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Analysis of two phenomenological descriptions of ground-state bands in spherical and closed-shell nuclei

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Recent results concerning the application of the centrifugal stretching and the variable-moment-of-inertia (VMI) models to the description of ground-state bands in near-harmonic nuclei are discussed. As an extension of previous work dealing with deformed nuclei, the solutions of the centrifugal-stretching model which apply to near-harmonic and closed-shell nuclei are obtained and compared with those of the VMI model. In both cases there is more than one solution which applies to these nuclei. The irregularities observed in a plot of energy ratios E_I/E_2 vs E_4/E_2 for $E_4/E_2 \sim 2$ are not well reproduced by either description but the predictions of the VMI model appear to correspond to the maximum and minimum values found experimentally. In the Te isotopes the transition between these limits takes place in a smooth way as the number of neutrons increases. The data of magic nuclei are only accurately reproduced by the VMI model. The ratios E_4/E_2 for these nuclei are found to be correlated with the extra binding due to closed shells.

[NUCLEAR STRUCTURE Complete solutions centrifugal stretching model. Comparison with VMI model for closed and near-closed shell nuclei.]

I. INTRODUCTION

In a recent paper, da Providencia and Urbano[1] show that an application of the generator coordinate method to the description of ground-state bands in even-even nuclei leads in a natural way to the formulas on which the centrifugal stretching (CS)[2] and variable moment-of-inertia (VMI)[3-6] models are based:

$$E_I(x) = \tfrac{1}{2}C(x - x_0)^2 + I(I+1)/2\mathcal{I}(x) \tag{1}$$

and

$$(dE/dx)_I = 0\,. \tag{2}$$

$E_I(x)$ is the excitation energy and C and x_0 are the parameters which characterize each band. Equation (2) determines, for each angular momentum I, the equilibrium value of the variable x. The moment of inertia $\mathcal{I}(x)$ is given by

$$\mathcal{I}(x) = (\text{const})x^n\,. \tag{3}$$

The CS model is obtained by setting $n=2$. In this case, x is the deformation β and Eq. (3) is that derived from the hydrodynamical model. By setting $n=1$ one obtains the VMI model. Since x is now treated as a general variable which may include pairing effects apart from the deformation of the surface,[3] the moment of inertia itself is taken as the variable of the model.[4] The parameters of the VMI model are C and $\mathcal{I}_0$.

According to da Providencia and Urbano, the application of the generator coordinate method to an axially deformed nucleus yields Eqs. (1), (2), and (3) with $n=1$, as in the VMI model. However, when the method is applied to spherical or near-harmonic, instead of deformed, nuclei, it yields $\mathcal{I}(x) \simeq x^2$ as in the CS model. In support of this result the authors point out that a comparison with the experimental data known for the Te isotopes with $A = 116$, 118, 120, 124, and 126 shows indeed that a slightly better agreement is achieved with the latter model.

Such a result seems, at first sight, to be at variance with the trend expected from an earlier analysis of the two models.[3] It was then shown that while well-deformed nuclei are equally well fitted in both cases, the VMI assumption gives a better description for less deformed and "softer" nuclei as the first term in Eq. (1) becomes important. The results of Ref. 3 were obtained with nuclei for which $2.2 < E_4/E_2 < 3.3$. In this range the VMI equation[4,5] yields a single real root with C and $\mathcal{I}_0$ positive. When allowing these parameters to take negative values, new real roots appear which have been shown to apply to both spherical[5,6] and magic nuclei.[5]

The fact that more than one solution applies to the Te isotopes as well as other nuclei makes a comparison of these two phenomenological descriptions somewhat more complicated than that suggested in Ref. 1 where only one root was taken into account.

While the description of ground-state bands in nuclei removed from magic numbers have stimulated considerable activity in the past years, the systematics of the level energies in bands observed more recently in spherical and closed-shell nuclei have not been as much explored yet although a good amount of experimental information is now available. It is the purpose of this note to report on the complete solutions of the CS model and to discuss the application of both this and the VMI model to the description of the ground-state bands of spherical and magic nuclei.

For two parameter models, as in the present case, a plot of E_I/E_2 against E_4/E_2 provides an advantageous way of comparing the model predictions with the data because it is parameter-independent. Such plots were first used by Mallmann[7] to show the regularities exhibited by ground-state bands of all even-even nuclei. In the light of the data available today one finds that the smooth change of these energy ratios does take place throughout the Periodic Table except for nuclei with energies lying approximately in the interval $1.8 < E_4/E_2 < 2.2$, the so called "spherical" region.[6] In these cases the behavior is erratic in marked contrast with that found for other values of E_4/E_2. Thus, while the E_4/E_2 values of the Te isotopes mentioned above, for example, are similar within 5%, the E_6/E_2 and E_8/E_2 ratios vary as much as 15%. Similar or larger variations are found in other near-harmonic nuclei. This fluctuation cannot be reproduced with two-parameter models since the predicted values of E_I/E_2 are a smooth function of E_4/E_2.

In the following it will be shown that in the "spherical" region:

(a) The CS model predicts a single ratio E_I/E_2.

(b) The VMI model predicts two values of E_I/E_2 which, within a few percent deviation, represent the maximum and minimum values observed experimentally in a manner reminiscent of the Schmidt lines for magnetic moments.

(c) If only one solution of the VMI model is taken into account, the over-all fit is slightly worse than that obtained with the CS model, in accordance with the results of da Providencia and Urbano[1] reported for the Te isotopes.

(d) In spite of the erratic pattern displayed by the level energy ratios, the "transition" from one of the VMI solutions to the other, for the Te isotopes, is a smooth function of the number of neutrons N.

In addition, concerning the sharp change observed in the systematics at $E_4/E_2 = 1.825$,[5,6] we find that both models predict a change at this value since different roots apply above and below it. However, for $1 < E_4/E_2 < 1.825$, the data are only reproduced by the VMI model with striking accuracy, showing only three exceptions.

Finally, for magic nuclei, the difference between measured binding energies and those calculated with the semiempirical mass formula is found to be correlated to the energy ratios E_4/E_2.

II. METHOD OF ANALYSIS

In the analysis of the CS solutions we shall follow a treatment analogous to that used previously[4,5] in the formulation of the VMI model. To simplify the notation we shall use primed and unprimed letters when referring to variables and parameters of the CS and VMI models, respectively.

The calculations within the CS scheme were carried out starting from Eq. (1) with $\mathcal{I}' = x'^2$, $(k'=1)$, and using Eq. (2) to determine the equilibrium value of x', for each angular momentum I. In the present case the equilibrium condition leads to a quartic equation for the variable x':

$$x'^4 - x_0' x'^3 = I(I+1)/C' \tag{4}$$

and the equation for the energy (see the analogous one for the VMI model in Ref. 5) is

$$e'^4 - 3e'^3 + (3 - 8d')e'^2 - (1 + 20d')e' + (1 + 16d')d' = 0\,, \tag{5}$$

where

$$e' = E'/\tfrac{1}{2}C'\mathcal{I}_0'$$

and

$$d' = I(I+1)/C'\mathcal{I}_0'^2 = \tfrac{1}{2}\sigma' I(I+1).$$

The softness parameter σ' is defined[4] as

$$\sigma' = \left(\frac{1}{\mathcal{I}'}\frac{d\mathcal{I}'}{dI}\right)_{I=0} = \frac{1}{\frac{1}{2}C'\mathcal{I}_0'^2} \tag{6a}$$

(The VMI formulas are mathematically equivalent[4] to those derived by Harris[8] from the cranking model. In Ref. 5 Harris's $C_H = \frac{1}{4}C$ is used instead of C.)

In the VMI formalism the analogous quantities are

$$e = E/\tfrac{1}{2}C\mathcal{I}_0^{\,2},$$

$$d = \tfrac{1}{2}\sigma I(I+1), \tag{6b}$$

and

$$\sigma = 1/2C\mathcal{I}_0^{\,3}.$$

The real roots of Eq. (5), which we denote α', β', and γ', are shown in Fig. 1 in solid line. Those corresponding to the VMI model (Ref. 5), the α, β, and γ roots, are also shown, in dashed line. Real roots do not exist for $d' < -27/256$ in the CS model. For $d' > 0$ there are two real solutions, α' and γ'. (The VMI model yields only one solution for d positive.)

The experimental data originally considered by Diamond, Stephens, and Swiatecki[2] are described by solution γ' with σ' near zero. For σ' large this solution corresponds to the Te data included in the analysis of Ref. 1 (see below). For $I=0$ the γ' root leads to $E_0'=0$ while the α' and β' roots lead to $E_0'(x)=\frac{1}{2}C'\mathcal{I}_0'$. A similar result is obtained with the VMI model. Thus the energy of the ground state in cases α' and β' is different from zero and the level energy is given by the difference $E_I'(x) - E_0'(x)$.

A band is characterized by a pair of values C' and $\mathcal{I}_0'$. Once these parameters are fixed not all points of the curves in Fig. 1 represent physical solutions but only those corresponding to certain values of the angular momentum I. For $K=0$ bands in even-even nuclei $I=0, 2, 4 \cdots$. This circumstance has been likened[5] to that of physical solutions on a Regge pole trajectory which describe bound states and resonances in high-energy physics. The two sequences of 0, 2, 4, and 6, shown in Fig. 1 on the curves α' and γ', represent two different bands. They are obtained in particular with the same pair $(C', \mathcal{I}_0')$. A combination of these solutions is therefore a solution. Only one of the many possible combinations leads to energy ratios E_4/E_2 within the interval of interest here, i.e. $1 < E_4/E_2 \lesssim 2.2$[9] and yields a sequence of values E_I uniformly increasing with I. This particular "mixed" solution is indicated in Fig. 1 with crosses, and it is labeled $\alpha'\gamma'$. It is appropriate to consider such a solution as one cannot find arguments derived from the model to neglect it. The justification of a phenomenological model and in particular, a solution of it, resides on its ability to account for the experimental data. Within the VMI formalism, it has been shown[5] that a mixed solution of this kind $(\alpha\beta\gamma)$ gives remarkable results concerning the bands in closed- and semi-closed-shell nuclei (see below).

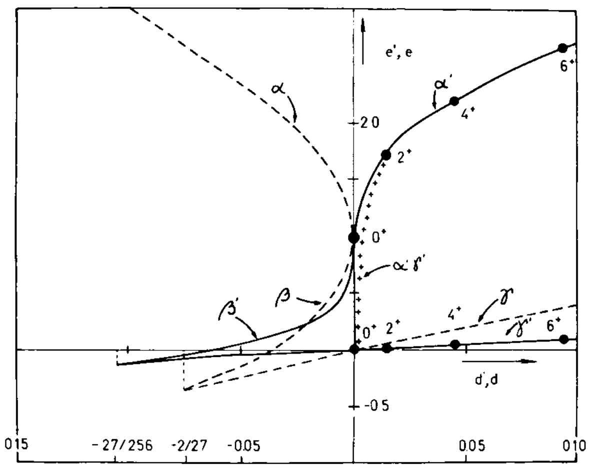

FIG. 1. The real roots α', β', and γ' of the quartic equation (5) of the CS model are shown in solid lines, while the real roots α, β, and γ corresponding to the VMI model are shown in dashed lines. The vertical axis represents energies in a dimensionless scale and the horizontal axis a quantity proportional to $I(I+1)$ and a constant containing the parameters. The dots represent the physical solutions obtained for a given pair $(\mathcal{I}_0', C')$ leading to a band $I=0$, 2, 4, and 6 on the α' root and on the γ' root. A combination of physical solutions of both branches also yields a band corresponding to a fixed $\mathcal{I}_0'$ and C' and therefore is a possible solution. The only combination which gives a sequence of energies increasing with I is that indicated with crosses and labeled $\alpha'\gamma'$.

The root β' diminishes with angular momentum and does not find application to any experimentally known band. Thus we shall not make use of this solution in the following analysis. Since part of the following discussion depends on the validity of the VMI β states it is appropriate to point out that in this case Eq. (2) corresponds to a maximum of the energy as function of the variable moment of inertia, instead of a minimum as in the other cases of solutions α and γ. At the same time the moment of inertia $\mathcal{I}$ obtained with this solution becomes negative. The nature of this solution is an intriguing point which undoubtedly deserves further study. For the moment one can only point to the agreement obtained with the experimental data when such β states are assumed to describe the first 2^+ states of magic nuclei.

III. RESULTS

In Fig. 2 the ratios E_I/E_2 for $I=6$ and 8 as functions of E_4/E_2 in the interval $1 \leqslant E_4/E_2 \leqslant 2.4$, are plotted. In the upper half, Fig. 2(a), all the solutions obtained with the CS model are shown. In Fig. 2(b) we present those corresponding to the VMI model.

Solid circles and triangles indicate the experimental values for $I=6$ and 8, respectively. In Fig. 2(a) they are labeled with numbers according to the list in Table I. Experimental energies, energy ratios, and references are also given in Table I. All experimental data available to us for which

$1 < E_4/E_2 < 2.23$, are included. Only representative points are shown for $E_4/E_2 > 2.23$. Doubtful assignments are indicated in parenthesis in Table I. Unless otherwise noted, the first excited I-spin state is the adopted member of the ground-state band.

The energy ratios E_4/E_2 obtained from the γ' root span the interval from the rigid rotator limit $20/6 = 3.333$ (not shown in Fig. 2) to the value $(20/6)^{1/2} = 1.825$ which coincides with the occurrence of a clear-cut discontinuity in the level energy systematics.[5,6] As shown in Fig. 2(a) the curves labeled α' and $\alpha'\gamma'$ connect smoothly with γ' and extend the interval of validity of the CS formulation down to the values $(20/6)^{1/3} = 1.491$ and 1, respectively. The latter curve corresponds to the mixed solution mentioned in the preceding section.

The common features and differences between the CS and VMI models become apparent from Fig. 2. In the VMI model the γ root leads to solutions which cover a more limited range; from 3.333 to $(20/6)^{2/3} = 2.231$. Those are continued smoothly by the curves obtained with the α root, which in turn reach the lower limit at the point of discontinuity 1.825, and by those labeled $\alpha\gamma$ which

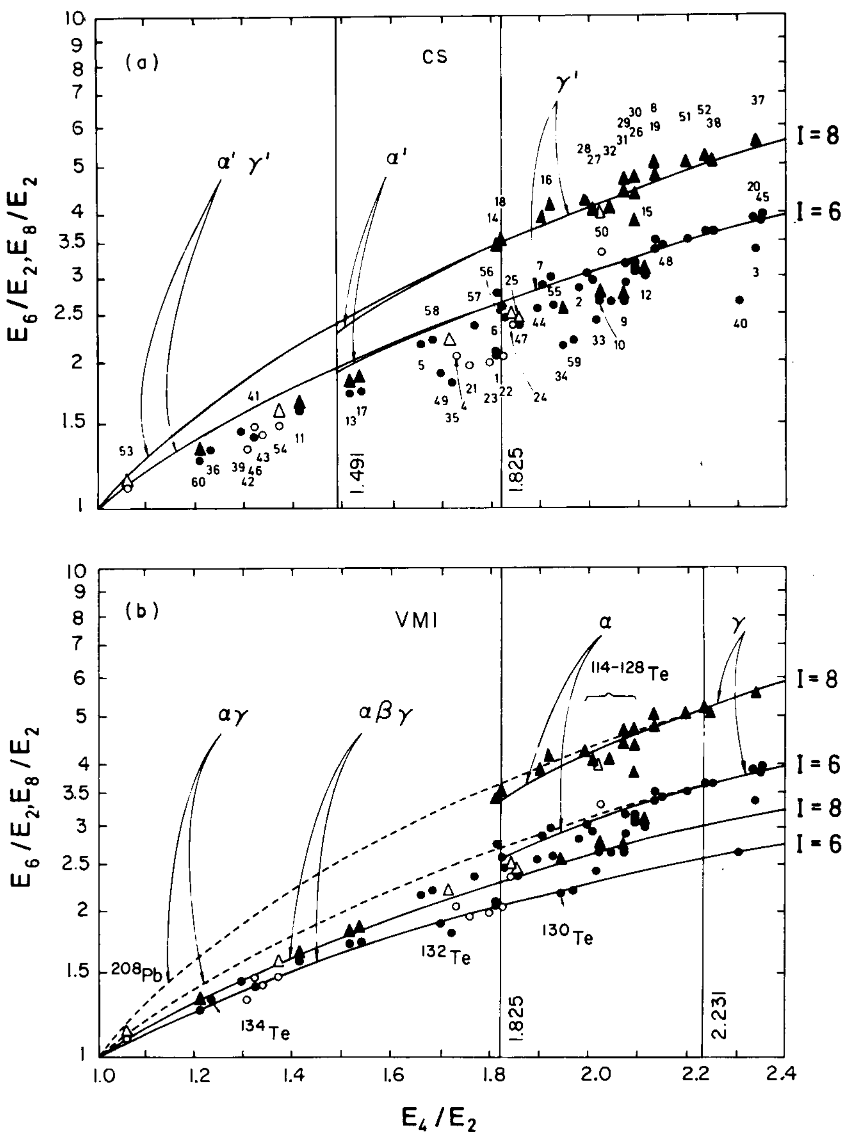

FIG. 2. Energy ratios E_I/E_2 as a function of E_4/E_2 for $I = 6$ and 8. The results of the CS and VMI model are shown in (a) and (b) respectively. Circles and triangles indicate experimental values of this ratio for $I = 6$ and 8 respectively. Open symbols represent doubtful assignments. The key to the numbers labeling the data points in the upper plot (a) is given in Table I. Only representative points are given for $E_4/E_2 > 2.231$.

TABLE I. Energies and energy ratios for the first I-spin excited states in even-even nuclei with $E_4/E_2 \lesssim 2.23$. Only representative cases are included with $E_4/E_2 > 2.23$.

I.D. Number	Nucleus	E_2 (keV)	E_4 (keV)	E_6 (keV)	E_8 (keV)	R_4	R_6	R_8	Refs.
1	$^{42}_{20}$Ca	1523	2750	3191		1.806	2.095		a
2	^{44}Ca	1156.95	2283.12	3285.02		1.973	2.839		b
3	$^{48}_{22}$Ti	983.4	2295.1	3332.5		2.334	3.389		c
4	^{50}Ti	1554	2677	(3201)		1.723	(2.060)		d
5	$^{52}_{24}$Cr	1434.19	2369.8	3113.8		1.652	2.171		e
6	$^{54}_{26}$Fe	1407.7	2538.9	2948		1.804	2.094		f
7	$^{72}_{34}$Se	862	1637	2467	3425	1.899	2.862	3.973	g, h
8	^{74}Se	635	1363	2231	3197	2.146	3.513	5.035	h, g
9	$^{86}_{38}$Sr	1078	2232	2859	2958	2.071	2.652	2.744	i
10	$^{88}_{40}$Zr	1056.9	2139.4	2810.7	2887.4	2.024	2.659	2.732	j, i
11	^{90}Zr	2186.2	3076.8	3447.78	3589.03	1.407	1.577	1.642	k
12	$^{90}_{42}$Mo	948	2003	2816	2877	2.113	2.970	3.035	i
13	^{92}Mo	1511	2284	2614	2762	1.512	1.730	1.828	l
14	^{94}Mo	870	1572	2421	2953	1.807	2.783	3.394	l, m
15	^{96}Mo	778	1628	2441	2979	2.093	3.138	3.829	l
16	^{98}Mo [n]	788	1510	2344	3272	1.916	2.975	4.152	l
17	$^{94}_{44}$Ru	1428	2183	2495	2640	1.529	1.747	1.849	l
18	^{96}Ru	832	1518	2150	2950	1.825	2.584	3.546	l
19	^{98}Ru	653	1398	2223	3127	2.141	3.404	4.789	l
20	^{102}Ru	476	1108	1875		2.328	3.939		l
21	$^{108}_{50}$Sn	1205	(2111)	(2365)		(1.752)	(1.963)		o
22	^{110}Sn	1212	(2199)	(2480)		(1.814)	(2.046)		o
23	^{112}Sn	1258	(2251)	(2552)		(1.789)	(2.029)		o
24	^{116}Sn	1293	2390	(3090)	(3257)	1.85	(2.39)	(2.52)	p, o, q
25	^{118}Sn	1233	2285		(2896)	1.85		(2.35)	p, o
26	$^{114}_{52}$Te	709.0	1484.2	2217.7	3089.3	2.093	3.128	4.357	r, s
27	^{116}Te	678.8	1359.4	2002.4	2773.2	2.003	2.950	4.085	r, s
28	^{118}Te	605.7	1206.5	1821.0	2574.1	1.992	3.006	4.250	r, s
29	^{120}Te	560	1161	1776	2653	2.073	3.171	4.738	s, t
30	^{122}Te	563.6	1180.5	1750.0	2669.0	2.095	3.105	4.736	u, s, t
31	^{124}Te	602.3	1247.9	1746.2	2664.3	2.072	2.899	4.424	u, s, t
32	^{126}Te	666.2	1361.3	1776.6	2766.6	2.043	2.667	4.153	u, s, t
33	^{128}Te	743.2	1497.1	1811.2		2.014	2.437		v, w
34	^{130}Te	839.5	1632.5	1814.5	2144.9	1.945	2.161	2.555	x, y
35	^{132}Te	974.6	1672.0	1775.2	(2158.4)	1.716	1.821	(2.215)	z
36	^{134}Te	1279.1	1576.1	1691.3		1.232	1.322		aa
37	$^{128}_{54}$Xe	443	1033	1737	2513	2.332	3.921	5.673	t
38	^{130}Xe	536.0	1204.2	1943.5	2696.0	2.247	3.626	5.030	t
39	^{136}Xe	1313.3	1695.1	1892.4		1.290	1.441		bb
40	$^{136}_{56}$Ba	818.3	1866.0	2206.1		2.280	2.696		cc
41	^{138}Ba	1435.7	1898.4	(2090.1)		1.322	(1.456)		dd
42	$^{140}_{58}$Ce	1596.6	2083.6	(2108.2)		1.305	(1.320)		ee
43	$^{142}_{60}$Nd	1576.0	(2101)	(2209)		(1.333)	(1.402)		ff
44	^{144}Nd	696.46	1314.44	1791.24		1.887	2.572		gg
45	$^{140}_{62}$Sm	531.0	1246.1	2132.5		2.347	4.016		hh
46	^{144}Sm	1660.0	2190.6	2323.2		1.319	1.399		ii, jj
47	^{146}Sm	747.20	1381.27	1811.79		1.849	2.425		kk
48	^{148}Sm	550.2	1180.1	1905.8		2.145	3.464		ll
49	$^{146}_{64}$Gd	1579.5	2658.0	2981.7	3093.2	1.683	1.888	1.958	ii, mm
50	^{150}Gd [nn]	638.05	1288.4	(2116.1)	(2554.4)	2.019	(3.317)	(4.003)	oo, pp
51	^{152}Gd	344.26	755.4	1227.4	1747.0	2.194	3.565	5.075	qq
52	$^{154}_{66}$Dy	334.7	747.0	1224.4	1748.2	2.232	3.658	5.223	rr
53	$^{208}_{82}$Pb	4086	4323	(4425)	(4608)	1.058	(1.083)	(1.128)	ss
54	^{210}Pb	795	(1091)	(1193)	(1273)	(1.372)	(1.501)	(1.601)	tt
55	$^{200}_{84}$Po	668	1279	1763		1.915	2.639		uu
56	^{202}Po	675.3	1245.0	1686.3		1.844	2.497		vv, uu, ww
57	^{204}Po	683.3	1198.5	1623.5		1.754	2.376		vv, xx

TABLE I (*Continued*)

I.D. Number	Nucleus	E_2 (keV)	E_4 (keV)	E_6 (keV)	E_8 (keV)	R_4	R_6	R_8	Refs.
58	^{206}Po	699.7	1175.9	1570.6		1.681	2.245		vv, xx
59	^{208}Po	685	1345	1522		1.964	2.222		yy
60	^{210}Po	1181.4	1426.7	1473.2	1556.9	1.208	1.247	1.318	zz

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are similar to $\alpha'\gamma'$ in Fig. 2(a).

An important difference between the two plots lies in the existence of the $\alpha\beta\gamma$ curves derived from the VMI formulas. The $\alpha\beta\gamma$ solution, which does not have an analog in the CS model, gives a very good account of the data for which $E_4/E_2 < 1.825$. In fact, it was because of this agreement that the existence of a sharp break in the systematics of the energy ratios E_I/E_2 at $E_4/E_2 = 1.825$ first became apparent.[5]

Table I lists 24 nuclei with $E_4/E_2 < 1.825$. One of these, ^{94}Mo ($E_4/E_2 = 1.807$, point 14 in Fig. 2), is fitted by the CS's α' root and should probably be assigned to the α root of the VMI model although it lies slightly outside its range of validity. The other 23 nuclei are not adequately described by the CS model except for ^{208}Pb (first point from the left, number 53 in the figure). Within their respective range of applicability, the $\alpha'\gamma'$ and α' solutions follow the trend of the data points but the deviations are large, of the order of 20% and 30% for $I=6$ and $I=8$, respectively. Similar deviations are observed for the $\alpha\gamma$ root of the VMI model [dashed line in Fig. 2(b)], but the fit obtained with the $\alpha\beta\gamma$ root is quite good although there are a few exceptions. Most notably ^{52}Cr, ^{204}Po, and ^{206}Po (numbers 5, 57, and 58) which are about 20% above the predicted value. These three values are of particular interest because they lessen the sharpness of the mentioned discontinuity and deserve further study. In this regard a measurement of the 8^+ state energies in these nuclei would be important. These states have been observed[10] in ^{204}Po and ^{206}Po as isomeric states but their energy could not be determined. The published results[10] suggest that the transition energy between the 8^+ and 6^+ states is very small. In this case the predicted $\alpha\beta\gamma$ value for $I=8$ would be closer to experiment than it is for $I=6$. With respect to ^{52}Cr it is worth noting that a second 4^+ state is known which lies close to the first 4^+. If the correct member of the ground-state band is the second 4^+ ($E_4/E_2 = 1.94$), then a good fit is obtained for this nucleus.[5]

In the same interval other smaller deviations (about 6%) from the $\alpha\beta\gamma$ curve are observed for ^{136}Xe, ^{140}Ce, ^{132}Te, and ^{50}Ti. Only 6^+ states are known in these four cases and in ^{140}Ce and ^{50}Ti the assignment is doubtful (see Table I). The remaining 16 cases are accurately reproduced with the VMI model. The predictions of the model have also been tested for $I=10$ by Lederer, Jaklevic, and Hollander.[11] With the exception of ^{204}Po, ^{206}Po, ^{132}Te, and ^{94}Mo the nuclei with $1 < E_4/E_2 < 1.825$ correspond to magic numbers and this interval has been appropriately identified as the "magic region."[6]

In the interval $1.825 < E_4/E_2 < 2.231$ (the latter value corresponds to the lower limit of the VMI γ root, but does not have a particular meaning in the CS model) the comparison is less obvious. One observes that the experimental energies do not follow a systematic trend as clear as that outside that interval. The family of the Te isotopes, from ^{114}Te to ^{130}Te, provides a good example of this erratic behavior. The experimental evidence seems to indicate that it is not possible to devise a two-parameter model such that a reasonable description of those states is attained with a single solution. In this regard the CS model is limited to the γ' solution while more than one value is obtained from the VMI model, corresponding to the α, the $\alpha\beta\gamma$, and the $\alpha\gamma$ roots.

The latter does not find application for $E_4/E_2 < 1.825$ and it differs very slightly from the α root for $E_4/E_2 > 1.825$. Thus we shall limit the discussion to the α and $\alpha\beta\gamma$ roots hereafter.

The analysis carried out by da Providencia and Urbano included the even-even nuclei ^{116,118,120,124,126}Te. As they point out, the γ' root of the CS model fits these nuclei better than the α root of the VMI model, although the deviations in both cases are appreciable. We have performed a similar comparison including all nuclei shown in Fig. 2 in the interval mentioned above. The results are presented in Table II. It is seen that in agreement with da Providencia and Urbano, the CS description works better than the VMI's α root, but if the $\alpha\beta\gamma$ root is also taken into account, then the VMI model gives a much improved fit although the irregularities of the data in this region, particularly for $I=6$, are still not satisfactorily accounted for. It is nevertheless noteworthy that, as was mentioned before, the VMI curves closely correspond to the limiting values found experimentally; apart from small deviations, all data are found between the two calculated values.

In summary, the analysis of the results of both the CS and VMI models shows significant differences between the two and we conclude that a much better description of ground-state bands in magic and spherical nuclei is achieved with the

TABLE II. Values of $Q = \sum^N [(E_I/E_2)_{\rm exp} - (E_I/E_2)_{\rm calc}]^2$ and of the percent standard error $\sigma = [Q/N(N-I)]^{1/2} \times 100$ for nuclei in the interval $1.825 \le E_4/E_2 \le 2.231$.

		$I=6$ (29 nuclei)		$I=8$ (22 nuclei)	
Model	Root	Q	σ (%)	Q	σ (%)
CS	γ'	2.76	5.83	12.09	16.18
VMI	α	3.26	6.33	15.07	18.06
	α $+\alpha\beta\gamma$	1.011	3.53	1.44	5.58

VMI model. This result becomes, therefore, an extension of that previously obtained[3] for deformed nuclei with $E_4/E_2 > 2.231$. On each end of the extended interval $1 < E_4/E_2 < 3.33$ a different solution applies and the experimental points lie neatly on two universal curves or branches for each spin I, rather than one,[7] except in the intermediate or spherical region. We can view the irregularities discussed above as an indication of a transition from one solution to the other which takes place in the intermediate region starting abruptly at $E_4/E_2 = 1.825$ and then gradually disappears as E_4/E_2 increases. This transition is further discussed in the following section.

IV. DESCRIPTION OF NEAR-HARMONIC NUCLEI IN TERMS OF A COMBINATION OF THE $\alpha\beta\gamma$ AND α SOLUTIONS

In the preceding section it was shown that the prediction of both the CS and VMI models do not fit the energies of some near-harmonic nuclei with a ratio $E_4/E_2 \sim 2$, but that the experimental values lie, in most cases, between the two values α and $\alpha\beta\gamma$ obtained with the VMI formalism. This circumstance suggests a description of these states in terms of a combination of both VMI values.[5,12] The simplest form is

$$(E_I/E_2)_{\exp} = a_I(E_I/E_2)_\alpha + b_I(E_I/E_2)_{\alpha\beta\gamma}, \quad (7)$$

with $a_I + b_I = 1$.

Taking into account the scattering of the data points in the "spherical" region and the lack of a pattern at least as clear as that observed in the magic and deformed regions, Eq. (7) may prove a useful tool to explore the "transition" from the upper (α root) to the lower ($\alpha\beta\gamma$) branch.

With this aim we have studied the coefficients a_I and b_I as a function of the atomic number Z and of the number of neutrons N for those cases where bands from several isotones or isotopes lying in the spherical interval are known. Table I lists a good number of isotones but most of them correspond to magic numbers ($N = 28$, 50, 82, and 126) and one then expects the constant value $b_I = 1$ for them. This is indeed observed with only one exception, ^{52}Cr. On the other hand, the near-harmonic isotones, ^{86}Sr and ^{88}Zr; ^{136}Ba and ^{132}Te, and ^{148}Sm and ^{150}Gd, are only two in each case and not much information can be obtained on the behavior of the coefficients. More data are available for the study of b_I as function of N. The Te isotopes, in particular, constitute the largest family of isotopes with known ground-state bands. The relevance of the Te isotopes in this connection has been pointed out by McDonald and Malmskog.[12] These authors have also noted that E_6/E_2 increases gradually with E_4/E_2 and decreases with the mass number A for $120 \leq A \leq 134$. This behavior, however, breaks down when the lighter isotopes are taken into account.

The variation of the coefficients observed for the Te isotopes for $I = 6$ is remarkably smooth as N increases. The coefficient b_6 is plotted vs N in Fig. 3. The size of the dots is larger than the error of each point. The curve reaches its minimum at $N = 66$ and increases uniformly as the shell is filled up to $N = 78$. From $N = 78$ to $N = 82$ the curve is flat. The lowest value corresponds to ^{118}Te, exactly half way between magic numbers $N = 50$ and 82, which suggests possibly a correlation between the coefficient b_6, or rather its complement a_6, and the amount of configuration mixing. This drop in the value of b_6 when the shell is half filled is evidenced also in three other cases. These are particularly interesting because three magic nuclei are involved: ^{44}Ca, ^{52}Cr, and ^{116}Sn.

Each one corresponds to a point half way between magic numbers. For ^{44}Ca, $N = 24$; for ^{52}Cr, $Z = 24$; and for ^{116}Sn, $N = 66$ as in ^{118}Te. It seems a striking coincidence that these nuclei represent just the only three exceptions to the rule $b_I \sim 1$ for magic nuclei. It would appear then that half-filled shells have a particularly sharp effect on the systematics of ground-state bands.

Since the rule $b_I \sim 1$ for magic nuclei seems to hold with the three exceptions just mentioned, it seems somewhat odd that nonmagic nuclei such as ^{130}Te and ^{132}Te also yield $b_I \sim 1$. Other cases of nonmagic nuclei lying on the $\alpha\beta\gamma$ curve are ^{208}Po

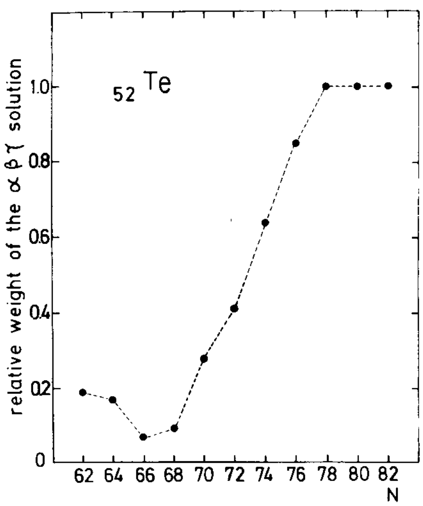

FIG. 3. The coefficient b_6 [Eq. (7)] representing the relative weight of the $\alpha\beta\gamma$ solution obtained when the experimental values are described in terms of a linear combination of the $\alpha\beta\gamma$ and α solutions of the VMI model, for the Te isotopes ($I = 6$).

and ^{136}Ba. It is tempting to associate this result with a case in which a subshell is closed and only a low j orbit separated by a reasonable energy gap (thus hindering configuration mixing) remains unoccupied. Such a scheme applies quite well to the case of ^{208}Po. Here $N=124$, the $f_{5/2}$ subshell is completed, and the $p_{1/2}$ orbit, which is relatively separated[13] from the $f_{5/2}$, is empty.

The odd-A data in the Te isotopes indicate that the last orbit to be filled before $N=82$ is $d_{3/2}$. In ^{130}Te, ^{132}Te, and ^{136}Ba the $h_{11/2}$ is completed. At the same time for $N>77$ the energy difference between the $h_{11/2}$ and $d_{3/2}$ orbits increases significantly. It seems worthwhile to search for other cases in order to find the extent to which this argument holds. ^{140}Sm also has $N=78$ but does not fit in this scheme as it lies on the upper branch in the deformed region [Fig. 2(b)]. In this case $Z=62$ and the relative positions of the orbits are probably different. Unfortunately these are not known, owing to the scarcity of the odd-A data around ^{140}Sm.

The Te data for $I=8$ are less illustrative and they have not been included in Fig. 3. One finds $b_8<0.2$ for $62\leqslant N\leqslant 74$ and it is 1 for $N=78$, 80, and 82. The energy of the 8^+ state in ^{128}Te$_{76}$, possibly the only intermediate case, is unfortunately now known to us. The transition from one branch to the other appears to be, in general, more abrupt for $I=8$ than for $I=6$. States with $I=8$ whose energies are clearly different from the VMI values (i.e., $b_8\sim 0.5$) are only found in a few cases like ^{90}Mo and ^{96}Mo.

The analysis of other isotopes of the elements Mo, Ru, Sn, and Sm are not as interesting as for the Te isotopes because fewer nuclei are known in each case. Moreover, in no case is the variation of b_I as gradual as for the Te isotopes. This is probably owing to the filling of an orbit with large angular momentum such as the $h_{11/2}$ in the latter case.

V. CORRELATIONS BETWEEN THE RATIO E_4/E_2 AND THE BINDING ENERGY DUE TO CLOSED SHELLS

The success of the $\alpha\beta\gamma$ solution in describing ground-state bands in magic nuclei stimulates the search for correlations between the model parameters and nuclear physical quantities. Such correlations, if found, are of interest because they serve the purpose of increasing our understanding of the physical meaning of the model parameters and opening new areas of application of the model.

In this connection the case of the $\alpha\beta\gamma$ solution is particularly attractive because less is known about the meaning of the parameters than for the other solutions. Thus, the definition of the softness parameter σ [Eq. 6(b)] allows one to relate the ratio E_4/E_2 to $\mathcal{I}_0$ and C. In the case of $\mathcal{I}_0$ and C positive (γ root) the physical meaning of these parameters is straightforward. Furthermore, it is supported by the analysis of the observed correlations between E_4/E_2 and several measured quantities such as the nuclear deformation, the γ- and β-band vibrational energy, and ratios of reduced transition probabilities, and between the moments of inertia $\mathcal{I}$ and intrinsic quadrupole moments.[4]

Similar correlations concerning the α and $\alpha\beta\gamma$ roots have not been found until now although some interpretations have been proposed for the meaning of the negative parameter $\mathcal{I}_0$ in the case of the α root.[6,14] The absolute value of $\mathcal{I}_0$ increases from zero to ∞ as E_4/E_2 varies from 2.231 to 1.825. Scharff-Goldhaber and Goldhaber[6] have interpreted this parameter as a measure of the "rigidity" of the spherical ground-state (in this case $\mathcal{I}_{I=0}\equiv 0$ and different from $\mathcal{I}_0$) against deformation. According to Harris's equations[8] a negative value $\mathcal{I}_0$ in a state with zero moment of inertia implies a finite angular velocity ω of the system. This yields the nonzero ground-state energy $E_0=\frac{1}{2}C\mathcal{I}_0{}^2$ mentioned in Sec. II. Thieberger[14] has proposed a simple classical model which reproduces the VMI-Harris formulas and illustrates the meaning of a negative $\mathcal{I}_0$.

In the following we report a correlation found between the ratio E_4/E_2 and the binding energy due to closed shells. This result concerns, therefore, magic nuclei and the $\alpha\beta\gamma$ root.

A relationship between these two quantities was suspected as ^{208}Pb, which is supposed to have the largest binding arising from shell structure among the nuclei shown in Fig. 2, and exhibits the lowest value of the ratio E_4/E_2.

The binding due to closed shells was computed as the difference

$$B_{cs}(A,Z)=B_{exp}(A,Z)-B_{se}(A,Z)\,, \tag{8}$$

where the experimental binding energies B_{exp} were obtained from the compilation of Everling *et al.*[15] and the semiempirical values B_{se} were calculated using the expression given by Wapstra.[16] Since the semiempirical formula[16] does not include shell effects (other than the odd-even mass difference) we may expect Eq. (8) to account reasonably well for the influence of completed shells. A plot of B_{cs} as a function of E_4/E_2 is shown in Fig. 4. All magic nuclei listed in Table I for which B_{exp} is known are included in the analysis. They span the range $E_4/E_2=1$ to 2 approximately, and they cover a wide interval of mass numbers from ^{42}Ca (point number 1 in Fig. 4) to ^{210}Po (point 60) and ^{210}Pb (point 54).

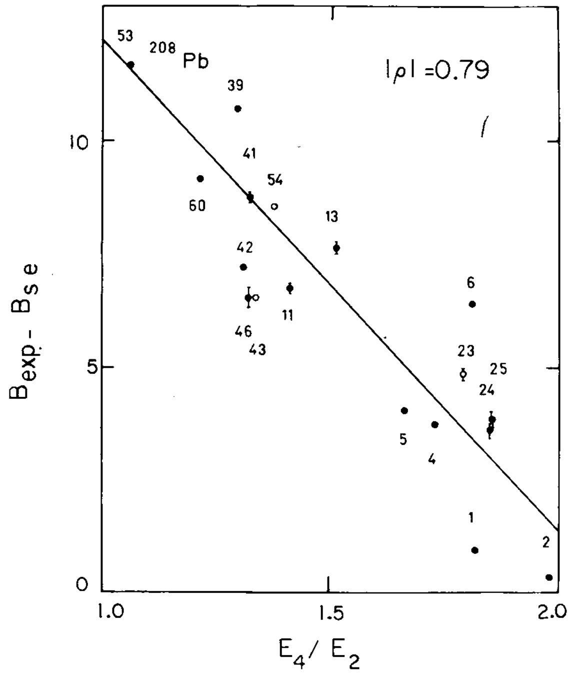

FIG. 4. The binding energy (in MeV) due to closed shells in magic nuclei as a function of E_4/E_2. The ordinate is computed as the difference between the experimental values and those calculated with the semiempirical mass formula. The correlation coefficient is $|\rho| = 0.79$.

The ordinates vary from 0 to 10 MeV approximately. Taking into account that these numbers are differences of numbers about 100 times larger, the correlation shown in Fig. 4 is remarkable. The correlation coefficient is found to be $|\rho| = 0.79$ with a probability less than 0.05% of being consistent with a zero correlation.

VI. SUMMARY

In this paper it has been shown that the VMI model gives a better description of ground-state bands in spherical and magic even-even nuclei than the CS model, and this result becomes, therefore, an extension of that previously obtained for deformed nuclei.

The spherical nuclei are not well fitted in general by either model. However, in a manner reminiscent of the Schmidt lines which bracket the experimental nuclear magnetic moments, the VMI curves appear to represent the limiting values found experimentally. The data of spherical nuclei are discussed in terms of a solution which is a linear combination of the VMI values. The coefficients of this linear combination are shown to be related to the shell structure and, at least in the case of the Te isotopes, to vary smoothly with the number of neutrons.

An interesting coincidence is pointed out: The only three semimagic nuclei which are not fitted by the $\alpha\beta\gamma$ solution correspond to a closed shell for one type of nucleon and an exactly half-filled shell for the other.

Finally, it is shown that the ratio E_4/E_2 for magic nuclei is definitely correlated to the extra binding energy arising from the closed shell.

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