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COMMENT ON A THERMODYNAMIC PROOF OF THE INEQUALITY BETWEEN ARITHMETIC AND GEOMETRIC MEAN

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The inequality has been argued to follow from an application of the first and second laws of thermodynamics. It is observed here that this conclusion is based on a non-independent set of physical premises, which makes it to be unjustified.

Landberg [1] has recently argued that the inequality

$$\frac{1}{n}\sum_{j=1}^{n} Z_j \geqslant (Z_1 Z_2 \ldots Z_n)^{1/n}\,, \tag{1}$$

where the Z_j's are positive real numbers and the equality sign holds only for $Z_1 = Z_2 = \ldots = Z_n$, follows very simply from an application of the first and second laws of thermodynamics.

Paralleling Landsberg [1], let us now consider a system S of n identical heat reservoirs whose absolute temperatures are denoted generically by T_α ($\alpha = 1, 2, \ldots, n$) and having each of them a positive heat capacity C which is a constant. At a certain instant the heat reservoirs are all put into thermal contact with each other and at a later time such thermal contact is suppressed. The question is: which is the final state that S has reached after the mutual thermal contact has ceased? Denoting by $\{T_\alpha^{(i)}\}$ the set of known initial temperatures and by $\{T_\alpha^{(f)}\}$ the set of unknown final temperatures, the first and second laws of thermodynamics assert that $\{T_\alpha^{(f)}\}$ must verify:

$$\sum_{\alpha=1}^{n} T_\alpha^{(i)} = \sum_{\alpha=1}^{n} T_\alpha^{(f)}\,, \tag{2a}$$

and

$$\ln \prod_{\alpha=1}^{n} \frac{T_\alpha^{(f)}}{T_\alpha^{(i)}} \geqslant 0\,. \tag{2b}$$

Every possible set $\{T_\alpha^{(f)}\}$ verifying eqs. (2) identifies a possible final state of the system S. Since the product $\Pi_{\alpha=1}^{n} T_\alpha^{(f)}$, subject to the condition $\Sigma_{\alpha=1}^{n} T_\alpha^{(f)}$ = constant, it has its maximum value if and only if all the T_α's are equal [‡1] (say to T_{eq}), the state where $T_\alpha = T_{eq}$ for all α is a final state from which no further evolution is possible: it is the equilibrium state of S for the given initial conditions $\{T_\alpha^{(i)}\}$. According to this, we see that the first and second laws of thermodynamics, plus the fact of C being positive, predict the existence of an equilibrium state of S for which $T_\alpha = T_{eq}$ for all α. Thus our position here is different from that of Landsberg [1] since this author assumes from the start the existence of such an equilibrium state. Landsberg's conclusion that (1) is a truth of pure mathematics directly accessible from established principles of science is then based on a non-independent set of physical premises: (i) the first and second laws, (ii) C is positive (which is implicit in Landsberg's argument) and (iii) S reaches an equilibrium state of common temperature T_{eq}. This non-independent set does in fact allow to derive inequality (1) [1] but this result is redundant which makes Landsberg's conclusion to be unjustified. We hold that inequality (1) is accessible only from pure mathematics and that it forms part of the starting knowledge of any science built in terms of quantitative concepts.

It is interesting to note that our above argument

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‡1 From this mathematical statement it is a straightforward matter to derive inequality (1).

depends crucially on the fact that C is positive. Without this condition our argument can not be carried out up to the point of showing the existence, as we did, of an equilibrium state for S. Thus, in such a case we can safely add, now as an independent physical premise, to the first and second laws that "S can reach an equilibrium state of equal temperature for all its component heat reservoirs". In turn, this (stability) condition leads immediately to show that C must be positive.

References

[1] P.T. Landsberg, Phys. Lett. 67A (1978) 1.