Definition of Nuclear Potentials from Double Dispersion Relations in Field Theory and Potential Scattering.

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Summary. — A method to describe the scattering of nucleons through a nuclear potential consistent with the results of field theory is presented. This potential results as a sum of ten potentials, five direct and five of the exchange type, each of them given by a superposition of Yukawa potentials. The weight functions for these superpositions are then shown to be obtainable from field theory through the use of double dispersion relations for the nucleon-nucleon system and of unitarity in potential scattering.

1. – Introduction.

It has been shown by N. KHURI (1) that the scattering amplitude obtained in potential theory from a superposition of Yukawa potentials, satisfies a special type of dispersion relation at fixed momentum transfer; this dispersion relation expresses the scattering amplitude directly related to the potential generating it. The properties of the scattering amplitude at fixed angle for superposition of Yukawa potentials of direct and exchange types has also been considered (2) and finally, BOWCOCK and MARTIN (3) were able to show for this type of potentials, that the scattering amplitude is analytic in the plane of the scattering angle inside the same ellipse obtained by LEHMANN (4) in field theory; they were also able to show that the scattering amplitudes satisfy Mandelstam representation (5) with known branch lines, and poles.

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(1) N. KHURI: *Phys. Rev.*, **107**, 1148 (1957).

(2) J. BOWCOCK and D. WALECKA: *Nucl. Phys.*, **12**, 371 (1959).

(3) J. BOWCOCK and A. MARTIN: *Nuovo Cimento*, **14**, 516 (1959).

(4) H. LEHMANN: *Nuovo Cimento*, **10**, 578 (1958).

(5) S. MANDELSTAM: *Phys. Rev.*, **112**, 1344 (1958).

CHARAP and FUBINI [6] have shown that one may define a potential for the scattering of scalar particles consistent with the results of field theory for bosons interacting through a charged pseudo-scalar meson field [7].

In the present paper we want to show that the same results are valid when they are generalized to take into account the fermionic nature of the nucleons. Then, to reproduce the results of field theory, one is led to the introduction of ten potentials, five of the direct and five of the exchange type. They are not independent, but related two by two by symmetry relations. These results are shown using Mandelstam representation for nucleon-nucleon scattering in field theory, as introduced by AMATI, LEADER and VITALE [8].

Next, we want to show how to construct these potentials from the results of field theory; to this end it is shown that the problem may be reduced to be formally identical with the already solved problem of superposition of just two potentials, one direct and one of the exchange type. Finally, a method for explicit calculation of the potentials is proposed.

2. – Scattering matrices in potential and field theories.

We will consider the elastic scattering of two nucleons through a charged pseudoscalar meson field. To fix ideas we may consider these nucleons to be a neutron (n) and a proton (p).

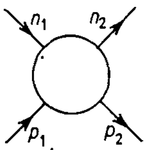

Fig. 1.

They have four-momenta p_1 and n_1 and p_2 and n_2 in the initial and final states, respectively. Energy-momentum conservation therefore reads

$$n_1+p_1 = n_2+p_2 \, .$$

This leaves three independent four-vectors to describe the process, out of which, only two independent scalars can be formed, because the particles are in the mass shell. It is convenient to define as usual (*)

$$(2.1)\qquad \begin{cases} s = -(p_1+n_1)^2 = -(p_2+n_2)^2 \, , \\ t = -(n_1-n_2)^2 = -(p_2-p_1)^2 \, , \\ u = -(n_1-p_2)^2 = -(n_2-p_1)^2 \, , \end{cases}$$

$$s+t+u = 4m^2 \, .$$

([6]) J. M. CHARAP and S. P. FUBINI: *Nuovo Cimento*, **14**, 540 (1959); **15**, 73 (1960). To be referred to as Ch. F. (I) and Ch. F. (II), respectively.

([7]) J. M. CHARAP and M. J. TAUSNER: *Nuovo Cimento*, **18**, 316 (1960).

([8]) D. AMATI, E. LEADER and B. VITALE: *Nuovo Cimento*, **17**, 68 (1960). To be referred to as A.L.V. (I).

(*) We are working with $A^2=\boldsymbol{A}^2-A_0^2$.

In the center of mass system of the two nucleons, with p and n as initial states, we have

$$(2.2)\qquad \begin{cases} s = \quad 4(m^2+\eta^2) = 4E^2\,, \\ t = -2\eta^2(1-\cos\theta)\,, \\ u = -2\eta^2(1+\cos\theta)\,, \end{cases}$$

where η is the modulus of the center of mass momentum of either particle, and θ is the scattering angle. Thus, s is the square of the c.m. energy, while t is minus the square of the momentum transfer.

It will be also convenient to introduce the four-vectors

$$(2.3)\qquad \begin{cases} P = \frac{1}{2}(p_1+p_2)\,, \\ N = \frac{1}{2}(n_1+n_2)\,, \\ \Delta = (n_2-n_1) = (p_1-p_2) \end{cases}$$

and the vector $\boldsymbol{n} = \boldsymbol{n}_1 \times \boldsymbol{n}_2$.

In the following, we will use the potential theory scattering amplitude T, defined through its relation to the scattering s matrix by

$$(2.5)\qquad s_{fi} = \delta_{fi} - 2\pi i\,\delta(w_f - w_i)T_{fi}\,,$$

where f and i denote the final and initial states, characterized by the momenta and other quantum numbers necessary to specify them. w is the kinetic energy.

If the potential through which particles interact is given as a superposition of Yukawa potentials, including a direct and an exchange part, that is

$$(2.6)\qquad V(r) = \int_{\mu}^{\infty} \varrho_D(\sigma)\,\frac{\exp[-\sigma r]}{4\pi r}\,\mathrm{d}\sigma + \int_{\mu}^{\infty} \varrho_E(\sigma)\,\frac{\exp[-\sigma r]}{4\mathrm{p} r}\,\mathrm{d}\sigma P_M\,,$$

which has a Fourier transform

$$(2.7)\qquad V(t,u) = \int_{\mu^2}^{\infty} \frac{\varrho_D(\sigma)}{\sigma^2 - t}\,\mathrm{d}\sigma + \int_{\mu^2}^{\infty} \frac{\varrho_E(\sigma)}{\sigma^2 - u}\,\mathrm{d}\sigma\,,$$

it is known (3) that the scattering amplitude satisfies a Mandelstam representation of the form

$$(2.8)\qquad T(s,t,u) = \left\{\frac{1}{\pi}\int_{\mu^2}^{\infty}\mathrm{d}t'\,\frac{\varrho_D(t')}{t'-t} + \frac{1}{\pi^2}\int_{4m^2}^{\infty}\mathrm{d}s'\int_{4\mu^2}^{\infty}\mathrm{d}t'\,\frac{F_D(s't')}{(t'-t)(s'-s)}\right\} + \left\{\begin{matrix} D \to E \\ t \to u \end{matrix}\right\}.$$

When working in field theory, we will use Feynman's covariant amplitude G, defined through its relation to the field theoretical S matrix by

$$S_{fi} = \delta_{fi} - 2\pi i \delta^4(p_i - p_f)\frac{m^2}{E^2}G_{fi}\,. \tag{2.9}$$

In the physical region of the variables it coincides with Goldberger, Nambu and Oehme's causal scattering amplitude M, for which dispersion relations are written. A.L.V. define Feynman's covariant amplitude by

$$S_{fi} = \delta_{\;i} + i\delta^4(p_f - p_i)\frac{m^2}{4\pi E^2}F_{\;i}\,. \tag{2.10}$$

Therefore

$$G_{fi} = F_{fi}/(2\pi)^3\,. \tag{2.11}$$

These authors show that it is possible to write a Mandelstam representation for the nucleon-nucleon problem, when the causal amplitude is decomposed in an appropriate way in terms of invariant operators. This decomposition can in general be written as

$$M_{fi} = \sum_{j=1}^{5} o_j(stu)\langle f\,O_j\,i\rangle\,, \tag{2.12}$$

where $o_j(stu)$ are scalar functions of s, t and u and the O_j are matrices in spin and isotopic spin space. Of the large variety of choices of permissible decompositions, to write a Mandelstam representation they find it convenient to use the so called perturbative invariants P_j, which allow them to write such representation proving it up to fourth order perturbation theory (See ref. [7]). Then (2.12) will read

$$M_{fi} = \sum_{j=1}^{5} p_j(s, t, u)(P_j)_{fi} \tag{2.13}$$

with

$$\begin{cases} P_1 = 1^n\cdot 1^p\,, & P_2 = (i\gamma^n\cdot P 1^p + i\gamma^p\cdot N 1^n)\,, \\ P_3 = (i\gamma^n\cdot P)\,(i\gamma^p\cdot N)\,, \quad P_4 = \gamma^n\cdot\gamma^p\,, & P_5 = \gamma_5^n\gamma_5^p\,. \end{cases} \tag{2.14}$$

Even then, it is necessary to introduce a new set of invariants, with simple behaviour under crossing, by means of the new combination

$$M_{fi} = \sum_{j=1}^{5}\;\sum_{T=0}^{1} c_j^T(s, tu)\langle f|\Lambda_T C_j|i\rangle\,. \tag{2.15}$$

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Here, a decomposition in terms of the possible isospin states has been made, by means of the projection operators in isospace

$$\Lambda_1 = \tfrac{1}{4}(3 + \boldsymbol{\tau}_n \cdot \boldsymbol{\tau}_p)\ ; \qquad \Lambda_0 = \tfrac{1}{4}(1 - \boldsymbol{\tau}_n \cdot \boldsymbol{\tau}_p). \tag{2.16}$$

Then, imposing $C_j \underset{(t\leftrightarrow u)}{\rightarrow} (-1)^j C_j$, one gets easily for the crossing condition

$$c_j^T(s, t, u) = (-1)^{j+T} c_j^T(sut)\ . \tag{2.17}$$

Clearly, T indicates the possible isospin states 0 or 1, in channel n-n. The relation between the P_j and C_j is known.

The Mandelstam representation is then written for each c_j^T as:

$$(2.18)\quad \left\{\begin{aligned}
c_1^T(stu) &= \left\{\frac{1}{4}\left[p_1^T(stu) + \frac{6m^2 - s - 2t}{m}\, p_2^T(stu) + (5m^2 - 2s - 2t)\, p_3^T\,(stu) -\right.\right. \\
&\qquad\left.\left. - 4p_4^T(stu) + p_5^T(stu)\right]\right\} - (-1)^T\{t \leftrightarrow u\} + \frac{D_1^T(t-u)}{s_D - s}\,, \\
c_2^T(stu) &= \left\{\frac{1}{4}\,[p_1^T(stu) - 2mp_2^T - (3m^2 - s)p_3^T + 2p_4^T - p_5^T]\right\} + \\
&\qquad + (-1)^T\{t \leftrightarrow u\} + \frac{D_2^T}{s_D - s}\,, \\
c_3^T(stu) &= \left\{\frac{1}{4}\,[p_1^T + 2mp_2^T + (m^2 - s)p_3^T - 2p_4^T - p_5^T]\right\} - \\
&\qquad - (-1)^T\{t \leftrightarrow u\} + \frac{D_3^T}{s_D - s}\,, \\
c_4^T(stu) &= \left\{\frac{1}{4}\left[p_1^T + \frac{(2m^2 - s)}{m}\, p_2^T + m^2 p_3^T + p_5^T\right]\right\} + (-1)^T\{t \leftrightarrow u\} + \frac{D_4^T}{s_D - s}\,, \\
c_5^T(stu) &= \left\{\frac{1}{4}\left[p_1^T + \frac{2m^2 + s + 2t}{m}\, p_2^T - (3m^2 - 2s - 2t)p_3^T + 4p_4^T + p_5^T\right]\right\} - \\
&\qquad - (-1)^T\{t \leftrightarrow u\} + \frac{D_5^T}{s_D - s}\,.
\end{aligned}\right.$$

with

$$\left\{\begin{aligned} p^1 &= 3p^+ + 2p^-, \\ p^0 &= 3p^+ - 6p^-, \end{aligned}\right.$$

and

$$p_j^{\pm}(stu) = \begin{pmatrix} 0 \\ -g^2\,\dfrac{\delta_{j5}}{2(\mu^2 - t)} \end{pmatrix} + \left[\frac{1}{\pi^2}\int_{4\mu^2}^{\infty} \mathrm{d}t' \int_{4m^2}^{\infty} \mathrm{d}s'\, \frac{F_{23}^j(s't')}{(s' - s)(t' - t)}\right] \mp (-1)^j[s \leftrightarrow u].$$

The signs $+$, $-$ as superscripts indicate the isotopic states in the n-$\bar{\text{n}}$ channel.

3. – The adiabatic limit of ALV's dispersion relations.

In the following we will consider that the energy of the incoming particles is such that $\eta^2 \ll m^2$. This we will call the adiabatic condition for the process. This must not be confused with the static condition, by which one considers $\mu^2 \ll m^2$. m is the nucleon mass; μ the meson mass.

To impose the adiabatic limit will mean to consider $\eta^2/m^2 \to 0$.

Dropping the contributions from the bound state to the causal amplitude, as it will not be of interest to us for the moment, we can write eqs. (2.15) in the form:

$$M_{fi} = \sum_{T=0}^{1} \sum_{j=1}^{5} [N_{j\alpha}(stu) C_\alpha p_T^j(stu) \Lambda_T + (-1)^T N_{j\alpha}^{(s,ut)} C_\alpha p_T^j(sut) \Lambda_T], \tag{3.1}$$

where sum over α is implied.

The scalar coefficients are given by comparison of (2.18) and (3.1) as:

$$\left\{\begin{array}{lll}
N_{1i} = \frac{1}{4}, \quad \text{for} \quad i = 1 \text{ to } 5, & & \\[2ex]
N_{21} = \dfrac{6m^2 - s - 2t}{2m}, & N_{22} = -\dfrac{m}{2}, & N_{23} = \dfrac{m}{2}, \\[2ex]
 & N_{24} = \dfrac{2m^2 - s}{4m}, & N_{25} = \dfrac{2m^2 + s + 2t}{4m}, \\[3ex]
N_{31} = \dfrac{5m^2 - 2s - 2t}{4}, & N_{32} = -\dfrac{3m^2 - s}{4}, & N_{33} = \dfrac{m^2 - s}{4}, \\[2ex]
 & N_{34} = \dfrac{m^2}{4}, & N_{35} = -\dfrac{3m^2 - 2s - 2t}{4}, \\[3ex]
N_{41} = -1, & N_{42} = \frac{1}{2}, & N_{43} = -\frac{1}{2}, \\
 & N_{44} = 0, & N_{45} = 1, \\[3ex]
N_{51} = \frac{1}{4}, & N_{52} = -\frac{1}{4}, & N_{53} = -\frac{1}{4}, \\
 & N_{54} = \frac{1}{4}, & N_{55} = \frac{1}{4}.
\end{array}\right. \tag{3.2}$$

We have only written the expression for $N_{ij}(stu)$, because the $N_j(sut)$'s are to be obtained from these by interchanging $t \leftrightarrow u$. Now, we want to consider physical processes, and for them $\cos\theta$ is in the region $(-1, 1)$. Therefore $|t|$ and $|u|$ are of the order of $2\eta^2$ (surely $|t|$, $|u| \leqslant 4\eta^2$), while $s \simeq 4m^2$. Then,

$$N_{ij}(stu) \simeq N_{ij}(sut) = N_{ij}(s) \tag{3.3}$$

because of the condition on the invariants $C_j \xrightarrow[(t\leftrightarrow u)]{} (-1)^j C_i$. Therefore, in the adiabatic limit we can write formula (3.1) as

$$M_{fi} = \sum_{j,T} N_{j\alpha} C_\alpha [p_j^T(s, tu) + (-1)^T p_j^T(s, ut)] \Lambda_T . \tag{3.4}$$

On the other hand, we have by eqs. (2.18), an expression for $p_j^{\pm}$ that can be expressed as

$$p_j^{\pm}(stu) = I_{23}^{j\pm}(s, t) \mp I_{12}^{j\pm}(u, t) \tag{3.5}$$

disregarding the pole for $j = 5$.

Then

$$\begin{aligned} p_j^1(stu) &= [3I_{23}^{j+}(s, t) + 2I_{23}^{j-}(s, t)] + [2I_{12}^{j-}(u, t) - 3I_{12}^{j+}(u, t)] = \\ &= [I_{23}^{j1}(s, t) + I_{12}^{j1}(u, t)] , \end{aligned} \tag{3.6}$$

$$\begin{aligned} p_j^0(s, tu) &= [3I_{23}^{j+}(s, t) - 6I_{23}^{j-}(s, t)] + [-3I_{12}^{j+}(u, t) - 6I_{12}^{j-}(u, t)] = \\ &= I_{23}^{j0}(s, t) + I_{12}^{j0}(u, t) . \end{aligned} \tag{3.6a}$$

Then we can write

$$p_j^T(stu) = \frac{1}{\pi^2}\int_{4\mu^2}^{\infty} dt' \int_{4m^2}^{\infty} ds' \frac{F_{23}^{jT}(s't')}{(s'-s)(t'-t)} + \frac{1}{\pi^2}\int_{4\mu^2}^{\infty} dt' \int_{4m^2}^{\infty} du' \frac{F_{12}^{jT}(u't')}{(u'-u)(t'-t)} , \tag{3.8}$$

$$p_i^T(sut) = \frac{1}{\pi^2}\int_{4\mu^2}^{\infty} du' \int_{4m^2}^{\infty} ds' \frac{F_{13}^{jT}(s'u')}{(s'-s)(u'-u)} + \frac{1}{\pi^2}\int_{4\mu^2}^{\infty} du' \int_{4m^2}^{\infty} dt' \frac{F_{12}^{jT}(u't')}{(u'-u)(t'-t)} . \tag{3.9}$$

Calling

$$G_j^T(stu) = \frac{1}{(2\pi)^3} N_{j\alpha} C_\alpha p_j^T(stu) , \tag{3.10}$$

$$G_j^T(sut) = \frac{1}{(2\pi)^3} N_{j\alpha} C_\alpha p_j^T(sut) , \tag{3.10a}$$

we obtain

$$G_j = G_j^1(stu)\Lambda_1 + G_j^0(stu)\Lambda_0 - G_j^1(sut)\Lambda_1 + G_j^0(sut)\Lambda_0 , \tag{3.11}$$

where each G_j satisfies a Mandelstam representation apart from known scalars factors.

Notice that in the adiabatic limit, where we said $|u|$ was much smaller than $4m^2$, the last term in expression (3.8), becomes practically independent of u so that it can be written

$$\int_{4\mu^2}^{\infty} \frac{\mathrm{d}t'\varphi_2^{jT}(t')}{t'-t} . \tag{3.12}$$

Similarly, considering $|t| \ll 4m^2$, one gets for the last term in eq. (3.9) an expression independent of t

$$\int_{4\mu^2}^{\infty} \frac{\mathrm{d}u'\varphi_1^{jT}(u')}{u'-u} . \tag{3.13}$$

If one replaces eqs. (3.12) and (3.13) into their places in eqs. (3.8) and (3.9), and the result in eqs. (3.10), one can easily get for the causal amplitude G_j of (3.11) an expression similar to the one given by (2.8) for the potential scattering matrix for a superposition of Yukawa potentials of direct and exchange type, if properly defined G_{D_j} and G_{E_j} are introduced.

Therefore, potentials V_D and V_E resulting from superpositions of Yukawa potentials could be introduced if one imposes to the results got from both theories to coincide at sufficiently low energies. This will guarantee the coincidence of the results for a certain range of energies. This is more easily done, however, if one works in one-dimensional, once substracted dispersion relations. This procedure allows as well to introduce a simple recipe for the construction of the various potentials.

4. – The identification of potential scattering and field-theoretic results.

It will be shown that this identification is possible only for energies below the meson production threshold.

Let us write for eq. (3.10) a one dimensional dispersion relation at constant t. We get, aside from scalar factors

$$G^{jT}(s, u)_t = \frac{1}{\pi}\int_{4m^2}^{\infty} \mathrm{d}s' \, \frac{\operatorname{Im} G_3^{jT}(s', t)}{s'-s} + \frac{1}{\pi}\int_{4m^2}^{\infty} \mathrm{d}u' \, \frac{\operatorname{Im} G_1^{jT}(u', t)}{u'-u} . \tag{4.1}$$

with

$$(4.1a)\qquad \operatorname{Im} G_3^{jT}(s', t) = \frac{1}{\pi}\int\limits_{4\mu^2}^{\infty} \frac{F_{23}^{jT}(s', t')}{t'-t}\,\mathrm{d}t' \quad \text{and} \quad \operatorname{Im} G_1^{jT}(u', t) = \frac{1}{\pi}\int\limits_{4\mu^2}^{\infty} \frac{F_{12}^{jT}(u', t')}{t'-t}\,\mathrm{d}t',$$

where by the reasons mentioned before, $\int\limits_{4m^2}^{\infty}\mathrm{d}u'(\;)/(u'-u)$ is really independent of u.

Similarly we get for (3.10a), at constant u

$$(4.2)\qquad G^{jT}(s, t))_u = \frac{1}{\pi}\int\limits_{4m^2}^{\infty}\mathrm{d}s'\,\frac{\operatorname{Im} G_3^{jT}(s', u)}{s'-s} + \frac{1}{\pi}\int\limits_{4m^2}^{\infty}\mathrm{d}t'\,\frac{\operatorname{Im} G_2^{jT}(t', u)}{t'-t},$$

with

$$(4.3)\qquad \operatorname{Im} G_3^{jT}(s', u) = \frac{1}{\pi}\int\limits_{4\mu^2}^{\infty}\mathrm{d}u'\,\frac{F_{13}^{jT}(s', u')}{u'-u} \quad \text{and} \quad \operatorname{Im} G_2^{jT}(t', u) = \frac{1}{\pi}\int\limits_{4\mu^2}^{\infty}\mathrm{d}u'\,\frac{F_{12}^{jT}(u', t')}{u'-u},$$

and $\int\limits_{4m^2}^{\infty}\mathrm{d}t'(\;)/(t'-t)$, independent of t.

Let us introduce

$$(4.4)\qquad G_{D_j}(s, u)_t = \tfrac{1}{2}[G_j^1(s, u)_t + G_j^0(s, u)_t]\,; \qquad G_{E_j}(s, u)_t = \tfrac{1}{2}[G_j^1(s, u)_t - G_j^0(s, u)_t]$$

and similar relations for $G_{D_j}(s, t)_u$ and $G_{E_j}(s, t)_u$.

Then (3.11) can be written

$$(4.5)\qquad G_j = [G_{D_j}(s, u)_t - G_{E_j}(s, t)_u] + [G_{E_j}(s, u)_t - G_{D_j}(s, t)_u]P^\tau,$$

where P is the familiar isotopic spin operator: $P^\tau = \tfrac{1}{2}(1+\boldsymbol{\tau}^n\cdot\boldsymbol{\tau}_p)$ with eigenvalues 1 and -1, for triplet and singlet states in isospace, respectively.

Let us try to compare this expression with a similar one in potential scattering. We will get this one by using a method similar to the method of Ch.F.II, adapted for fermions. Consider first a system of distinguishable particles, with two charge states.

From eqs. (2.6) and (2.8), we know that for a potential made up of a superposition of direct and exchange Yukawa potentials, both direct and exchange parts satisfy a Mandelstam representation.

Let us now consider a potential of the form $V(r) = V_D + V_E P^\tau$. It is known that for a potential of this type, it is possible to separate Schrödinger's equation into two independent ones, by means of the projection operator (2.16)

$$(4.6)\qquad \Lambda_1 = \tfrac{1}{4}(3 + \boldsymbol{\tau}_n\cdot\boldsymbol{\tau}_p) = \tfrac{1}{2}(1 + P^\tau)\,; \qquad \Lambda_0 = \tfrac{1}{2}(1 - P^\tau).$$

Each of the uncoupled Schrödinger's equations is written in terms of the potentials

$$V_{\binom{1}{0}} = V_D(\pm)V_E\,. \tag{4.7}$$

The resulting total scattering amplitude is then the sum of the amplitudes deriving from each potential

$$T(s, tu) = T_1(s, tu)\Lambda_1 + T_0(s, tu)\Lambda_0\,, \tag{4.8}$$

which, by virtue of (4.6) and (4.7) and, introducing

$$T_D(s, tu) = \tfrac{1}{2}(T_1 + T_0)\,, \qquad T_E(s, tu) = \tfrac{1}{2}(T_1 - T_0) \tag{4.10}$$

can be written

$$T(s, tu) = T_D(stu) + T_E(stu)\, P^\tau. \tag{4.11}$$

Each of the amplitudes T_+, T_-, T_D, T_E, satisfy a Mandelstam representation, if they are generated by potentials made up as superpositions of Yukawa potentials.

If we now consider that the system of particles is that of two undistinguishable fermions, the total scattering amplitude will be given by

$$\begin{aligned} T = T(s, tu) - T(s, ut) = T_1(s, tu)\Lambda_1 + T_0(s, tu)\Lambda_0 - \\ - [T_1(s, ut)\Lambda_1 - T_0(s, ut)\Lambda_0]\,, \end{aligned} \tag{4.12}$$

that is

$$T = [T_D(s, tu) - T_E(s, ut)] + [T_E(s, tu) - T_D(s, ut)]P^\tau \quad (^*). \tag{4.13}$$

Now, if we write for $T_{\binom{D}{E}}(s, tu)$ a dispersion relation at fixed momentum transfer, and for $T_{\binom{D}{E}}(s, ut)$ a dispersion relation at fixed u, we get an expression exactly analogous to (4.5), namely

$$T = [T_D(s, u)_t - T_E(s, t)_u] + [T_E(s, u)_t - T_D(s, t)_u]P^\tau\,, \tag{4.14}$$

(*) Of course from here we get for proton-neutron scattering: $T_{np} = T_D(stu) - T_E(sut)$ while for proton-proton scattering: $T_{pp} = T_{np}(stu) - T_{np}(sut)$.

where

$$(4.15)\qquad \begin{cases} T_D(s,u)_t = \dfrac{1}{\pi}\displaystyle\int_{4m^2}^{\infty} \mathrm{d}s' \,\dfrac{\operatorname{Im} T_D(s',t)}{s'-s} + V_D(t)\,, \\[2ex] T_E(s,t)_u = \dfrac{1}{\pi}\displaystyle\int_{4m^2}^{\infty} \mathrm{d}s' \,\dfrac{\operatorname{Im} T_E(s',u)}{s'-s} + V_E(u)\,, \end{cases}$$

with

$$(4.16)\qquad \operatorname{Im} T_D(s't) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} \mathrm{d}t' \,\frac{F_D(s't')}{t'-t}\,; \qquad \operatorname{Im} T_E(s',u) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} \mathrm{d}u' \,\frac{F_E(s'u')}{u'-u}\,,$$

from which, performing one subtraction we get

$$(4.17)\qquad T_D(\eta^2,u)_t = T_D(0,t) + \frac{\eta^2}{\pi}\int_0^{\infty} \mathrm{d}\eta'^2 \,\frac{\operatorname{Im} T_D(\eta'^2,t)}{\eta'^2(\eta'^2-\eta^2-i\varepsilon)}\,,$$

$$(4.18)\qquad T_E(\eta^2,t)_u = T_E(0,u) + \frac{\eta^2}{\pi}\int_0^{\infty} \mathrm{d}\eta'^2 \,\frac{\operatorname{Im} T_E(\eta'^2,u)}{\eta'^2(\eta'^2-\eta^2-i\varepsilon)}\,.$$

On the other hand, we know that for a potential superposition of Yukawa potentials Khuri has shown that it is possible to write a dispersion relation at fixed-momentum transfer in terms of the potentials generating the scattering amplitude. This form of dispersion relation is just the first of eqs. (4.15); from this and eq. (4.17) one gets

$$(4.19)\qquad V_D(t) = T_D(0,t) - \frac{1}{\pi}\int_0^{\infty} \frac{\operatorname{Im} T_D(\eta'^2,t)}{\eta'^2}\,\mathrm{d}\eta'^2\,.$$

A similar relation is valid for the exchange part

$$(4.20)\qquad V_E(u) = T_E(0,u) - \frac{1}{\pi}\int_0^{\infty} \frac{\operatorname{Im} T_E(\eta'^2,u)}{\eta'^2}\,\mathrm{d}\eta'^2\,.$$

In field theory, as results from eqs. (4.1) and (4.1*a*), each $p_{\binom{D}{E}_j}(s,u)_t$ satisfy a dispersion relation of the form

$$(4.21)\qquad p_{\binom{D}{E}}(s,u)_t = \frac{1}{\pi}\int_{4m^2}^{\infty} \mathrm{d}s' \,\frac{\operatorname{Im} p_{3\binom{D}{E}_j}(s',t)}{s'-s} + \frac{1}{\pi}\int_{4m^2}^{\infty} \mathrm{d}u' \,\frac{\operatorname{Im} p_{1\binom{D}{E}_j}(u',t)}{u'-u}\,,$$

performing one subtraction, we get

$$(4.22)\qquad p_{\binom{D}{E}_j}(\eta^2, u)_t = p_{\binom{D}{E}_j}(0)_t + \\ + \frac{1}{\pi}\int_0^\infty \mathrm{d}\eta'^2 \frac{\eta^2 \operatorname{Im} p_{3\binom{D}{E}_j}(\eta'^2, t)}{\eta'^2(\eta'^2 - \eta^2 - i\varepsilon)} - \frac{\eta^2}{\pi}\int_{4m^2}^\infty \mathrm{d}u' \frac{\operatorname{Im} p_{1\binom{D}{E}_j}(u', t)}{(u'+t)(u'+\eta^2+t)}\,.$$

If we confine ourselves to the elastic region, we can express (4.22) as

$$(4.23)\qquad p_{\binom{D}{E}_j}(\eta^2, u)_t = p_{\binom{D}{E}_j}(0)_t + \frac{1}{\pi}\int_0^\infty \mathrm{d}\eta'^2 \frac{\eta^2 \operatorname{Im} p_{\binom{D}{E}_{j\mathrm{el}}}(\eta'^2, t)}{\eta'^2(\eta'^2 - \eta^2 - i\varepsilon)} + O(\eta^2/\eta^2_{\mathrm{thr}})\,,$$

where η_{thr} is the center of mass momentum corresponding to the threshold for pion production: $\eta^2_{\mathrm{thr}} = m\mu + \mu^2/4$. $O(\eta^2/\eta^2_{\mathrm{thr}})$ means terms of the order of $\eta^2/\eta^2_{\mathrm{thr}}$. (See Ch. F. I).

We also know that the elastic part of G satisfies the relation

$$(4.24)\qquad \operatorname{Im} G_{\binom{D}{E}_{j\mathrm{el}}} = \pi \sum_k \left[G^+_{D_j\mathrm{el}} * G_{\binom{D}{E}_{k\mathrm{el}}} + G^+_{E_j\mathrm{el}} * G_{\binom{D}{E}_{k\mathrm{el}}}\right],$$

valid in the elastic region (where $E = m$), resulting from the unitarity condition applied to the S matrix. Together with eqs. (4.23), they determine completely the field-theoretic scattering amplitudes in the elastic region, in terms of $p_{D_j}(0)_t$ and $p_{E_j}(0)_t$. Suppose we introduce a direct potential scattering amplitude T_{D_j} and an exchange one $T_{E_j}(\eta^2 tu)P^\tau$ for each term of the decomposition of G into G_{D_j} and $G_{E_j}P^\tau$, and define the set of functions $t_{\binom{D}{E}_j}(\eta^2, tu)$ through

$$(4.25)\qquad T_{\binom{D}{E}_j}(\eta^2 tu) = t_{\binom{D}{E}_j} P_j\,,$$

P_j being the perturbative invariants.

Then, the scattering matrix s, associated with these potential scattering amplitudes, will satisfy unitarity imposing on them the relation

$$(4.26)\qquad \operatorname{Im} T_{\binom{D}{E}_j} = \pi \sum_k \left[T^+_{D_j} * T_{\binom{D}{E}_k} + T^+_{E_j} * T_{\binom{D}{E}_k}\right].$$

Again, this is a set of ten coupled, non linear integral equations. Now, with each of the scattering amplitudes, we can associate a potential $V_{D_j}(t)$ and $V_{E_j}(t)P^\tau$. As these are completely undetermined, we can define them as a superposition of Yukawa potentials for each j, direct or exchange, with spectral functions given by the relations

(4.27) $$V_{D_j}(t) = t_{D_j}(\eta^2, u)_t - \frac{1}{\pi}\int_0^\infty \frac{\operatorname{Im} t_{D_j}(\eta'^2, t)}{\eta'^2 - \eta^2 - i\varepsilon}\, d\eta'^2\,,$$

(4.28) $$V_{E_j}(t) = t_{E_j}(\eta^2, u)_t - \frac{1}{\pi}\int_0^\infty \frac{\operatorname{Im} t_{E_j}(\eta'^2, t)}{\eta'^2 - \eta^2 - i\varepsilon}\, d\eta'^2\,.$$

This means that the functions $t_{\binom{D}{E}_j}$ satisfy the relations

(4.29) $$t_{\binom{D}{E}_j}(\eta^2, u)_t = t_{\binom{D}{E}_j}(0)_t + \frac{\eta^2}{\pi}\int_0^\infty \frac{\operatorname{Im} t_{\binom{D}{E}_j}(\eta'^2, t)}{\eta'^2(\eta'^2 - \eta^2 - i\varepsilon)}\, d\eta'^2\,,$$

and therefore

(4.30) $$V_{\binom{D}{E}_j}(t) = t_{\binom{D}{E}_j}(0)_t - \frac{1}{\pi}\int_0^\infty \frac{\operatorname{Im} t_{\binom{D}{E}_j}(\eta'^2, t)}{\eta'^2}\, d\eta'^2\ (^*)\,.$$

If we do this, we see that eqs. (4.26) and (4.29) completely determine the potential scattering amplitudes $T_{D_j}(\eta^2, u)_t$ and $T_{E_j}(\eta^2, u)_t$. We see that eqs. (4.26) and (4.29) are formally identical to eqs. (4.23), (4.24) of field theory. Therefore, if we further impose to the potentials generating the scattering amplitudes to be of such a shape as to make these equal to the field-theoretic scattering amplitudes at zero energy, that is

(4.31) $$\begin{cases} T_D(0, u)_t = G_D(0, u)_t\,, \\ T_E(0, u)_t = G_E(0, u)_t\,, \end{cases}$$

which of course means,

(4.32) $$\begin{cases} t_D(0, u)_t = p_D(0, u)_t\,, \\ t_E(0, u)_t = p_E(0, u)_t\,, \end{cases}$$

then, one gets

(4.33) $$[p_{\binom{D}{E}_j}(s, u)_t]_{\text{el}} = t_{\binom{D}{E}_j}(s, u)_t + O(\eta^2/\eta^2_{\text{thr}})\,.$$

(*) Similar relations are valid for $t_{\binom{D}{E}_j}(\eta^2, t)_u$ and $p_{\binom{D}{E}_j}(\eta^2, t)_u$, giving the potentials $V_{D_j}(u)$ and $V_{E_j}(u)$ and the amplitudes $T_{\binom{D}{E}_j}(\eta^2, t)_u$.

So, the results got from potentials of the form (4.30) will reproduce the field-theoretic results for energies sufficiently below threshold.

5. – Explicit construction of the potentials.

For the moment let us only consider the direct parts of the potentials and amplitudes, as exactly analogous relations hold for the exchange parts.

If the scattering amplitude at zero energy is known in field theory, as we get from formulae (4.30) and (4.32)

$$V_{D_j}(t) = p_{D_j}(0, u)_t - \frac{1}{\pi}\int\limits_0^\infty \frac{\operatorname{Im} t_{D_j}(\eta'^2, t)}{\eta'^2}\, d\eta'^2 , \tag{5.1}$$

we can construct V_{D_j} if $\operatorname{Im} t_{D_j}$ is known.

Now $p_{D_j}(0, u)_t$ can be got from the Mandelstam representation for $p_{D_j}(s, ut)$, if we write for it a one dimensional dispersion relation at fixed energy and then set the value of η^2 equal to zero. We get

$$p_{D_j}(0, u)_t = \frac{1}{\pi}\int\limits_{4\mu^2}^\infty dt' \frac{\operatorname{Im} p_{2D_j}(t', 0)}{t' - t} + \frac{1}{\pi}\int\limits_{4m^2}^\infty du' \frac{\operatorname{Im} p_{1D_j}(u', 0)}{u' + t} , \tag{5.2}$$

where, η^2 being equal to zero, we have set $u = -t$. In this equation $\operatorname{Im} p_2$ is the absorptive part of the causal amplitude for channel two, $(p_1 + \overline{p}_2 \to \to n_2 + \overline{n}_1)$, while $\operatorname{Im} p_1$ belongs to channel one $(p_1 + \overline{n}_2 \to \overline{n}_1 + p_2)$. As we are interested in values of $|t| \ll 4m^2$ and the absorptive parts are of the form

$$\operatorname{Im} p_{D(1,2)} = \sum F_n(x) \tag{5.3}$$

with

$$F_n(x) = 0 \quad \text{for} \quad x < (nM^2) , \tag{5.3}$$

M being the mass of particles in the intermediate state, the main contribution will come only from the first term, and from this one, only the pionic intermediate states will be important. Therefore, what is required to know from field theory, are the spectral functions for channel two for one, two, etc., pions in the intermediate states. This fact is related to our decomposition of the potentials in direct and exchange tipes. In the direct type of interaction, which groups together all graphs of the type of Fig. 1, channel one cannot give rise to pions in an intermediate state if we choose to de-

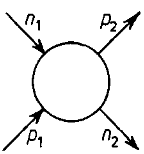

Fig. 2.

scribe the process by neutral pions; if we consider the exchange process, it will group all graphs of the type of Fig. 2; in a charged theory, graphs of type 1, will represent an even number of charged pions interchanged, while graphs of type 2, will represent an odd number of charged pions interchanged. Therefore in the direct part of the interaction we don't expect pions in the intermediate states for channel $(p\bar{n})$ and no pions in the intermediate state for the interaction $(p\bar{p})$ in the exchange parts.

To obtain the needed scattering amplitudes, one can make use, for instance of the results got by ALV.I, (their functions ϱ_j given in their formula (3.43)) which takes into account up to two pions in the intermediate states.

Once the scattering amplitudes are got from field theory, to solve the problem completely, one has to solve yet the system of eqs. (4.26), and (4.29). This would allow the knowledge of each $\operatorname{Im} t_{\binom{D}{E}_j}$ necessary to construct the potentials. We saw that as it stands, eq. (4.26) represents a system of ten coupled non linear integral equations. If we write the unitarity condition in terms of T_{1j}, T_{0j} instead of using the $T_{\binom{D}{E}_j}$'s then, Schrödinger's equations is separated in terms of the potentials (V_1, V_0) and the unitarity condition applies separately for the T_{1j}'s and the T_{0j}'s. Thus we get two systems of equations, consisting each in a set of five non linear coupled integral equations. Let us see if we can solve these systems. Suppose we want to obtain the observable cross-sections related to the scattering amplitudes. We know that the differential scattering cross-sections can be obtained from the potential scattering amplitude, that coincides with the field theoretic scattering amplitude at energies sufficiently below threshold, if we compute, for instance for $T=1$:

$$|(\bar{u}_{p_2}\bar{u}_{n_2}|T_1|u_{p_1}u_{n_1})|^2 , \tag{5.5}$$

where u_{p_1}, u_{n_1} and $\bar{u}_{p_2}$, $\bar{u}_{n_2}$ are the spinors for the incoming and outgoing particles respectively. The perturbative invariants are of the form $P_j = P_j^n P_j^p$, where the n and p operators are to be saturated between $\bar{u}_{n_2}$ and u_{n_1}, and $\bar{u}_{p_2}$ and u_{p_1}, respectively.

In the Appendix, it is shown that if we compute expression (5.5), and then order the result, we obtain an expression of the form

$$\begin{aligned}(5.6)\quad |T|^2 = {} & O(m^4)[t_3^+t_3] + O(m^3)[t_2^+t_3 + t_2^+t_3] + O(m^2)[t_2^+t_2^+ + t_1^+t_3 + t_3^+t_1 + t_3^+t_4 + t_4^+t_3] + \\ & + O(m)[t_2^+t_4 + t_4^+t_2 + t_1^+t_2 + t_2^+t_1] + O(m^0)[t_4^+t_4 + t_1^+t_4 + t_4^+t_1 + t_1^+t_1] + O(1/m^4)[t_5^+t_5] .\end{aligned}$$

Suppose the scattering amplitudes t_j are functions only of η^2 and θ, with a given dependence on m independent of j, then we can consider the nonrelativistic limit of this expression. We see that to a first approximation, the only term contributing to the cross-section is proportional to $[t_3^+ t_3]$. Therefore, if we set Schrödinger's equation in terms of the potential v_3 alone, we will get from it a scattering amplitude that will reproduce the results got in a first non relativistic approximation. But this scattering amplitude will then satisfy a unitarity relation of the form

$$P_3 \operatorname{Im} T_3(\eta^2, t) = \pi T_3^+ * T_3 \tag{5.7}$$

because the perturbative invariants are hermitian. Therefore eq. (5.7) together with (4.29) and (4.32) completely solves the problem in this first approximation, because, we already know how to solve a system like this unambiguously, from the results of Ch.F.I (Section **5**).

If we go to the next approximation, we see that now we have a mixture of potentials three and two acting. The unitarity relation that we now obtain is

$$P_2 \operatorname{Im} t_2(\eta^2, t) = \pi[T_2^+ * T_2 + T_2^+ * T_3], \tag{5.8}$$

where now T_3 and $\operatorname{Im} T_3$ are already known from the first approximation. Therefore, eqs. (5.8) and (4.29) and (4.32) can again be solved by perturbative methods.

The next step would be to consider terms of order m^2/m^4, but we see from (5.6) that the cross-section obtained to this order includes the contribution from potential v_1, but this is not all, for we also find a term coming from the mixture of potentials three and four. This mixture of potentials goes all the way down to the terms of order $1/m^2m^4$; this means that to solve completely the problem we should try to device a method for solving the system of three simultaneous equations, coupled and of integral type, coming from the scattering amplitudes corresponding to potentials V_1, V_4 and V_5. We don't know of any such method. Anyhow it is reasonable to expect that the results got up to second order in our nonrelativistic approximation, could describe accurately the nucleon nucleon interaction for a reasonable range of energies.

To identify the potentials obtained by this method with the usual direct, tensor, spin-orbit, spin-spin and spin orbit-spin orbit potentials, one must compute the matrix elements of the perturbative invariants in terms of positive energy spinors.

When this is done, one finds that, the principal contributions coming from potentials associated with P_2 and P_3, the interaction should be accurately described by central, spin-orbit and s.o.-s.o. parts, with a contribution from the tensor potential of the same order of magnitude in powers of $1/m$, than the s.o.-s.o. type. This does not agree with the results got phenomenologically by GAMMEL and THALER [9], for instance, who fit the scattering data by means of a central plus tensor plus spin orbit potential, without considering the possibility of a spin orbit-spin orbit term. This matter undoubtedly requires further investigation [*]. A phenomenological potential containing all five possible types would be most desirable to this purpose.

The potentials that can be obtained in the way proposed, would be correct only outside a region of radius $1/3\mu$, as up to two pions in the intermediate states would be considered. The inner region must be still considered phenomenologic.

On the other hand, if phenomenologic potentials of the described type were obtained, phenomenologic spectral functions for the nucleon-antinucleon channel two could be obtained, thus allowing a check on the theory by using Chew's method [10] for the determination of bound states and phase shifts.

It is worth making a last remark about the method proposed. The potentials obtained present the same formal structure as Gupta's potentials [11], in the sense that they contain a part explicitely proportional to the isotopic spin operator.

At the same time, even though we have used a process of non relativistic limit in trying to uncouple the unitarity condition, they are free from the inconsistencies of the static approximation, as defined in Section **3**, pointed out by Gupta in thes ame paper and by Ch.F.I for scalar particles interacting through a neutral meson field, because these inconsistencies derive from the application of the limit $1/m \to 0$ to the terms of a field-theoretic perturbative expansion of the interaction.

About the result obtained, notice that a spin orbit-spin orbit potentila is as able to produce the S and D waves mixture wanted in the ground state of the deuteron, as a tensor one.

* * *

When this paper was being completed, I learnt from Dr. AMATI that he and his collaborators, were working at this problem, trying to apply their results from A.LV.I, to the determination of the potentials. I ignore if our methods agree.

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(9) J. L. GAMMEL and R. M. THALER: *Phys. Rev.*, **103**, 1874 (1956); **107**, 291, 1337 (1957).

(*) This is done in a forthcoming paper.

(10) G. F. CHEW: *Dispersion relations and unitarity as the basis for a dynamical theory of strong interactions*, University of California U.C.R.L.-9289 (1960).

(11) S. GUPTA: *Phys. Rev.*, **117**, 1146 (1960).

APPENDIX

a) The evaluation of the cross-sections.

To compute eq. (5.5), one must consider the contribution of terms of the form

$$(\text{A.1})\qquad t_j t_k^+(\bar{u}_{n_2} P_j^n u_{n_1})(u_{n_1}^+ P_k^{n+} \bar{u}_{n_2}^+)(\bar{u}_{p_2} P_j^p u_{p_1})(u_{p_1}^+ P_k^{p+} \bar{u}_{p_2}^+)\,,$$

where sum over the final spin states and average over the initial spin states, both for positive energy, is to be understood. Therefore, we must consider all possible terms of the form

$$(\text{A.2})\qquad t_j t_k^+ \operatorname{Tr}(P_j^n \Lambda_{+n_1} P_k^n \Lambda_{+n_2}) \operatorname{Tr}(P_j^p \Lambda_{+p_1} P_k^p \Lambda_{+p_2}) = t_j t_k^+ \pi_{jk}\,,$$

where $P_k^{(n,p)'} = \gamma_0 P_i^{(n,p)} \gamma_0$, and $\Lambda_{+(n,p)}$ represent the positive energy projection operators

$$(\text{A.3})\qquad \begin{cases} \Lambda_{+n} = \dfrac{\hat{n} + m}{2m}\,, \\[2ex] \Lambda_{+p} = \dfrac{\hat{p} + m}{2m}\,. \end{cases}$$

One easily gets

$$(\text{A.4}a)\qquad \operatorname{Tr}\left(1^n \frac{\hat{n}_1 + m}{2m} 1^n \frac{\hat{n}_2 + m}{2m}\right) = 1 + \frac{n_1 \cdot n_2}{m^2}\,,$$

$$(\text{A.4}b)\qquad \operatorname{Tr}\left(1^n \frac{\hat{n}_1 + m}{2m} \gamma^n \cdot P \frac{\hat{n}_2 + m}{2m}\right) = \frac{\operatorname{Tr}}{4m}(\gamma_\alpha^n \gamma_\mu^n 2N_\alpha P_\mu) = \frac{2N \cdot P}{m}\,,$$

$$(\text{A4.}c)\qquad \operatorname{Tr}\left(1^n \frac{\hat{n}_1 + m}{2m} \gamma_\varrho^n \frac{\hat{n}_2 + m}{2m}\right) = \frac{\operatorname{Tr}}{4m}(\gamma_\alpha^n \gamma_\varrho^n 2N_\alpha) = \frac{2}{m} g_{\alpha\varrho} N_\alpha\,,$$

$$(\text{A.4}d)\qquad \operatorname{Tr}(1^n \Lambda_{+n_1} \gamma_5^n \Lambda_{+n_2}) = \frac{\operatorname{Tr}}{4m^2}(\gamma_\alpha^n \gamma_5^n \gamma_\beta^n n_{1\alpha} n_{2\beta}) =$$

$$= -\frac{1}{4m^2}(-2i\sigma_\beta^n n_{2\beta} n_{10} + 2i\sigma_\alpha^n n_{1\alpha} n_{20}) = \frac{i}{2m^2} \operatorname{Tr} \boldsymbol{\sigma}^n \cdot \boldsymbol{\Delta} n_{10} = 0\,,$$

$$(\text{A.5}a)\qquad \operatorname{Tr}(\gamma_\alpha P_\alpha \Lambda_{+n_1} \gamma_\varrho P_\varrho \Lambda_{+n_2}) = \frac{1}{m^2}(g_{\alpha\sigma} g_{\beta\varrho} - g_{\beta\sigma} g_{\alpha\varrho} + g_{\varrho\sigma} g_{\alpha\beta}) P_\alpha n_{1\beta} P_\varrho n_{2\sigma} + P^2 =$$

$$= \frac{1}{m^2}(2P \cdot n_1 P \cdot n_2 - n_1 \cdot n_2 P^2) + P^2\,,$$

(A.5b) $$\mathrm{Tr}\,(\gamma_\alpha P_\alpha \Lambda_{+n_1}\gamma_\varrho\Lambda_{+n_2}) = \frac{1}{m^2}\,(P\cdot n_2 g_{\beta\varrho}n_{1\beta} + P\cdot n_1 g_{\beta\varrho}n_{2\beta}) + (m^2 - n_1\cdot n_2)g_{\beta\varrho}P_\beta,$$

(A.5c) $$\mathrm{Tr}\,(\gamma_\alpha P_\alpha \Lambda_{+n_1}\gamma_5\Lambda_{+n_2}) =$$
$$= \frac{2i}{4m}\,[\boldsymbol{\sigma}^n\cdot(\boldsymbol{n}_2 - \boldsymbol{n}_1)P_0 + \boldsymbol{\sigma}^n\cdot\boldsymbol{P}(n_1 - n_2)_0] = \mathrm{Tr}\,\frac{i}{2m}\,\boldsymbol{\sigma}\cdot\boldsymbol{\Delta}P_0 = 0\,,$$

(A.6a) $$\mathrm{Tr}\,(\gamma_\alpha\Lambda_{+n_1}\gamma_\varrho\Lambda_{+n_2}) = \frac{1}{m^2}\,[g_{\alpha\sigma}\,n_{2\sigma}g_{\beta\varrho}n_{1\beta} + g_{\varrho\sigma}g_{\alpha\beta}n_{2\sigma}n_{1\beta} + (m^2 - \boldsymbol{n}_1\cdot\boldsymbol{n}_2)g_{\alpha\varrho}]\,,$$

(A.6b) $$\mathrm{Tr}\,(\gamma_\alpha\,\Lambda_{+n_1}\gamma_5\Lambda_{+n_2}) =$$
$$= \frac{i}{2m}\,[\boldsymbol{\sigma}^n\cdot(\boldsymbol{n}_2 - \boldsymbol{n}_1)\delta_{\alpha 0} + (n_1 - n_2)_0\,\sigma_\alpha] = \mathrm{Tr}\,\frac{i}{2m}\,\boldsymbol{\sigma}^n\cdot\boldsymbol{\Delta}\delta_{\alpha 0} = 0\,,$$

(A.7) $$\mathrm{Tr}\,(\gamma_5\,\Lambda_{+n_1}\gamma_5\Lambda_{+n_2}) = \frac{\mathrm{Tr}}{4m^2}\,[\gamma_\beta\gamma_\sigma n_{1\beta}n_{2\sigma} - m^2] = \frac{1}{m^2}\,(n_1\cdot n_2 - m^2).$$

From here one can easily check that, the order of magnitude of the different factors in powers of m is:

$$\begin{array}{lllll}
\pi_{11}\to O(1)\,, & \pi_{22}\to O(m^2)\,, & \pi_{33}\to O(m^4)\,, & \pi_{44}\to O(1)\,, & \pi_{55}\to O(1/m^4),\\
\pi_{12}=\pi_{21}\to O(m)\,, & \pi_{23}\to O(m^3)\,, & \pi_{34}\to O(m^2)\,, & \pi_{45}=0 & \\
\pi_{13}\to O(m^2)\,, & \pi_{24}\to O(m)\,, & \pi_{35}=0 & & \\
\pi_{14}\to O(1)\,, & \pi_{25}=0 & & & \\
\pi_{15}=0\,. & & & &
\end{array}$$

From here, one immediately gets eq. (5.6).

b) The non relativistic limit of the invariants.

Let us take as positive energy spinors

(A.9) $$u_{n_i} = \begin{pmatrix} 1 \\ (\boldsymbol{\sigma}\cdot\boldsymbol{n}_i)/(E+m)\end{pmatrix} \quad\text{and}\quad \bar{u}_{n_i} = 1 \quad -\frac{\boldsymbol{\sigma}\cdot\boldsymbol{n}_i}{E+m}\,,$$

where the normalization factor $[1+\boldsymbol{p}^2/(E+m)^2]^{-\frac{1}{2}}$ has been taken equal to 1. Therefore we obtain

(A.10) $$u_{n_2}u_{n_1} = 1 - \frac{\boldsymbol{\sigma}^n\cdot\boldsymbol{n}_2\boldsymbol{\sigma}^n\cdot\boldsymbol{n}_1}{(E+m)^2} \simeq 1 + i\,\frac{\boldsymbol{\sigma}^n\cdot\boldsymbol{n}}{(E+m)^2}\,, \qquad \text{with } \boldsymbol{n} = \boldsymbol{n}_1\times\boldsymbol{n}_2,$$

$$\text{(A.11)}\qquad \bar{u}_{p_2}u_{p_1}\simeq 1+i\,\frac{\boldsymbol{\sigma}^p\cdot\boldsymbol{n}}{(E+m)^2}\,,$$

$$\text{(A.12)}\qquad \bar{u}_{n_2}\gamma^n\cdot P u_{n_1}=P_0+\frac{\boldsymbol{\sigma}^n\cdot\boldsymbol{n}_2\boldsymbol{\sigma}^n\cdot\boldsymbol{n}_1}{(E+m)^2}P_0+\frac{\boldsymbol{\sigma}^n\cdot\boldsymbol{n}_2\boldsymbol{\sigma}^n\cdot\boldsymbol{P}}{E+m}+\frac{\boldsymbol{\sigma}^n\cdot\boldsymbol{P}\boldsymbol{\sigma}^n\cdot\boldsymbol{n}_1}{E+m}=$$

$$=P_0\left[1+\frac{\boldsymbol{n}_1\cdot\boldsymbol{n}_2}{(E+m)^2}+\frac{2\boldsymbol{P}^2}{(E+m)P_0}\right]-i\boldsymbol{\sigma}^n\cdot\boldsymbol{n}P_0\left[\frac{1}{(E+m)^2}-\frac{1}{P_0(E+m)}\right]\simeq$$

$$\simeq A-iB\boldsymbol{\sigma}^n\cdot\boldsymbol{n}\,,$$

$$\text{(A.13)}\qquad \bar{u}_{p_2}\gamma^p\cdot N u_{p_1}=A-iB\boldsymbol{\sigma}^p\cdot\boldsymbol{n}\,,$$

$$\text{(A.14)}\qquad \bar{u}_{n_2}\boldsymbol{\gamma}^n u_{n_1}=\frac{\boldsymbol{\sigma}^n\cdot\boldsymbol{n}_2\boldsymbol{\sigma}^n}{E+m}+\frac{\boldsymbol{\sigma}^n\boldsymbol{\sigma}^n\cdot\boldsymbol{n}_1}{E+m}=\frac{1}{E+m}[2\boldsymbol{N}+i\boldsymbol{\sigma}^n\times\boldsymbol{\Delta}]\,,$$

$$\text{(A.15)}\qquad \bar{u}_{p_2}\boldsymbol{\gamma}^p u_{p_1}=\frac{1}{E+m}[2\boldsymbol{P}-i\boldsymbol{\sigma}^p\times\boldsymbol{\Delta}]\,,$$

$$\text{(A.16)}\qquad \bar{u}_{n_2}\gamma_0^n u_{n_1}=1+\frac{\boldsymbol{\sigma}^n\cdot\boldsymbol{n}_2\boldsymbol{\sigma}^n\cdot\boldsymbol{n}_1}{(E+m)^2}\simeq 1-i\,\frac{\boldsymbol{\sigma}^n\cdot\boldsymbol{n}}{(E+m)^2}\,,$$

$$\text{(A.17)}\qquad \bar{u}_{p_2}\gamma_0^p\bar{u}_{p_1}=1-i\,\frac{\boldsymbol{\sigma}^p\cdot\boldsymbol{n}}{(E+m)^2}\,,$$

$$\text{(A.18)}\qquad \bar{u}_{n_2}\gamma_5^n u_{n_1}=-i\,\frac{\boldsymbol{\sigma}^n\cdot\boldsymbol{n}_2}{E+m}+i\,\frac{\boldsymbol{\sigma}^n\cdot\boldsymbol{n}_1}{E+m}=-i\,\frac{\boldsymbol{\sigma}^n\cdot\boldsymbol{\Delta}}{E+m}\,,$$

$$\text{(A.19)}\qquad \bar{u}_{p_2}\gamma_5^p u_{p_1}=i\,\frac{\boldsymbol{\sigma}^p\cdot\boldsymbol{\Delta}}{E+m}\,.$$

From here we can calculate the non relativistic limit of P_2:

$$\text{(A.20)}\qquad \bar{u}_{n_2}\gamma^n\cdot P\,u_{n_1}\bar{u}_{p_2}1^p u_{p_1}+\bar{u}_{p_2}\gamma^p\cdot N u_{p_1}\bar{u}_{n_2}1^n u_{n_1}=$$

$$=[A-iB\boldsymbol{\sigma}^n\cdot\boldsymbol{n}]\left[1+i\,\frac{\boldsymbol{\sigma}^p\cdot\boldsymbol{n}}{(E+m)^2}\right]+[\sigma^n\to\sigma^p]=$$

$$=2A+iC(\boldsymbol{\sigma}^n+\boldsymbol{\sigma}^p)\cdot\boldsymbol{n}+\frac{2B}{(E+m)^2}\boldsymbol{\sigma}^n\cdot\boldsymbol{n}\boldsymbol{\sigma}^p\cdot\boldsymbol{n}\,,$$

and the relativistic limit of P_3

$$\text{(A.21)}\qquad \bar{u}_{n_2}\gamma^n\cdot P\,u_{n_1}\bar{u}_{p_2}\gamma^p\cdot N u_{p_1}=[A-iB\boldsymbol{\sigma}^n\cdot\boldsymbol{n}][A-iB\boldsymbol{\sigma}^p\cdot\boldsymbol{n}]=$$

$$=A^2-iAB(\boldsymbol{\sigma}^n+\boldsymbol{\sigma}^p)\cdot\boldsymbol{n}-B^2\boldsymbol{\sigma}^n\cdot\boldsymbol{n}\boldsymbol{\sigma}^p\cdot\boldsymbol{n}\,.$$

We see that both invariants contain terms that assign to the associated potential, central, spin orbit, and spin orbit-spin orbit types of interactions.

If we search for the tensor coupling, we see that it appears by the first time in potentials V_4, because the non relativistic limit of P_4 is

$$\bar{u}_{n_2}\gamma^n_\mu u_{n_1}\bar{u}_{p_2}\gamma^p_\mu u_{p_1} = \left[1 - i\,\frac{\boldsymbol{\sigma}^n\cdot\boldsymbol{n}}{(E+m)^2}\right]\left[1 - i\,\frac{\boldsymbol{\sigma}^p\cdot\boldsymbol{n}}{(E+m)^2}\right] -$$

$$-\frac{1}{(E+m)^2}\left[2\boldsymbol{N} + i\boldsymbol{\sigma}^n\times\boldsymbol{\Delta}\right]\cdot\left[2\boldsymbol{P} - i\boldsymbol{\sigma}^p\times\boldsymbol{\Delta}\right] \simeq$$

$$\simeq 1 + \frac{i}{(E+m)^2}(\boldsymbol{\sigma}_p + \boldsymbol{\sigma}^n)\cdot\boldsymbol{n} + \frac{1}{(E+m)^2}\left[\boldsymbol{\sigma}^n\cdot\boldsymbol{\Delta}\boldsymbol{\sigma}^p\cdot\boldsymbol{\Delta} - \boldsymbol{\sigma}^n\cdot\boldsymbol{\sigma}^p\Delta^2\right] - \frac{\boldsymbol{\sigma}^n\cdot\boldsymbol{n}\boldsymbol{\sigma}^p\cdot\boldsymbol{n}}{(E+m)^4}\,.$$

RIASSUNTO (*)

Si presenta un metodo per descrivere lo scattering dei nucleoni tramite un potenziale nucleare che si accorda con i risultati della teoria dei campi. Questo potenziale risulta somma di dieci potenziali, 5 diretti e 5 di scambio, ciascuno dei quali è dato da una sovrapposizione di potenziali di Yukawa. Si mostra poi che le funzioni di peso per queste sovrapposizioni si possono ottenere dalla teoria dei campi con l'uso delle relazioni di doppia dispersione per il sistema nucleone-nucleone e con l'uso dell'unitarietà per lo scattering del potenziale.

(*) *Traduzione a cura della Redazione.*