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On the Quantization of Tensor Fields with Zero Mass.

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Summary. — A general formulation of quantum field theory for massless tensor fields, compatible with the supplementary conditions, is given. The free field (or the interacting field in the interaction representation) is expressed in terms of its components « along » the transversal polarization tensors defined in the text. These components are taken as canonical co-ordinates, on which the canonical commutation relations are imposed. The corresponding commutation relations of the field components are given.

1. – Introduction.

The field of an elementary particle with vanishing mass, satisfies the equations [1]:

$$\partial_\nu \partial_\nu A_{\nu_1 \dots \nu_s} = 0 \,, \tag{1.1}$$

$$\partial_{\nu_1} A_{\nu_1 \dots \nu_s} = 0 \,, \tag{1.2}$$

$$A_{\nu_1 \nu_1 \nu_3 \dots \nu_s} = 0 \,. \tag{1.3}$$

The field quantities $A_{\nu_1 \dots \nu_s}$ are the components of a symmetric tensor of rank s. (1.1) is the field equation and (1.2), (1.3) are the supplementary conditions which ensure that the theory is based on an irreducible representation of a unique spin s.

In the usual quantization procedure, the field components are taken as canonical variables and the canonical commutation relations are imposed.

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(1) H. Umezawa: *Quantum Field Theory* (1956).

However, these commutation relations will not be compatible with the supplementary conditions (1.2) and (1.3). Therefore, a subsidiary condition on the state vector is adopted instead of equations (1.2) and (1.3). Nevertheless, this procedure is not universally adhered to [2].

We want to approach the problem of the quantization from a different point of view. We intend to take (1.2) and (1.3) as constraint equations and so we are going to extract the independent components from the field tensor. As a consequence, a generalization of a method for the vectorial field due to J. G. VALATIN [3] will result. Of course, the commutation relations between the independent components of the field will be compatible with the supplementary conditions.

2. – Decomposition of the field tensor.

Starting with the symmetric and traceless tensor of rank s, $\boldsymbol{A}$, satisfying the equations (1.1), (1.2), (1.3), let us define by recurrence the following tensors:

$$A^{(0)} = n_{\nu_1} \dots n_{\nu_s} A_{\nu_1 \dots \nu_s}$$

$$A^{(0)}_{\nu_1 \dots \nu_s} = \partial^{-s} \partial_{\nu_1} \dots \partial_{\nu_s} A^{(0)}$$

$$A^{(r)}_{\nu_1 \dots \nu_r} = n_{\nu_{r+1}} \dots n_{\nu_s} (A_{\nu_1 \dots \nu_s} - \sum_{t=0}^{r-1} A^{(r)}_{\nu_1 \dots \nu_s}) \tag{2.1}$$

$$A^{(r)}_{\nu_1 \dots \nu_s} = \partial^{-(s-r)} \partial_{\nu_{r+1}} \dots \partial_{\nu_s} A^{(r)}_{\nu_1 \dots \nu_r} + (\text{symm}) \tag{2.2}$$

n_ν is an arbitrary time-like unit vector: $n_\nu n_\nu = -1$. « (symm) » means terms necessary to obtain a symmetric tensor. $\partial = n_\nu \partial_\nu$ and ∂^{-1} is the inverse operator (COESTER and JAUCH [4]).

All these tensors satisfy (1.1), (1.2) and (1.3). We also have:

$$n_{\nu_1} A^{(r)}_{\nu_1 \dots \nu_r} = 0 \qquad r \neq 0 \tag{2.3}$$

$$n_{\nu_{r+1}} \dots n_{\nu_s} A^{(r)}_{\nu_1 \dots \nu_s} = A^{(r)}_{\nu_1 \dots \nu_r}. \tag{2.4}$$

From (2.1) with $r = s$, we have:

$$A_{\nu_1 \dots \nu_s} = \sum_{t=0}^{s} A^{(t)}_{\nu_1 \dots \nu_s}. \tag{2.5}$$

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[2] See, for example: T. KIMURA: *Prog. Theor. Phys.*, **16**, 555 (1956); S. OZAKI: *Prog. Theor. Phys.*, **14**, 511 (1955).

[3] J. G. VALATIN: *Danske Vidensk. Selsk. Mat. Fys. Medd.*, **26**, No. 13 (1951).

[4] F. COESTER and J. M. JAUCH: *Phys. Rev.*, **78**, 149 and 827 (1950).

The tensor $A^{(r)}_{\nu_1\ldots\nu_r}$ $(r \neq 0)$ being orthogonal to ∂_ν and to n_ν, has the same number of independent components as a symmetric and traceless tensor in two dimensions. This number is two. Of course, $A^{(0)}$ being scalar, has only one component. From (2.2), we deduce that $A^{(r)}_{\nu_1\ldots\nu_s}$ posseses the same number of independent components as $A^{(r)}_{\nu_1\ldots\nu_r}$. Then, equation (2.5) is a decomposition of the field tensor in $s+1$ tensors with a total of $2s+1$ independent components. The tensor $A^{(0)}_{\nu_1\ldots\nu_s}$ is purely longitudinal, while $A^{(s)}_{\nu_1\ldots\nu_s}$ is purely transversal.

3. – Polarization tensors.

Let us take two derivation operators α_ν and β_ν, in such a way that:

$$\begin{aligned} n_\nu\alpha_\nu &= 0 \qquad & n_\nu\beta_\nu &= 0 \\ \alpha_\nu\partial_\nu &= 0 \qquad & \beta_\nu\partial_\nu &= 0 \\ \alpha_\nu\alpha_\nu &= 1 \qquad & \beta_\nu\beta_\nu &= 1 \end{aligned}$$

$$\alpha_\nu\beta_\nu = 0 \,.$$

(They are not unique).

Now, let us define by recurrence, the following tensor operators

$$\alpha_{\nu_1\ldots\nu_{r+1}} = \alpha_{\nu_1\ldots\nu_r}\alpha_{\nu_{r+1}} - \beta_{\nu_1\ldots\nu_r}\beta_{\nu_{r+1}} \tag{3.1}$$

$$\beta_{\nu_1\ldots\nu_{r+1}} = \alpha_{\nu_1\ldots\nu_r}\beta_{\nu_{r+1}} + \beta_{\nu_1\ldots\nu_r}\alpha_{\nu_{r+1}} \,. \tag{3.2}$$

They are symmetric and traceless. Furthermore, they are orthogonal to n_ν and ∂_ν, and

$$\alpha_{\nu_1\ldots\nu_r}\beta_{\nu_1\ldots\nu_r} = 0 \,.$$

Let us call them the transversal polarization tensors.

Two independent components of $A^{(r)}_{\nu_1\ldots\nu_r}$ $(r \neq 0)$ may be obtained by means of

$$\alpha_{\nu_1\ldots\nu_r}A^{(r)}_{\nu_1\ldots\nu_r} \qquad \text{and} \qquad \beta_{\nu_1\ldots\nu_r}A^{(r)}_{\nu_1\ldots\nu_r} \,.$$

Then, we can put:

$$A^{(r)}_{\nu_1\ldots\nu_r} = \alpha_{\nu_1\ldots\nu_r}Q^{(2r)} + \beta_{\nu_1\ldots\nu_r}Q^{(2r+1)} \tag{3.3}$$

$$A^{(0)} = Q^{(1)} \,. \tag{3.4}$$

By means of (2.2) we can obtain the tensors $A^{(r)}_{\nu_1\ldots\nu_s}$.

The $2s+1$ scalar quantities $Q^{(\varrho)}$ ($\varrho = 1, \ldots, 2s+1$) are free variables which may be taken as canonical co-ordinates of the field. They satisfy the equation

$$\partial_\nu \partial_\nu Q^{(\varrho)} = 0 , \tag{3.5}$$

without any supplementary condition.

4. – Gauge transformations.

If the tensor $A_{\nu_1 \ldots \nu_{s-1}}$ satisfies (1.1), (1.2) and (1.3), then it follows from

$$G_{\nu_1 \ldots \nu_s} = \partial_{\nu_s} A_{\nu_1 \ldots \nu_{s-1}} + (\text{symm}) \tag{4.1}$$

that

$$B_{\nu_1 \ldots \nu_s} = A_{\nu_1 \ldots \nu_s} + G_{\nu_1 \ldots \nu_s} , \tag{4.2}$$

also satisfies the same equations. The transformation $\boldsymbol{A} \to \boldsymbol{B}$ is called a Gauge transformation.

The decomposition of A is (cf. eq. (2.5))

$$A_{\nu_1 \ldots \nu_{s-1}} = \sum_{t=0}^{s-1} A^{(t)}_{\nu_1 \ldots \nu_{s-1}} .$$

Then

$$G_{\nu_1 \ldots \nu_s} = \sum_{t=0}^{s-1} G^{(t)}_{\nu_1 \ldots \nu_s} . \tag{4.3}$$

The tensor $\boldsymbol{G}$ has no totally transversal component:

$$G^{(s)}_{\nu_1 \ldots \nu_s} = 0 .$$

Thus, if a gauge transformation is made on $\boldsymbol{A}$, all but one of its tensor components are changed. The part unaffected by a gauge transformation is $A^{(s)}_{\nu_1 \ldots \nu_s}$, i.e. $Q^{(2s)}$ and $Q^{(2s+1)}$. They are actually the free components of $\boldsymbol{A}$. The other components are needed to describe the « direct » or « longitudinal » interaction between the sources of the field.

5. – Quantization.

The quantities $Q^{(\varrho)}$ form a system of independent co-ordinates satisfying (3.5). Therefore, we impose on them the « canonical » commutation relations

$$[Q^{(\varrho)}(x), Q^{(\varrho')}(x')] = i \cdot \delta^{\varrho\varrho'} \cdot D(x - x') , \tag{5.1}$$

$$\delta^{\varrho\varrho'} = 0 \quad \varrho \neq \varrho' ; \qquad \delta^{\varrho\varrho} = \pm 1 .$$

(We leave open the assignment of sign. See ([3]) for the electromagnetic field) (3.3), (3.4) and (5.1) give

$$(5.2)\qquad [A^{(r)}_{\nu_1\ldots\nu_r}(x),\, A^{(r')}_{\mu_1\ldots\mu_{r'}}(x')] = i\cdot\delta^{rr'}\cdot(\alpha_{\nu_1\ldots\nu_r}\alpha_{\mu_1\ldots\mu_r}+\beta_{\nu_1\ldots\nu_r}\beta_{\mu_1\ldots\mu_r})\cdot D(x-x') = i\cdot\delta^{rr'}\cdot d_{\nu_1\ldots\nu_r;\,\mu_1\ldots\mu_r}D(x-x')\,.$$

It is easily seen that $d_{\nu_1\ldots\nu_r;\,\mu_1\ldots\mu_r}$ is independent of the choice of the polarization vectors α_ν and β_ν. In fact, from (3.1) and (3.2) we can derive the recurrence relations:

$$d_{\nu_1\ldots\nu_{r+1};\,\mu_1\ldots\mu_{r+1}} = d_{\nu_1\ldots\nu_r;\,\mu_1\ldots\mu_r}\,d_{\nu_{r+1};\,\mu_{r+1}} - e_{\nu_1\ldots\nu_r;\,\mu_1\ldots\mu_r}\,e_{\nu_{r+1};\,\mu_{r+1}}$$

$$e_{\nu_1\ldots\nu_{r+1};\,\mu_1\ldots\mu_{r+1}} = d_{\nu_1\ldots\nu_r;\,\mu_1\ldots\mu_r}\,e_{\nu_{r+1};\,\mu_{r+1}} + e_{\nu_1\ldots\nu_r;\,\mu_1\ldots\mu_r}\,d_{\nu_{r+1};\,\mu_{r+1}}\,.$$

With

$$e_{\nu_1\ldots\nu_r;\,\mu_1\ldots\mu_r} = \alpha_{\nu_1\ldots\nu_r}\beta_{\mu_1\ldots\mu_r} - \beta_{\nu_1\ldots\nu_r}\alpha_{\mu_1\ldots\mu_r}\,,$$

$d_{\nu;\mu}$ and $e_{\nu;\mu}$ are actually independent of α_ν and β_ν; they are the fundamental tensors (symm. and antisymm. resp.) in a plane perpendicular to n_ν and ∂_ν.

Defining now,

$$(5.3)\qquad d^{(r)}_{\nu_1\ldots\nu_s;\,\mu_1\ldots\mu_s} = \partial^{-2(s-r)}\partial_{\nu_{r+1}}\ldots\partial_{\nu_s}\partial_{\mu_{r+1}}\ldots\partial_{\mu_r}d_{\nu_1\ldots\nu_r;\,\mu_1\ldots\mu_r} + (\text{symm})$$

we have (cf. (2.2) and (5.2)):

$$(5.4)\qquad [A^{(r)}_{\nu_1\ldots\nu_s}(x),\, A^{(r')}_{\mu_1\ldots\mu_s}(x')] = i\cdot\delta^{rr'}\cdot d^{(r)}_{\nu_1\ldots\nu_s;\,\mu_1\ldots\mu_s}D(x-x')\,.$$

Finally, summing over r and r' we obtain

$$(5.5)\qquad [A_{\nu_1\ldots\nu_s}(x),\, A_{\mu_1\ldots\mu_s}(x')] = i\cdot\bar{d}_{\nu_1\ldots\nu_s;\,\mu_1\ldots\mu_s}D(x-x')\,,$$

$$(5.6)\qquad \bar{d}_{\nu_1\ldots\nu_s;\,\mu_1\ldots\mu_s} = \sum_r \delta^{rr}d^{(r)}_{\nu_1\ldots\nu_s;\,\mu_1\ldots\mu_s}\,,$$

(5.4) are the commutation relations to be imposed on the field tensor. They are compatible with the supplementary conditions (1.2) and (1.3).

6. – Examples.

In the case $s=1$, taking $-\delta^{00}=\delta^{11}=1$, the commutation relations (5.5) are reduced to those of J. VALATIN ([3]) for the electromagnetic field,

$$(6.1)\qquad \begin{cases} d^{(0)}_{\mu;\,\nu} = \partial^{-2}\partial_\mu\partial_\nu\,, \\ d^{(1)}_{\mu;\,\nu} = \delta_{\mu\nu} - \partial^{-2}\partial_\mu\partial_\nu - n_\mu\partial_\nu\partial^{-2} - n_\nu\partial_\mu\partial^{-1}\,, \\ \bar{d}_{\mu;\,\nu} = \delta_{\mu\nu} - 2\partial^{-2}\partial_\mu\partial_\nu - n_\mu\partial_\nu\partial^{-1} - n_\nu\partial_\mu\partial^{-1}\,. \end{cases}$$

In the case $s = 2$, we have

$$A^{(0)}_{\mu\nu} = \partial^{-2}\partial_\mu\partial_\nu A^{(0)}\,; \qquad A^{(0)} = Q^{(1)}$$

$$A^{(1)}_{\mu\nu} = \partial^{-1}\partial_\mu A^{(1)}_\nu + \partial^{-1}\partial_\nu A^{(1)}_\mu\,; \qquad A^{(1)}_\mu = \alpha_\mu Q^{(2)} + \beta_\mu Q^{(3)}$$

$$A^{(2)}_{\mu\nu} = \alpha_{\mu\nu} Q^{(4)} + \beta_{\mu\nu} Q^{(5)}$$

(6.2) $$A_{\mu\nu} = \partial^{-2}\partial_\mu\partial_\nu Q^{(1)} + \partial^{-1}(\partial_\mu\alpha_\nu + \partial_\nu\alpha_\mu)Q^{(2)} + \\ + \partial^{-1}(\partial_\mu\beta_\nu + \partial_\nu\beta_\mu)Q^{(3)} + \alpha_{\mu\nu}Q^{(4)} + \beta_{\mu\nu}Q^{(5)}\,,$$

(6.3) $$[A^{(0)}_{\mu\nu}(x),\, A^{(0)}_{\varrho\sigma}(x')] = i\cdot\delta^{00}\partial^{-4}\partial_\mu\partial_\nu\partial_\varrho\partial_\sigma D(x-x')\,,$$

(6.4) $$[A^{(1)}_{\mu\nu}(x),\, A^{(1)}_{\varrho\sigma}(x')] = i\cdot\delta^{11}\partial^{-2}(\partial_\mu\partial_\nu d^{(1)}_{\varrho;\sigma} + (\text{symm}))\,D(x-x')\,,$$

(6.5) $$[A^{(2)}_{\mu\nu}(x),\, A^{(2)}_{\varrho\sigma}(x')] = i\cdot\delta^{22}(\alpha_{\mu\nu}\alpha_{\varrho\sigma} + \beta_{\mu\nu}\beta_{\varrho\sigma})D(x-x') = \\ = i\cdot\delta^{22}(d^{(1)}_{\mu;\varrho}d^{(1)}_{\nu;\sigma} + d^{(1)}_{\mu;\sigma}d^{(1)}_{\nu;\varrho} - d^{(1)}_{\mu;\nu}d^{(1)}_{\varrho;\sigma})\,D(x-x')\,.$$

The last commutation relation coincides with that of T. KIMURA ([5]) for the incoming transversal field if the trace part is substracted from his result.

([5]) T. KIMURA: *Prog. Theor. Phys.*, **16**, 555 (1956), p. 566, eq. (4.10).

RIASSUNTO (*)

Si dà una formulazione generale della teoria quantistica dei campi, per campi di tensori privi di massa, compatibili con le condizioni supplementari. Il campo libero (o il campo interagente nella rappresentazione per mezzo dell'interazione) si esprime in termini delle sue componenti « secondo » i tensori di polarizzazione trasversali definiti nel testo. Tali componenti si prendono come coordinate canoniche alle quali si impongono le relazioni di commutazione canoniche. Si danno le corrispondenti relazioni di commutazione delle componenti del campo.

(*) *Traduzione a cura della Redazione*

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