Low-Energy Scattering of Charged Particles.

R. O. BARRACHINA and C. R. GARIBOTTI (*)
Instituto Balseiro (**) - 8400-*Bariloche, Argentina*

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Summary. – The off-energy-shell T-matrix for two charged particles is studied in the low momentum limit ($k \to 0$). The T-matrix for a Coulomb interaction (T_C) is usually considered as the limit of the amplitude for a screened potential (T_s) when the screening is removed. We show that this statement is not true for small enough energies. For an attractive interaction T_C and T_s differ significantly when $k \to 0$. T_C behaves as $k^{-\frac{1}{2}}$, while T_s keeps its k^{-1} behaviour even when the screening is turned off. We note that this is an effect which would be observed in ion-atom collisions when one electron is ejected from the atom and captured into a continuum state of the ion.

We consider the T-matrix element in the momentum representation

$$t = \langle \boldsymbol{p}_2 | T\left(\frac{k^2}{2m} + i\varepsilon\right) | \boldsymbol{p}_1 \rangle \tag{1}$$

for real $p_1 = |\boldsymbol{p}_1|$, $p_2 = |\boldsymbol{p}_2|$ and k. For a pure Coulomb interaction this amplitude does not approach a well-defined limit on the half energy shell $p_1 = k$ or $p_2 = k$. As was pointed out by FORD [1,2] the T-matrix is discontinuous at the energy shells. In order to analyse this situation, many authors [1-4] have introduced screened Coulomb potentials and studied the behaviour of the T-matrix as the screening is turned off. The screening is characterized by a parameter R, which gives the range of the force. A large change in the magnitude of the T-matrix takes place within a narrow strip around the shells $p_1 = k$ or $p_2 = k$ whose width is of order $1/R$. When R goes to infinity and k is fixed, such that $kR \gg 1$, the screened potential amplitude converges to the Coulombian one.

(*) Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina.
(**) Comisión Nacional de Energía Atómica and Universidad Nacional de Cuyo.
(1) W. F. FORD: *Phys. Rev. B*, **133**, 1616 (1964).
(2) W. F. FORD: *J. Math. Phys.*, **7**, 626 (1966).
(3) J. C. Y. CHEN and A. C. CHEN: *Advances in Atomic and Molecular Physics*, Vol. **8** (New York, N.Y., 1972).
(4) M. KOLSRUD: *J. Phys. A*, **11**, 1271 (1978).

In atomic-collision physics Coulomb interactions between the particles are assumed, however the experimental conditions impose a finite natural cut-off on the interaction and the condition $kR \gg 1$ fails for small enough values of k. In spite of being of actual interest, the zero energy limit ($k \to 0$) of the screened Coulomb T-matrix has not been researched. In this paper we show that the order of the double limit $k \to 0$, $R \to \infty$ is critical and the present results might give rise to some interesting experimental consequences.

In the following we shall be concerned particularly with the analysis of the zero-energy limit of the « half »-shell amplitude

$$t = \langle \Psi_{\boldsymbol{k}}^- | V | \boldsymbol{p} \rangle \qquad (\boldsymbol{p} \neq \boldsymbol{k}) , \tag{2}$$

where $\Psi_{\boldsymbol{k}}^-$ is the final scattering state of momentum $\boldsymbol{k}$. According to the final-state interaction theory [5], the zero energy limit is dominated by the inverse of the s-wave Jost function $f_0(k, R)$, which is called the enhancement factor,

$$t \approx \frac{1}{f_0(k, R)} \langle \boldsymbol{k} | V | \boldsymbol{p} \rangle \qquad (k \text{ small}) \tag{3}$$

and is given by the following integral expression [6] (atomic units):

$$f_0(k, R) = 1 + \frac{2m}{k} \int_0^\infty \exp[ikr] V_R(r) \varphi_{0k} r \, \mathrm{d}r . \tag{4}$$

Here V_R is a screened Coulomb potential of range R and φ_{0k} is the $l = 0$ regular solution of the corresponding radial equation. When k is small, φ_{0k} approachs zero linearly in k, and

$$f_0(k, R) \propto \frac{1 - iak}{a} \qquad (k \text{ small}) , \tag{5}$$

where a is a constant. On the other hand, if we suppose that the two-body interaction is given by a pure Coulomb potential $V = Z/r$, the final scattering state $\Psi_{\boldsymbol{k}}^-$ is a Coulomb wave. Its normalization factor is the complex conjugate of the Coulomb factor:

$$f_C(\alpha) = \exp[-\pi\alpha/2] \Gamma(1 + i\alpha) , \tag{6}$$

$$\alpha = \frac{mZ}{k} . \tag{7}$$

This factor $f_C(\alpha)$, factored out from the T-matrix element, is the corresponding enhancement factor and forecasts an energy dependence which is obviously quite different from the zero energy limit previously described. In fact $f_C^2(k) \sim 2m\pi Z/k$ (k small). This rough discussion shows that the two limits $k \to 0$ and $R \to \infty$ do not commute.

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(5) K. M. WATSON: *Phys. Rev.*, **88**, 1163 (1952); J. GILLESPIE: *Final-State Interaction* (San Francisco, Cal., 1964).
(6) J. R. TAYLOR: *Scattering Theory* (New York, N.Y., 1972).

Further, into this problem let us consider a particular exponential screening given by the Hulthèn potential

$$V_{\mathrm{H}}(r) = (Z/R)/(\exp[r/R] - 1)^{-1}\,. \tag{8}$$

The corresponding s-wave regular solution of the radial equation is

$$\varphi_{0k}(r) = kR\varrho\, {}_2F_1(1 + i\beta,\, 1 - i(\beta + 2kR);\, 2;\, \varrho)\exp[ikr]\,, \tag{9}$$

$$\varrho = 1 - \exp[-r/R]\,, \tag{10}$$

$$\beta = kR\left[\left(1 + \frac{2\alpha}{kR}\right)^{-\frac{1}{2}} - 1\right]. \tag{11}$$

Replacing in the integral equation for the s-wave Jost function, we have

$$f_0(k, R) = 1 - 2mZR\int\limits_0^1 (1-\varrho)^{-2ikR}\, {}_2F_1(1 + i\beta,\, 1 - i(\beta + 2kR);\, 2;\, \varrho)\,\mathrm{d}\varrho\,. \tag{12}$$

This gives an analytic expression for $f_0(k)$

$$f_0(k, R) = {}_2F_1(-i\beta,\, i(\beta + 2kR);\, 1;\, 1)\,, \tag{13}$$

which can be written in terms of gamma-functions ([7])

$$f_0(k, R) = \frac{\Gamma(1 - 2ikR)}{\Gamma(1 + i\beta)\Gamma(1 - i(\beta + 2kR))}\,. \tag{14}$$

This expression depends on R through the product kR, so the ordering of the double limiting process $k \to 0$ and $R \to \infty$ is important. In fact, when $2|\alpha| \ll kR$ it can be easily verified that

$$\frac{1}{f_0(k, R)} \approx f_{\mathrm{C}}(n)\exp[-i\alpha\ln(2kR)] \qquad (2|\alpha| \ll kR)\,. \tag{15}$$

This result agrees with Kolsrud ([4]) conjecture which states that « the absolute values of screened-Coulomb T-matrices may converge in the limit of vanishing screening to the results of the standard short-range theory. The condition is simply that the values of the momenta should be fixed before the limiting process ».

The absolute value of the s-wave Jost function may be written as

$$|f_0(k, R)|^2 = \left|\frac{\sinh(\pi\beta)\sinh(\pi(\beta + 2kR))}{\pi\alpha\sinh(2\pi kR)}\right|. \tag{16}$$

In the limit $2|\alpha| \ll kR$, it precisely displays the behaviour predicted by eq. (15). On the other hand, when $2|\alpha| \gg kR$ the energy behaviour of the s-wave Jost function differs significantly from that described by the Coulomb factor.

([7]) M. Abramowitz and I. Stegum: *Handbook of Mathematical Functions* (New York, N.Y., 1965).

In fact, when k goes to zero the following approximation is valid:

$$|f_0(k, R)|^2 \approx -\frac{R}{2mZ}\,\frac{1+bk^2}{b} \tag{17}$$

with

$$b(R) = -\left[\frac{\pi R}{\sinh\left(\pi\sqrt{2mZR}\right)}\right]^2, \tag{18}$$

which is an expected result (compare with eq. (5)).

In particular, for an atractive Hulthén potential $Z < 0$, the factor b becomes

$$b(R) = \left[\frac{\pi R}{\sin\left(\pi\sqrt{2m|Z|R}\right)}\right]^2. \tag{19}$$

It has a finite value except when a zero-energy resonance occurs. In this case the factor $b(R)$ is infinite and the s-wave Jost function approachs zero linearly in k, *e.g.* the « half » off-shell amplitude behaves as k^{-1}, meanwhile the Coulomb amplitude behaves as $k^{-\frac{1}{2}}$. These results are not unique to the Hulthén potential and can be obtained starting from other cut-off Coulomb potentials.

We conclude that the zero-energy limit of the screened Coulomb T-matrix shows a behaviour similar to that found by FORD at the energy shells: When the range R is large and k is greater than a quantity of the order of $1/R$, the absolute value of the T-matrix does not differ from that of a pure Coulomb potential. However, when k approaches zero the energy dependence of the screened Coulomb T-matrix is quite different from the zero-energy limit of the pure Coulomb T-matrix. In the authors' opinion, almost all the analyses of processes with Coulomb interaction in which two particles constitute a small-energy continuum state must be re-examined in view of these results.

A very small-energy charged beam (electrons or ions) would allow us to observe deviations from a Coulomb behaviour in the direct-scattering ion-ion cross-section. However, it seems experimentally very difficult to produce a monoenergetic beam with the appropriate characteristics. A particular process where the results of our analysis could be verified is the electron capture to projectile's continuum effect (ECC) in ion-atom collisions ([8]). In that case the ejected electron can move with arbitrarily small velocities relative to the projectile. This ECC effect is observable as a sharp cusp in the double differential electron cross-section. The shape of this peak might depend on whether the interaction should be considered screened or not.

Another interesting feature of the present result is as a partial proof of Kolsrud conjecture.

The foregoing analysis can be extended out of the energy half-shell, although similar conclusions are expected.

([8]) J. MACEK: *Phys. Rev. A*, **1**, 235 (1970); C. R. GARIBOTTI and J. MIRAGLIA: *Phys. Rev. A*, **21**, 572 (1980); *J. Phys. B*, **14**, 863 (1981).