STRONG COUPLING EXPANSION OF THE LIOUVILLE MODEL

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Received 21 January 1983

Using a strong coupling expansion we show that in two dimensions the Liouville model has a continuous energy spectrum and that the mean value of the field is infinite. We also apply the technique to the one-dimensional system and to a simple quantum mechanical problem where those properties are known to be true.

Polyakov's treatment of the string model [1] has motivated great interest in the quantization of the two-dimensional Liouville model [2]. Its lagrangian is

$$\mathcal{L} = (1/2\beta^2)(\nabla\phi)^2 + (\mu/\beta)^2 e^{\phi} - z\phi \,, \tag{1}$$

for $z = 0$; we have generalized the model with this extra parameter for reasons which will become clear later. For the string model it is also necessary to give the topology of the surface [3] and to add boundary terms [4]. However as was pointed out recently the system defined by this lagrangian in the whole plane might have some interesting properties such as a continuous energy spectrum [5,6] and no ground state [5].

This model cannot be treated with perturbation theory neither in β nor in μ^2, the first because of the term ϕ/β coming from the expansion of the exponential and the second because of infrared divergencies [5]. These problems do not show up if we consider β large and μ/β fixed, namely in a strong coupling expansion.

In this letter we describe very briefly a strong coupling expansion [7] and in this context we show that the field ground state expectation value $\langle\phi\rangle$ diverges and that the model has a continuous energy spectrum. We also consider a simpler model where the exponential in eq. (1) is replaced by an infinite barrier. In one dimension both systems are exactly solvable when $z = 0$ exhibiting the properties that we mentioned before.

The strong coupling expansion for continuum field theories was proposed in ref. [7] and applied later to a $O(N)$-invariant scalar theory [8], generalized to fermions [9] and used for quantum electrodynamics [10]. We review briefly how to make this expansion; starting with the generating functional in euclidean metric

$$Z(J) = \int \prod_x \mathrm{d}\phi_x \exp\left(-\int \mathcal{L}\,\mathrm{d}^D x + \int J(x)\,\phi(x)\,\mathrm{d}^D x\right), \tag{2}$$

we take the kinetic energy term as a perturbation and solve the functional integral exactly for the rest of the lagrangian. For our system this amounts to considering β^2 large with μ/β fixed. Writing

$$Z(J) = \exp\left(-\frac{1}{2}\int \mathrm{d}^D x\,\mathrm{d}^D y\,\frac{\delta}{\delta J(x)}\,D(x-y)\,\frac{\delta}{\delta J(y)}\right)$$
$$\times \prod_x \mathrm{d}\phi_x \exp\left(-\frac{\mu^2}{\beta^2}\int \mathrm{d}^D x(e^{\phi} - z\phi)\right) \tag{3}$$
$$\times \exp\left(\int J\phi\,\mathrm{d}^D x\right),$$

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where $D(x)$ is given by

$$D(x) = -\beta^{-2}\partial^2\delta^D(x)\,, \tag{4}$$

by expanding the first and last exponentials we immediately find diagrammatical rules with a "propagator" given by $D(k^2) = k^2/\beta^2$ and s-point vertices which are the sth moments of the second exponential. Double counting occurs when integrating over the position of these vertices if two or more of them coincide. It can be proved that this problem is avoided by taking the connected part of the moments. For details we refer to the original papers [7].

To give a meaning to this expansion we need to discretize the D-dimensional space and regularize the delta Dirac functions. Taking $\delta^D(x-y) = \delta_{x,y}/a^D$ (a^D is the unit cell volume) eq. (4) reads

$$D(k^2) = (ka)^2/\beta^2 a^{D+2}\,. \tag{5}$$

We first study the infinite barrier problem. In this case the s-point vertices b_s have a simple expression

$$b_s = \frac{(\mathrm{d}^s/\mathrm{d}l^s)\int_{-\infty}^0 \mathrm{d}y \exp(za^D y + ly)|_{l=0}}{\int_{-\infty}^0 \mathrm{d}y \exp(za^D y)} = \frac{(-)^s s}{(za^D)^s}\,. \tag{6}$$

To evaluate their connected parts B_s we first consider the function

$$f(\eta) = \sum_{s=0}^{\infty} \frac{b_s \eta^s}{s} = (1 + \eta/za^D)^{-1}\,, \tag{7}$$

then they are given by the coefficients of the expansion of $\ln f(\eta)$, namely

$$B_s = (-)^s (s-1)/(za^D)^s\,. \tag{8}$$

Once we have the strong coupling expansion of $Z(J)$ the connected Green's functions are obtained from

$$G_{\mathrm{c}}^{(N)}(x_1, ..., x_N) = \frac{1}{a^{ND}} \frac{\delta^N \ln Z(J)}{\delta J(x_1) ... \delta J(x_N)}\bigg|_{J=0}\,. \tag{9}$$

It is easy to find the general form of the series for any of these functions. For the kth order we have the following factors

(i) $(\beta^2 a^{D+2})^{-1}$ for each of the k propagators,

(ii) $(za^D)^{-(2k+N)}$ coming from the product of all the connected vertices,

(iii) $(a^D)^{2k+N}$ from the expansion of the last exponential in eq. (3),

(iv) a^{-ND} from eq. (9).

In particular for the field ground state expectation value we obtain a series of the form

$$\langle\phi\rangle = a^{-D} \sum_{k=0}^{\infty} \mathrm{d}_k \frac{a^{D(2k+1)}}{(\beta^2 a^{D+2})^k (za^D)^{2k+1}}\,, \tag{10}$$

which can be written as a series in powers of only one dimensionless variable $x = (\beta^2 z^2 a^{D+2})^{-1}$

$$\langle\phi\rangle = ex^\delta \sum_{k=0}^{\infty} \alpha_k x^k = ex^\delta S(x)\,. \tag{11}$$

Here $\delta = D/(D+2)$ and $e = (\beta^{2D} z^{D-2})^{1/(D+2)}$.

The continuum limit is not straightforward [7], as $a \to 0$ x becomes large and we have to use extrapolation techniques to obtain the asymptotic value C of $x^\delta S(x)$.

In one dimension

$$\langle\phi\rangle = C(\beta^2/z)^{1/3}\,, \tag{12}$$

which gives the exact dependence in β and z [11]; all the approximations are contained in the evaluation of the constant C. Let us notice that $\langle\phi\rangle$ diverges as $z \to 0$.

Similarly we can write a series expansion for the propagator and find the lowest energy levels. Denoting the sum of the one-particle irreducible strong coupling diagrams by σ it can be shown [8] that the Fourier transform of the propagator is given by

$$G_{\mathrm{c}}^{(2)}(k^2) = a^D/(k^2/\beta^2 a^D + 1/\sigma)\,. \tag{13}$$

σ^{-1} is a polynomial in $(ka)^2$ of the form

$$\sigma^{-1} = B_2^{-1}\,[1 + A_0(x) + A_1(x)(ka)^2 + ...]\,, \tag{14}$$

with coefficients $A(x)$ which are only functions of x and D. The degree of this polynomial increases with the order of the strong coupling expansion but terms proportional to $(ka)^4$ appear only in the order $(1/\beta^2)^4$.

The pole of the propagator is located at

$$(\Delta E_1)^2 = k_{\mathrm{M}}^2 = a^{-2}\{[1 + A_0(x)]/[x + A_1(x)]\} = M_0^2 F(x)\, x^{2/D+2}\,, \tag{15}$$

where we went back to Minkowski metric with $k_{\mathrm{M}}^2 = -k^2$, $M_0 = (z\beta)^{2/D+2}$ contains all the dependence in z and β and $\lim_{x\to\infty} x^{2/D+2} F(x) = \gamma_1$ is just a number to be evaluated by extrapolation techniques. As before the functional form in z and β is given directly to us by the strong coupling expansion. Had we evaluated

σ to higher orders we would have found a polynomial of a higher degree in $(ka)^2$ but all the poles would have been of the form

$$\Delta E_n = M_0 \gamma_n \,. \tag{16}$$

Again for $D = 1$ this is in agreement with the exact result [11]. As $z \to 0$ we can see that the energy spectrum becomes continuous. The energy differences ΔE_n of the ground state with any other level are zero, the only way to have a non-trivial spectrum is to take simultaneously the limits $z \to 0$ and $n \to \infty$ in such a way that $z^{2/3}\gamma_n = q$ is kept constant. The integer n, which labels the discrete spectrum when $z \neq 0$, is traded by the positive number q which labels the continuum spectrum.

The values of the constants C and γ_1 were estimated by means of Padé approximants. In the first case we obtained (up to order $1/\beta^{10}$) the sequence of three approximants

$$C^{(1)} = 0.554\,, \quad C^{(2)} = 0.635, \quad C^{(3)} = 0.685\,.$$

For γ_1 we calculated up to order $1/\beta^8$

$$\gamma_1^{(1)} = 1.51\,, \quad \gamma_1^{(2)} = 1.52\,.$$

The exact values are $C = 1.236$ and $\gamma_1 = 1.389$. We see that the first sequence moves in the right direction although we cannot say whether it will converge to the correct value or not. For the energy gap we have a better agreement.

Similar considerations apply to the Liouville model in one and two dimensions. However in this case the connected s-point vertices B_s are more complicated and consequently the final results do not have a simple functional form as in eqs. (11) or (15). Now we have

$$b_s = \frac{(\mathrm{d}^s/\mathrm{d}l^s)\int_{-\infty}^{\infty}\mathrm{d}y\,\exp[-(\mu^2/\beta^2)a^D e^y + za^D y + ly]|_{l=0}}{\int_{-\infty}^{\infty}\mathrm{d}y\,\exp[-(\mu^2/\beta^2)a^D e^y + za^D y]}\,, \tag{17}$$

and after a little algebra we find [12]

$$B_1 = \ln(\beta^2 z/\mu^2) - 1/2za^D + g^{(1)}(za^D)\,, \tag{18a}$$

$$B_s = (-)^s[(s-1)/2(za^D)^s]\,[1 + 2za^D/(s-1)] + g^{(s)}(za^D)\,, \quad s > 1\,. \tag{18b}$$

The functions $g^{(s)}(x)$ are analytic in $x = 0$. Let us remark that the most divergent term for small za^D is of the same form as the connected vertices we found for the infinite barrier problem [eq. (8)].

Again it is easy to find the general form of the kth order in the strong coupling expansion of the Green's functions. The only change is in rule (ii), due to the more complex expression of the B_s now we have terms with all negative powers of (za^D) from 0 to $-(2k+1)$.

Because of this change, we find that apart from terms of the form obtained in eqs. (11) and (15) there appears a series in negative powers of e. Instead of eq. (11) we now have

$$\langle\phi\rangle = \ln(\beta^2 z/\mu^2) + e x^\delta S_0(x) + S_1(x) + e^{-1}x^{-\delta}S_2(x) + \dots\,, \tag{19}$$

where the $S_l(x)$ are power series of x.

For $D = 1$ this expression simplifies in the limit $z \to 0$ yielding only two terms (we restore the $\hbar$)

$$\langle\phi\rangle = \ln(\beta^2 z/\mu^2) + \hbar^{2/3}C_0(\beta^2/z)^{1/3}\,. \tag{20}$$

In two dimensions and with z fixed eq. (19) gives a series in powers of $1/\beta$

$$\langle\phi\rangle = \ln(\beta^2 z/\mu^2) + \hbar^{1/2}\sum_{l=0}^{\infty}(\hbar^{1/2}\beta)^{-l}C_l\,. \tag{21}$$

As before the C_l are numbers obtained by extrapolating the series $x^{\delta(1-l)}S_l(x)$ for $x \to \infty$.

Both terms in eq. (20) give a divergent contribution. The last one takes into account that the potential well widens as z decreases, this effect was also present in the infinite barrier problem. The first term is classical and it is the position of the minimum of the potential $V(\phi) = (\mu^2/\beta^2)\exp(\phi) - z\phi$.

In eq. (21) we see that the divergence coming from the classical term is also present in two dimensions. This suggests an effective potential of the form ($z = 0$)

$$V(\phi) = (M_{\mathrm{R}}^2/\tilde{\beta}^2)\exp(\tilde{\beta}\phi)\,, \tag{22}$$

where M_{R} is the renormalized mass (in two dimensions the theory is super-renormalizable) and $\tilde{\beta}$ is determined by the quantum corrections in $\langle\phi\rangle$.

This form of the potential is in agreement with the one given by D'Hoker and Jackiw [5]. They checked it in the loop expansion while we worked in a $1/\beta$ expansion with μ/β fixed.

The energy difference between any excited level and the ground state is given by a series of the form

$$(\Delta E_n)^2 = M_0^2 x^{2/D+2}\sum_{l=0}^{\infty}F_l(x)x^{-l\delta}e^{-l}\,,$$
$$\Delta E_{\mathrm{M}} = M_0\gamma_n\,, \tag{23}$$

and again the system has a continuous energy spectrum as $z \to 0$.

The strong coupling expansion has proved to be a useful method to show some properties of the Liouville model. Here we have considered only the whole two-dimensional space, however we think that this technique could also be applied to more realistic topologies for the string model.

We are glad to thank Professor R. Jackiw for useful information and discussions about the Liouville model while he was at the Centro Atómico Bariloche, Argentina, under a Multinational Program of the Organization of the American States. One of us (GA) would like to acknowledge discussions with W. Bardeen and C. Zachos and thanks for the hospitality received at Fermilab.

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