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SOME STEADY AND UNSTEADY HEAT CONDUCTION SOLUTIONS IN ORTHOTROPIC CIRCULAR CYLINDERS

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Simple, approximate solutions are obtained in the case of prescribed axially symmetric boundary conditions. The results are compared in one example with values obtained using a finite element code and good agreement is shown to exist. The problem is of interest in several fields of engineering and applied science, for instance when performing thermal analysis of nuclear fuel elements which in some instances exhibit thermally orthotropic characteristics.

1. Introduction

Orthotropic materials find wide application in modern technology: from nuclear, oceanic and space technology to printed circuit boards in complex electronic packages.

On the other hand, a rather limited amount of literature is available on the solution of thermal problems in orthotropic solids. at least when compared with the large number of excellent textbooks, papers and technichal reports dealing with materials of isotropic characteristics.

The present paper deals with the development of approximate, simple solutions of some boundary value problems in orthotropic circular cylinders. In some instances the results are verified using a finite element code. *

2. Unsteady state heat conduction problem in an orthotropic circular plate

It is assumed that the heat conduction coefficients k_x, k_y (see fig. 1) do not vary with temperature. The problem is then governed, in the case of an unsteady, two-dimensional phenomenon, by the partial differential equation [1]

$$k_x \frac{\partial^2 T}{\partial x^2} + k_y \frac{\partial^2 T}{\partial x^2} = c_p \rho \frac{\partial T}{\partial t}. \tag{1}$$

Let the initial and boundary conditions be

$$T(x, y, 0) = T_0 \left[1 - \left(\frac{r}{a}\right)^2\right]^p, \tag{2a}$$

$$T[L(x, y) = 0, t] = 0, \tag{2b}$$

where $L(x, y) = 0$ is the functional relation which defines the boundary of the domain, and p is an integer ($p > -1$).

Making

$$T(x, y, t) = T_1(x, y)\, \tau_1(t) \tag{3}$$

and substituting into (1) one obtaines, following the classical procedure of separation of variables,

$$\frac{1}{T_1}\left(k_x \frac{\partial^2 T_1}{\partial x^2} + k_y \frac{\partial^2 T_1}{\partial y^2}\right) = c_p \rho \frac{\tau_1'}{\tau_1} = -\beta_n^2. \tag{4}$$

The function $\tau(t)$ is simply

$$\tau = A \exp\left(\frac{-\beta_n^2}{c_p \rho} t\right), \tag{5}$$

where A is an arbitrary constant.

On the other hand, the function $T_1(x, y)$ is a

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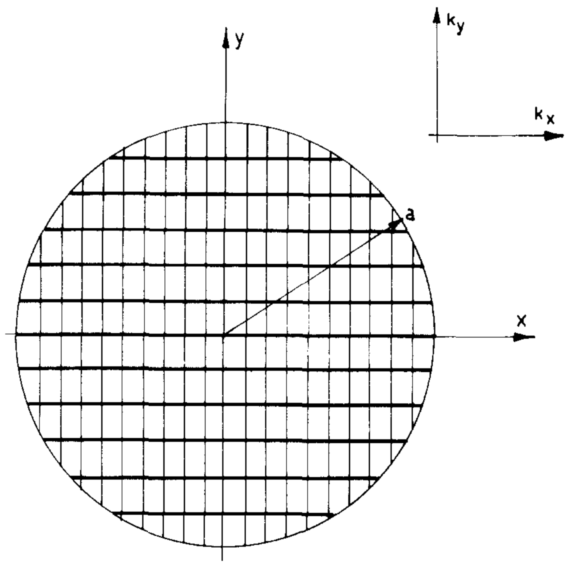

Fig. 1. Thermally orthotropic circular plate.

solution of the partial differential equation

$$k_x \frac{\partial^2 T_1}{\partial x^2} + k_y \frac{\partial^2 T_1}{\partial y^2} + \beta_n^2 T_1 = 0 \,, \tag{6}$$

which may be expressed as

$$\nabla_0^2 T_1 + \beta_n^2 T_1 = 0 \tag{7}$$

where ∇_0^2 is the orthotropic Laplacian operator

$$\nabla_0^2 = k_x \frac{\partial^2}{\partial x^2} + k_y \frac{\partial^2}{\partial y^2} \,.$$

It is important to point out that initial or boundary value problems similar to the present one are quite simple when dealing with rectangular domains (a double Fourier series approach is in general applicable for many problems of practical significance).

Ref. [2] deals with approximate solutions of unsteady heat conduction problems in plates of "exotic" boundary shape in the case when $p = 0$.

It will be shown now that the approach developed in [2] is valid in the case of orthotropic circular configurations when the temperature field is defined by eqs. (1) and (2).

Obviously, approximate analytical or numerical techniques must be used if one wishes to find the solution of (6) in the case of nonrectangular domains.

The first step consists in determining the separation constants β_n.

It is shown in calculus of variations that the solution of the partial differential equation (6) is equivalent to the minimization of the functional [2]

$$J[T_1] = \iint_D \left[k_x \left(\frac{\partial T_1}{\partial x} \right)^2 + k_y \left(\frac{\partial T_1}{\partial y} \right)^2 - \beta_n^2 T_1^2 \right] \mathrm{d}x\, \mathrm{d}y \tag{8}$$

subject to the boundary condition

$$T_1 [L(x, y) = 0] = 0 \,. \tag{9}$$

The simplest approximation which satisfies (9) is probably the expression

$$T_1 \simeq T_{1a} = [a^2 - (x^2 + y^2)] \sum_{n=1}^{N} \sum_{m=0}^{M} A_{nm} (x^2 + y^2)^{n-1} \times x^{2m} y^{2m} \,. \tag{10}$$

Substituting (10) into (8) and using the minimization condition

$$\frac{\partial J[T_{1a}]}{\partial A_{nm}} = 0 \,, \tag{11}$$

one obtains a linear system of equations in the A_{nm}'s. From the nontriviality requirement one obtains a secular determinant whose roots are the desired eigenvalues β_n.

It should be clear at this point that expression (10) makes the algorithmic procedure quite simple from the point of view of calculating the separation constants. When expressing the temperature field in its final form, it is considerably more expedient to use the Fourier–Bessel expansion which has the following advantages:

(a) it is the exact solution in the case of a circular, isotropic plate, and

(b) the coefficients are obtained by means of well-known orthogonality relations.

Accordingly, as a first order approximation one takes:

$$T(x, y, t) \simeq T_a(r, t) = \sum_{n=1}^{N} B_n J_0(\alpha_n r) \exp\left(\frac{-\beta_n^2}{c_p \rho} t \right), \tag{12}$$

where J_0 is the Bessel function of the first kind and of order zero; the α_n's are the roots of $J_0(x)$ and the

B_n's are given by [3]

$$B_n = T_0 \frac{2^{p+1}}{\alpha_n^{p+1}} \Gamma(p+1) \frac{J_{p+1}(\alpha_n)}{J_1^2(\alpha_n)} , \tag{13}$$

where $\Gamma(p+1)$ is the gamma function of argument $(p+1)$.

Expression (13) is obtained making $t = 0$ in eq. (12) and using the initial condition (2a) and Sonine's integral [3].

Making $p = 0$ in (12), eq. (13) yields the well-known expression

$$B_n = \frac{2T_0}{\alpha_n J_1(\alpha_n)} , \tag{14}$$

and for $p = 1$

$$B_n = T_0 \frac{4}{\alpha_n^2} \frac{J_2(\alpha_n)}{J_1^2(\alpha_n)} . \tag{15}$$

Since

$$J_2(\alpha_n) = \frac{2}{\alpha_n} J_1(\alpha_n) \tag{16}$$

replacing (16) into (15) yields

$$B_n = T_0 \frac{8}{\alpha_n^3 J_1(\alpha_n)} . \tag{17}$$

3. Steady state temperature distribution in an orthotropic circular cylinder

The heat conduction problem is now governed by the linear partial differential equation

$$k_x \frac{\partial^2 T}{\partial x^2} + k_y \frac{\partial^2 T}{\partial y^2} + k_z \frac{\partial^2 T}{\partial z^2} = 0 \tag{18}$$

subject to the boundary conditions (see fig. 2)

$$T(x, y, z)|_{z=0} = T_0 \left[1 - \left(\frac{r}{a}\right)^2\right]^p , \tag{19a}$$

$$T(x, y, z)|_{z=L} = 0 , \tag{19b}$$

$$T[a^2 - (x^2 + y^2) = 0, z] = 0. \tag{19c}$$

Applying the method of separation of variables one writes

$$T(x, y, z) = T_1(x, y)\, Z(z) \tag{20}$$

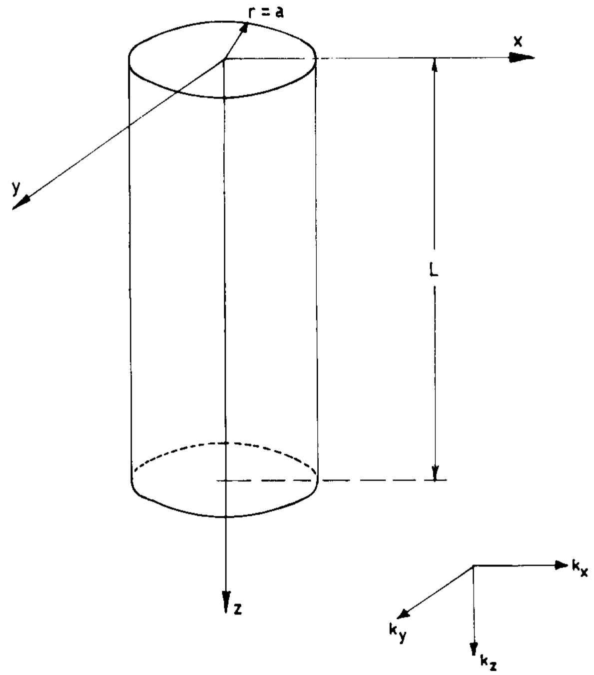

Fig. 2. Thermally orthotropic circular cylinder.

and substituting into (18) results in the differential equations

$$k_x \frac{\partial^2 T_1}{\partial x^2} + k_y \frac{\partial^2 T_1}{\partial y^2} + \beta_n^2 T_1 = 0 \tag{21a}$$

and

$$\frac{d^2 Z}{dz^2} - \frac{\beta_n^2}{k_z} Z = 0 . \tag{21b}$$

The solution of (21b) subject to the requirement expressed in (19b) is simply

$$Z(z) = A_n \sinh \frac{\beta_n}{\sqrt{k_z}} (L - z) . \tag{22}$$

Using now the same procedure followed in the preceding section one writes

$$T(x, y, z) \simeq T_a(r, z) = \sum_{n=1}^{N} B_n J_0(\alpha_n r) \left[\sinh \frac{\beta_n}{\sqrt{k_z}} (L - z)\right] . \tag{23}$$

It is important to point out that the separation constants β_n are the same eigenvalues determined in the previous section.

Making $z = 0$ in (23) and using the boundary condition (19a) one gets

$$B_n = T_0 \frac{2^{p+1}}{\alpha_n^{p+1}} \frac{\Gamma(p+1)}{\sinh(\beta_n/\sqrt{k_z})\, L} \frac{J_{p+1}(\alpha_n)}{J_1^2(\alpha_n)} . \qquad (24)$$

If the rod is infinitely long the solution of (21b) is

$$Z(z) = A_n \exp\left(\frac{-\beta_n}{\sqrt{k_z}} z\right) \qquad (25)$$

and the approximate expression for the temperature distribution is

$$T(x, y, z) \simeq T_a(r, z)$$

$$= \sum_{n=1}^{N} B_n J_0(\alpha_n r) \exp\left(\frac{-\beta_n}{\sqrt{k_z}} z\right) , \qquad (26)$$

where the B_n's are given by relation (13).

4. Numerical results and discussion

The separation constants β_n^2 are calculated making $N = 2$ and $M = 0$ in expression (10).

The minimization condition (11) yields a two-by-two determinantal equation

$$\begin{vmatrix} (1+\lambda)\,\pi - \dfrac{\beta_n^2 a^2}{k_x} 1.047197 & (1+\lambda)\, 1.047197 - \dfrac{\beta_n^2 a^2}{k_x} 0.261799 \\ (1+\lambda)\, 1.047197 - \dfrac{\beta_n^2 a^2}{k_x} 0.261799 & (1+\lambda)\, 1.047197 - \dfrac{\beta_n^2 a^2}{k_x} 0.104719 \end{vmatrix} = 0 , \qquad (27)$$

where $\lambda = k_y/k_x$

Table 1 shows the values of $\beta_n^2 a^2/k_x$ for $k_y/k_x =$ 1, 2 and 3. It is observed that for $k_y/k_x = 1$ (thermally isotropic material) the two eigenvalues are in good agreement with the exact results.

Figs. 3 and 4 depict the temperature variation as a function of the dimensionless variable $\tau = k_x t/c_p \rho a^2$ for the two points of the plate and for $k_y/k_x = 2$ and 3, respectively.

Table 1
Eigenvalues determined using eq. (27)

k_y/k_x	$\dfrac{\beta_1^2 a^2}{k_x}$		$\dfrac{\beta_2^2 a^2}{k_x}$	
	Exact	Approximate	Exact	Approximate
1	5.7832	5.784	30.4712	36.883
2	–	8.676	–	55.325
3	–	11.568	–	73.767

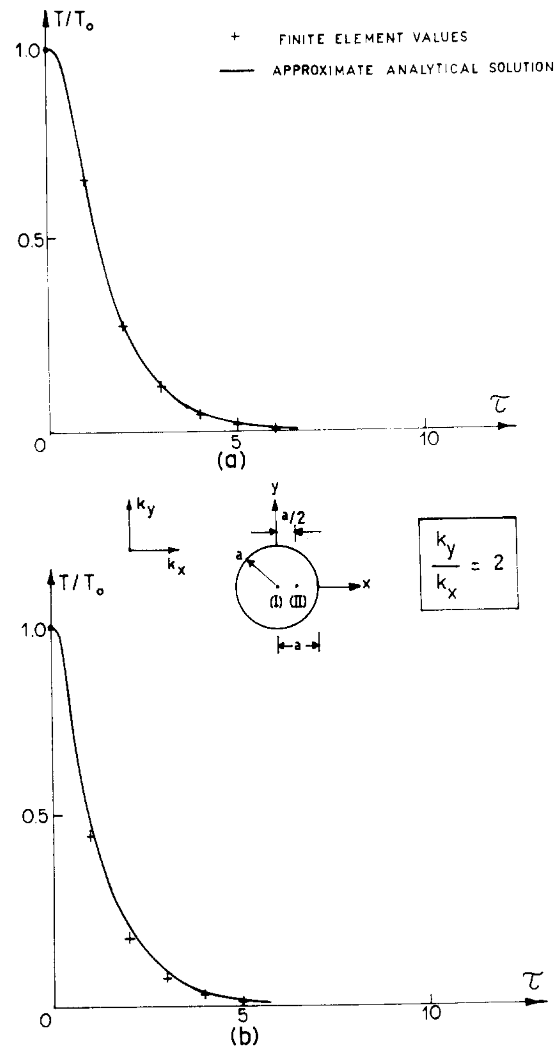

Fig. 3. Temperature variation: (a) at the center, and (b) at point (I).

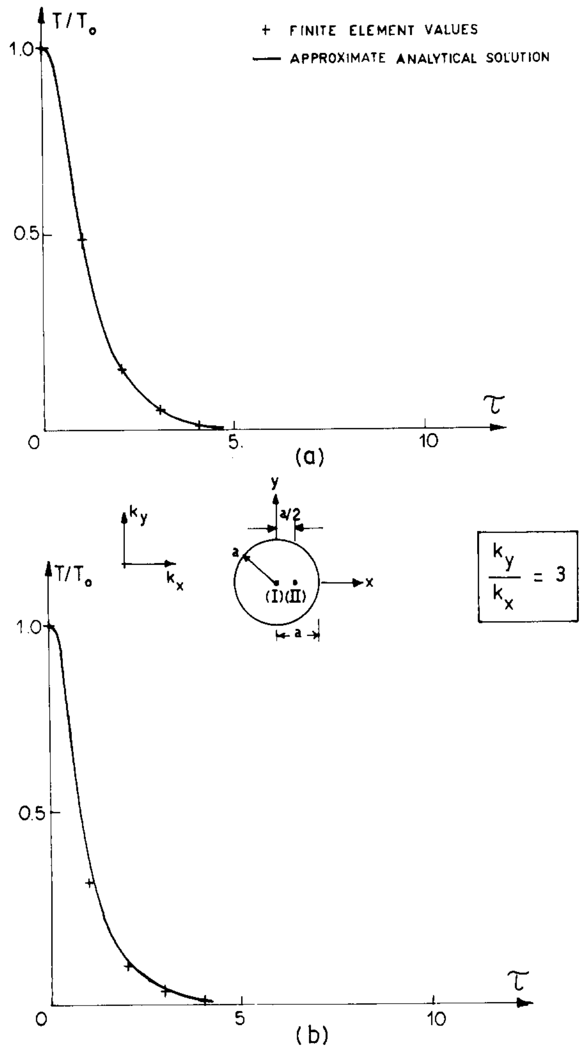

Fig. 4. Temperature variation: (a) at the center and (b) at point (II).

Also shown on these graphs are values obtained using the finite elements method. In all these situations the parameter p has been taken equal to zero [see eq. (2a)]. The agreement is good from an engineering viewpoint.

Fig. 5 shows the variation of the dimensionless variable T/T_0 plotted as a function of τ at the center of a thermally orthotropic circular plate for $p = 1$ and $k_y/k_x = 2$ and 3.

It is important to emphasize the fact that for relatively large values of the exponent only the first term of the truncated series expansion predominates and calculations become quite simple. It is also interesting to notice the fact that the present approximate procedure leads to a unified formulation, ideal from a designer's viewpoint.

It is important to point out that the analytical approximations involved do not take into account the angular variation of the temperature field. Since $k_x \neq k_y$ it is rather obvious that the temperature variable is a function of both polar coordinates, r and θ.

However, from the point of view of many practical applications and in view of the fact that the boundary of the domain and the prescribed boundary and/or initial conditions possess radial symmetry, it is reasonable to expect that the approximate procedure developed herein will yield an "average"-sort of value with respect to θ for a given value of the radial variable.

The same approach is valid if the prescribed initial or boundary condition is given in terms of a polynomial of the type

$$P(r) = \sum a_s r^s . \tag{28}$$

It is important to point out that the approximations can be improved if one adds additional coordinate functions which take into account the angular variation of the temperature field.

5. Conclusions

Considering the common use of composite materials which in many instances can be modelled by orthotropic materials, it may prove advantageous to the design engineer to count on a simple functional relation, such as the first term of expansion (12), which provides reasonably accurate results from a practical viewpoint and as a preliminary design aid.

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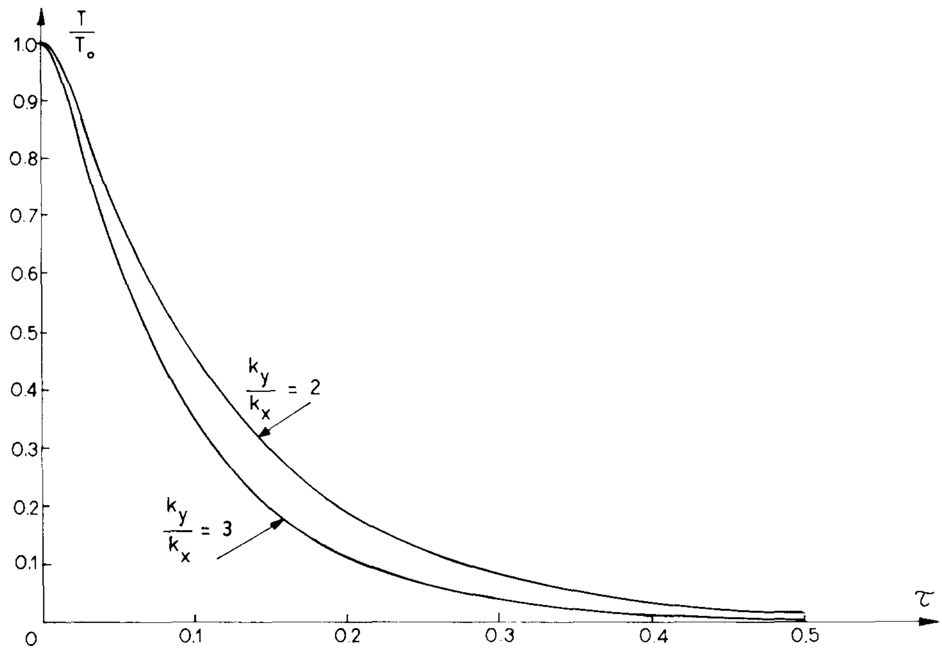

Fig. 5. Temperature variation at the center of a thermally orthotropic circular plate: $T(r, \theta, 0) = T_0[1 - (r/a)^2]$.

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