{(\varrho)}\,,\end{aligned}\right. \tag{4.2}$$

where

$$A^{(\varrho)}=p^{(\varrho)}_{\nu_1\dots\nu_s}A_{\nu_1\dots\nu_s}\,. \tag{4.3}$$

The $2s+1$ scalar quantities $A^{(\varrho)}$ given by (4.3) may be taken as the free components of the field tensor A, and (4.2) expresses the latter as a function of the former. The presence of the polarization tensors in (4.2) assures the fulfilment of (4.1).

Some physical significance can be ascribed to the decomposition (4.2). The index ϱ is related to the spin orientation of the represented particle, but we are not going to elaborate on this point.

[8] R. E. BEHRENDS and C. FRONSDAL: *Phys. Rev.*, **106**, 345 (1957).
[9] R. E. BEHRENDS and C. FRONSDAL: *Phys. Rev.*, **106**, 345 (1957), Appendix.

5. – Lagrangians and field equations.

For the sake of simplicity we will consider only real tensor fields. A natural Lorentz invariant Lagrangian for such a field is:

$$L = -\tfrac{1}{2}(m^2 A_{\nu_1 \dots \nu_s} A_{\nu_1 \dots \nu_s} + \partial_\mu A_{\nu_1 \dots \nu_s} \partial_\mu A_{\nu_1 \dots \nu_s}) . \tag{5.1}$$

This Lagrangian is essentially unique in the free field case. In order for the equation of motion to be deduced from the action integral, only the independent components of A must be varied. This is necessary if we want to be consistent with the constraints (4.1).

The action integral is

$$\mathcal{A} = -\int \mathrm{d}^4 x L .$$

By using (4.2) and (5.1), we have [10],

$$\mathcal{A} = -\frac{1}{2}\int \mathrm{d}^4 x \,(m^2 \sum_\varrho p^{(\varrho)}_{\nu_1 \dots \nu_s} A^{(\varrho)} \sum_\sigma p^{(\sigma)}_{\nu_1 \dots \nu_s} A^{(\sigma)} + \dots) ,$$

$$\mathcal{A} = -\frac{1}{2}\int \mathrm{d}^4 x \,(m^2 \sum_{\varrho\sigma} A^{(\varrho)} p^{(\varrho)}_{\nu_1 \dots \nu_s} p^{(\sigma)}_{\nu_1 \dots \nu_s} A^{(\sigma)} + \dots) .$$

With the orthonormality property (2.5), we obtain

$$\mathcal{A} = -\frac{1}{2}\int \mathrm{d}^4 x \,(m^2 \sum_\varrho A^{(\varrho)} A^{(\varrho)} + \dots) = \sum_\varrho \int \mathrm{d}^4 x L^{(\varrho)}, \tag{5.2}$$

$$L^{(\varrho)} = -\tfrac{1}{2}(m^2 A^{(\varrho)} A^{(\varrho)} + \partial_\mu A^{(\varrho)} \partial_\mu A^{(\varrho)}) . \tag{5.3}$$

The equations of motion are now easily deduced from (5.2). The independent variations of the free components give the $2s+1$ equations

$$(\partial_\mu \partial_\mu - m^2) A^{(\varrho)} = 0 \tag{5.4}$$

or equivalently (cf. (4.2))

$$(\partial_\mu \partial_\mu - m^2) A_{\nu_1 \dots \nu_s} = 0 . \tag{5.5}$$

The equations of the theory are then the constraint equations (4.1) and the equations of motion (5.5) (or (5.4)).

([10]) It can be proved by a Fourier transformation that the polarization tensors may be viewed as hermitian operators.

In the presence of interactions, the Lagrangian is

$$L = \sum_{\varrho} L^{(\varrho)} + L^{(\mathrm{int})} ,$$

where $L^{(\mathrm{int})}$ has the form

(5.6) $$L^{(\mathrm{int})} = - A_{\mu_1 \dots \mu_s} F_{\mu_1 \dots \mu_s} .$$

F is a tensorial function of the field representing the source. We may now proceed exactly as before, but to speed up deductions we put

$$A_{\mu_1 \dots \mu_s} = P_{\mu_1 \dots \mu_s;\, \nu_1 \dots \nu_s} B_{\nu_1 \dots \nu_s} ,$$

where B is a tensor upon which no constraint is imposed.

The projection operator allows only the free components of A to follow the variations of all the 4^s components of B.

The action integral of the interaction is

$$\mathcal{A}^{\mathrm{int}} = - \int \mathrm{d}^4 x \, A_{\mu_1 \dots \mu_s} F_{\mu_1 \dots \mu_s} = - \int \mathrm{d}^4 x \, P_{\mu_1 \dots \mu_s;\, \nu_1 \dots \nu_s} B_{\nu_1 \dots \nu_s} \cdot F_{\mu_1 \dots \mu_s} .$$

It follows from the variation of all the components of B and the hermitic character of P [11], that

(5.7) $$(\partial_\mu \partial_\mu - m^2) A_{\mu_1 \dots \mu_s} = - P_{\mu_1 \dots \mu_s;\, \nu_1 \dots \nu_s} F_{\nu_1 \dots \nu_s} .$$

Of course, (5.7) is compatible with the constraint (4.1). (5.7) means that only that part of the source which belongs to the space of P interacts with A.

6. – Quantization.

When a Lagrangian has the form

(6.1) $$L = \tfrac{1}{2} (m^2 Q Q + \partial_\mu Q \partial_\mu Q)$$

the canonical rules imply the commutation relations

(6.2) $$[Q(x), Q(x')] = - i \Delta(x - x') .$$

([11]) Cfr. footnote ([10]).

This is our case because the partial Lagrangian (5.3) has the form (6.1). Therefore

$$(6.3)\qquad [A^{(\varrho)}(x),\, A^{(\sigma)}(x')] = -\, i\,\delta^{\varrho\sigma}\,\Delta(x-x')\,.$$

Multiplying (6.3) with $p^{(\varrho)}_{\mu_1\ldots\mu_s}p^{(\sigma)}_{\nu_1\ldots\nu_s}$ and summing over ϱ and σ, we have (cf. (4.2)):

$$(6.4)\qquad [A_{\mu_1\ldots\mu_s}(x),\, A_{\nu_1\ldots\nu_s}(x')] = -\, i\sum_{\varrho} p^{(\varrho)}_{\mu_1\ldots\mu_s}p^{(\varrho)'}_{\nu_1\ldots\nu_s}\,\Delta(x-x')\,.$$

In (6.4), $p^{(\varrho)'}_{\nu_1\ldots\nu_s}$ operates on the x' variable of $\Delta(x-x')$, but neither the defining equations nor the properties (cfr. (2.1), (2.4)) of $p^{(\varrho)}_{\nu_1\ldots\nu_s}$ are changed if we transform ∂_μ into $-\partial_\mu$. Consequently

$$(6.5)\qquad [A_{\mu_1\ldots\mu_s}(x),\, A_{\nu_1\ldots\nu_s}(x')] = -\, i\,P_{\mu_1\ldots\mu_s;\,\nu_1\ldots\nu_s}\,\Delta(x-x')\,,$$

(4.1), (5.8) and (6.5) show that the projection operators enter the theory in a rather fundamental way.

7. – Example. Vector field.

(3.4) gives the projection operator for the vector field. The corresponding commutation relations are

$$(7.1)\qquad [A_\mu(x),\, A_\nu(x')] = -\, i(\delta_{\mu\nu} + p^{-2}p_\mu p_\nu)\,\Delta(x-x')\,.$$

The invariant function of Pauli and Jordan satisfies the Klein-Gordon equation; *i.e.*

$$(7.2)\qquad p^{-2}\Delta(x-x') = \frac{1}{m^2}\Delta(x-x')\,.$$

Then, (7.1) is equivalent to

$$(7.3)\qquad [A_\mu(x),\, A_\nu(x')] = -\, i\left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{m^2}\right)\Delta(x-x')\,.$$

We see now that the free field commutation relations do not differ from those usually imposed. It is nevertheless worth mentioning that this is not the case with the vacuum expectation value of the chronological product.

In fact, from

$$\langle 0|\, T[A^{(\varrho)}(x),\, A^{(\sigma)}(x')]\,|0\rangle = -\, i\,\Delta_F(x-x')\,,$$

proceeding as in Sect. **6**, we shall arrive to

$$\langle 0|T[A_{\mu_1\dots\mu_s}(x),\, A_{\nu_1\dots\nu_s}(x')]|0\rangle = -iP_{\mu_1\dots\mu_s;\,\nu_1\dots\nu_s}\Delta(x-x') \tag{7.4}$$

which for the vector field is

$$\begin{cases} \langle 0|T[A_\mu(x),\, A_\nu(x')]|0\rangle = -i(\delta_{\mu\nu} + p^{-2}p_\mu p_\nu)\,\Delta_F(x-x') = \\ \qquad = -i(\delta_{\mu\nu} - p^{-2}\partial_\mu\,\partial_\nu)\,\Delta_F(x-x')\,. \end{cases} \tag{7.5}$$

But now the operator p^{-2} must be retained because Δ_F does not satisfy the homogeneous Klein-Gordon equation.

The presence of the projection operator in (7.4) is related to the fact that not only the field operator, but also the propagator must obey the constraint equations.

Appendix

In Sect. **2** we defined the polarization tensors by means of the recurrence formulae:

$$\begin{cases} p^{(2r-1)}_{\mu_1\dots\mu_s} = p^{(2r-2)}_{\mu_1\dots\mu_{s+1}}a_{\mu_{s+1}} + p^{(2r-3)}_{\mu_1\dots\mu_{s+1}}b_{\mu_{s+1}}\,, \\ p^{(2r)}_{\mu_1\dots\mu_s} = p^{(2r-2)}_{\mu_1\dots\mu_{s+1}}b_{\mu_{s+1}} - p^{(2r-3)}_{\mu_1\dots\mu_{s+1}}a_{\mu_{s+1}}\,. \end{cases} \tag{A.1}$$

A second application of (A.1) gives

$$\begin{cases} p^{(2r-1)}_{\mu_1\dots\mu_s} = p^{(2r-4)}_{\mu_1\dots\mu_{s+1}}a_{\mu_{s+1}\mu_{s+2}} + p^{(2r-5)}_{\mu_1\dots\mu_{s+2}}b_{\mu_{s+1}\mu_{s+2}}\,, \\ p^{(2r)}_{\mu_1\dots\mu_s} = p^{(2r-4)}_{\mu_1\dots\mu_{s+2}}b_{\mu_{s+1}\mu_{s+2}} - p^{(2r-5)}_{\mu_1\dots\mu_{s+2}}a_{\mu_{s+1}\mu_{s+2}}\,. \end{cases} \tag{A.2}$$

Where

$$\begin{cases} a_{\mu\nu} = a_\mu b_\nu + a_\nu b_\mu\,, \\ b_{\mu\nu} = b_\mu b_\nu - a_\mu a_\nu\,. \end{cases} \tag{A.3}$$

Step by step, the following formulae are arrived to

$$\begin{cases} p^{(2r-1)}_{\mu_1\dots\mu_s} = p^{(0)}_{\mu_1\dots\mu_{s+r}}a_{\mu_{s+1}\dots\mu_{s+r}}\,, \\ p^{(2r)}_{\mu_1\dots\mu_s} = p^{(0)}_{\mu_1\dots\mu_{s+r}}b_{\mu_{s+1}\dots\mu_{s+r}}\,, \end{cases} \tag{A.4}$$

with

$$\begin{cases} a_{\nu_1\dots\nu_r} = a_{\nu_1\dots\nu_{r-1}}b_{\nu_r} + b_{\nu_1\dots\nu_{r-1}}a_{\nu_r}\,, \\ b_{\nu_1\dots\nu_r} = b_{\nu_1\dots\nu_{r-1}}b_{\nu_r} - a_{\nu_1\dots\nu_{r-1}}a\,. \end{cases} \tag{A.5}$$

For $p^{(0)}$ we have

$$N p^{(0)}_{\mu_1 \dots \mu_t} = c_{\mu_1} \dots c_{\mu_t} + \alpha_1 (P_{\mu_1;\mu_2} c_{\mu_3} \dots c_{\mu_t} + \text{symm.}) + \alpha_2 (P_{\mu_1;\mu_2} P_{\mu_3;\mu_4} c_{\mu_5} \dots c_{\mu_t} + \text{symm.}) + \\ + \dots + \left\{ \begin{array}{l} \alpha_{t/2} P_{\mu_1;\mu_2} \dots P_{\mu_{t-1};\mu_t} \\ \alpha_{(t-1)/2} P_{\mu_1;\mu_2} \dots P_{\mu_{t-2};\mu_{t-1}} c_{\mu_t} \end{array} + \text{symm.} \right\} \begin{array}{l} (t = \text{even}), \\ (t = \text{odd}). \end{array}$$

$P_{\mu;\nu}$ is the projection operator (3.4). $\alpha_1, \alpha_2, \dots$ are constants to be determined so as to nullify the trace of $p^{(0)}_{\mu_1 \dots \mu_t}$. N is the normalization constant, «symm.» means symmetrizing terms.

Obviously, $p^{(0)}_{\mu_1 \dots \mu_t}$ is symmetric, traceless and perpendicular to d_μ.

RIASSUNTO (*)

Nel presente lavoro si considerano come equazioni restrittive da imporre all'intensità del campo le condizioni occorrenti a descrivere una particella elementare per mezzo di una grandezza irriducibile nel gruppo di Lorentz. Si ricavano le componenti indipendenti in modo covariante. Le equazioni di campo si deducono da un lagrangiano in cui solo le componenti libere possono variare. Si danno anche le relazioni di commutazione compatibili con le limitazioni. Nella Sez. 1 l'intensità di campo è un tensore; in un prossimo lavoro considereremo tensori spinoriali e spinori di Dirac semplici pei quali proporremo equazioni d'onda del second'ordine.

(*) *Traduzione a cura della Redazione.*

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