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APPLICATION OF CONFORMAL MAPPING TO THE DETERMINATION OF UNSTEADY TEMPERATURE DISTRIBUTION IN THERMALLY ORTHOTROPIC PLATES

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Very few papers and reports are available on the solution of heat conduction problems in anisotropic solids. This situation is of particular interest in some nuclear engineering applications (for instance, fuel elements possess, in general, anisotropic characteristics). The present study deals with the solution of an unsteady heat conduction problem in domains of complicated boundary shape, considering a particular case of anisotropy: a thermally orthotropic material. It is shown that the conformal mapping technique coupled with the Ritz method leads to a unified solution of the title problem for an arbitrary orientation for the axes of orthotropy with respect to the directions of the sides of the plates.

1. Introduction

Orthotropic materials are commonly used in modern technology, from nuclear engineering to space and ocean applications, but only a few technical papers and reports treat heat conduction problems in orthotropic solids. Furthermore, very few unsteady state heat conduction problems dealing with thermally orthotropic configurations, possess a straightforward "classical" exact solution. Consider a rectangular plate where the material axes do not coincide with the direction of the sides of the plate, see fig. 1a.

The problem is then governed by the partial differential equation [1]

$$k_{x_1}\frac{\partial^2 T}{\partial x_1^2}+k_{y_1}\frac{\partial^2 T}{\partial y_1^2}=c_p\rho\frac{\partial T}{\partial t}\,. \tag{1}$$

Let the initial and boundary conditions be

$$T(x_1,y_1,0)=T_0\,, \tag{2a}$$

$$T[L(x_1,y_1)=0,t]=0\,, \tag{2b}$$

where $L(x_1,y_1)=0$ is the functional relation which defines the boundary of the domain.

Apparently, an exact solution of the differential system defined by eqs. (1) and (2) is only possible if the material axes coincide with the x, y directions, see fig. 1b. In this case, it can be easily shown that the exact, analytical solution is given by the expression [2]

$$T(X,Y,t)=16T_0\pi^{-2}\sum_{n=1,3,\ldots}^{\propto}\;\sum_{m=1,3,\ldots}^{\propto}((nm)^{-1}$$

$$\times\sin(n\pi X/a)\sin(m\pi Y/b))$$

$$\times\left\{\exp\ -\pi^2[n^2+(k_y/k_x)(ma/b)^2]\frac{k_x t}{ac_p\rho}\right\}\,. \tag{3}$$

The differential system defined by eqs. (1) and (2) does not seem to posses an exact solution in the case of a circular configuration. As shown in ref. [2], a simple, approximate solution is easily obtained by making use of the Ritz method.

The present study constitutes an extension of ref. [2] since it is shown here that the same approximate analytical solution is valid in the case of a square shape when the material axes are rotated of $\frac{1}{4}\pi$ with respect

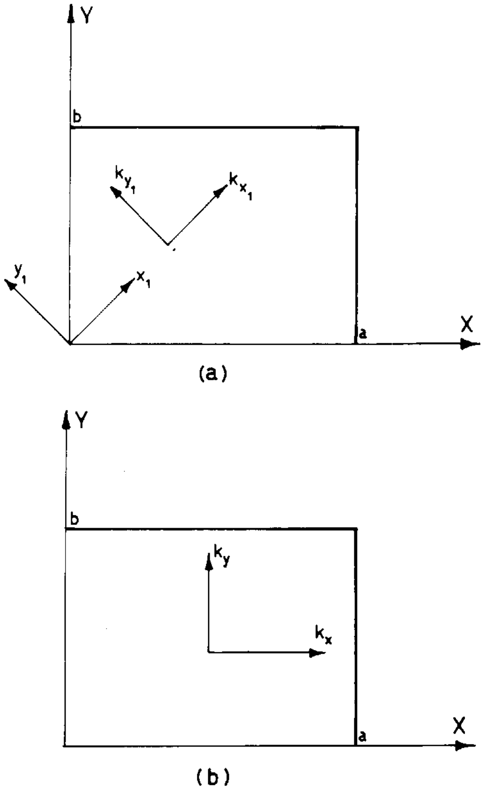

Fig. 1. Thermally orthotropic rectangular plate. (a) Material axes do not coincide with the directions of the principal axes; (b) both systems of axes coincide.

to the directions of the sides of the square (fig. 2a).

The same consideration is valid in the case of an octogonal plate when the angle of rotation is equal to $\frac{1}{8}\pi$ (fig. 2b).

The results obtained are in good agreement with values calculated using a finite element code *.

2. Development of the approximate analytical solution

Making

$$T(x, y, t) = T_1(x, y)\,\tau_1(t) \tag{4}$$

and substituting in eq. (1) one obtains, following the classical procedure of separation of variables,

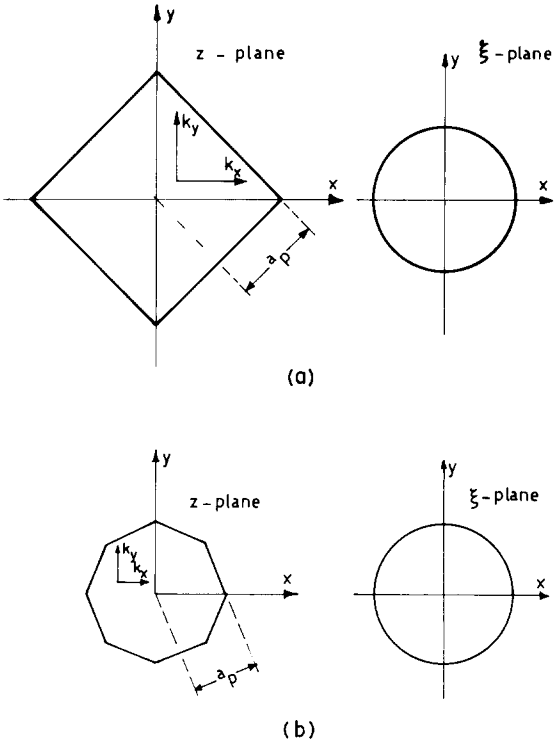

Fig. 2. Configurations under study in the present investigation.

$$\frac{1}{T_1}\left(k_x \frac{\partial^2 T_1}{\partial x^2} + k_y \frac{\partial^2 T_1}{\partial y^2}\right) = c_p\rho \frac{\tau_1'}{\tau_1} = -\beta^2 . \tag{5}$$

The function $\tau_1(t)$ is simply

$$\tau_1 = A \exp(-\beta^2/c_p\rho)\, t , \tag{6}$$

where A is an arbitrary constant.

On the other hand, the function $T_1(x, y)$ is a solution of the partial differential equation

$$k_x \frac{\partial^2 T_1}{\partial x^2} + k_y \frac{\partial^2 T_1}{\partial y^2} + \beta^2 T_1 = 0 , \tag{7}$$

which may be expressed as

$$\nabla_0^2 T_1 + \beta^2 T_1 = 0 , \tag{8}$$

where ∇_0^2 is the orthotropic Laplacian operator [2]

$$\left(\nabla_0^2 = k_x \frac{\partial^2}{\partial x^2} + k_y \frac{\partial^2}{\partial y^2}\right) .$$

It is shown by the calculus of variations that the

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* Developed at Centro Atómico Bariloche, CNEA, by F. Basombrio and B. Cruz.

solution of eq. (7) is equivalent to the minimization of the functional [3]

$$J[T_1] = \iint\limits_{D} \left[k_x\left(\frac{\partial T_1}{\partial x}\right)^2 + k_y\left(\frac{\partial T_1}{\partial y}\right)^2 - \beta^2 T_1^2\right] \mathrm{d}x\,\mathrm{d}y\,, \tag{9}$$

subject to the boundary condition

$$T_1[L(x,y) = 0] = 0\,. \tag{10}$$

Let

$$z = x + \mathrm{i}y = f(\zeta)\,, \qquad \zeta = \xi + \mathrm{i}\eta \tag{11}$$

be the functional relation which transforms the given, complicated shape in the z-plane, onto a unit circle in the ζ-plane. Admittedly, finding $z = f(\zeta)$ is not an easy task, but it is known for many simply- and doubly-connected shapes of practical significance [4,5].

Substituting eq. (11) into eq. (9), one obtains

$$\begin{aligned} J[T_1] = k_x \iint\limits_{C} &\left[2\,\mathrm{Re}\left[\left(\frac{\partial T_1}{\partial \zeta}\right)^2 \frac{1}{f'^2(\zeta)}\right]\right. \\ &\left. + 2\,\frac{\partial T_1}{\partial \zeta}\frac{\partial T_1}{\partial \bar\zeta}\,\frac{1}{\|f'(\zeta)\|}\right] \|f'(\zeta)\|\,\mathrm{d}\xi\,\mathrm{d}\eta \\ - k_y \iint\limits_{C} &\left[2\,\mathrm{Re}\left[\left(\frac{\partial T_1}{\partial \zeta}\right)\frac{1}{f'^2(\zeta)}\right]\right. \\ &\left. - 2\,\frac{\partial T_1}{\partial \zeta}\frac{\partial T_1}{\partial \bar\zeta}\,\frac{1}{\|f'(\zeta)\|}\right] \|f'(\zeta)\|\,\mathrm{d}\xi\,\mathrm{d}\eta - \beta^2 \iint\limits_{C} T_1^2 \\ &\times \|f'(\zeta)\|\,\mathrm{d}\xi\,\mathrm{d}\eta\,. \end{aligned} \tag{12}$$

In the case of the regular polygonal shapes, such as those shown in fig. 2, one has

$$z = a_\mathrm{p} A_s \int\limits_0^\zeta [(1-\zeta^s)^{2/s}]^{-1}\,\mathrm{d}\zeta\,, \tag{13}$$

where a_p is the apothem, s is the order of polygon and A_s is a parameter which has been tabulated in the open literature [6].

The simplest approximation which satisfies the transformed boundary condition

$$T_1(\zeta,\bar\zeta)|_{|\zeta|=1} = 0\,, \tag{14}$$

is probably the expression

$$T_1 \simeq T_{1a}(\zeta,\bar\zeta) = \sum_{n=1}^{N} A_n[1 - (\zeta\bar\zeta)^n]\,. \tag{15}$$

Substituting eq. (15) in eq. (12) and using the minimization condition

$$\frac{\partial J[T_{1a}]}{\partial A_n} = 0\,, \qquad n = 1, 2, \ldots, N\,, \tag{16}$$

one obtains a linear system of equations in the A_n's.

From the non-triviality condition one obtains a secular determinant whose roots are β_n^2's the desired eigenvalues.

It should be clear at this point that expression (15) makes the algorithmic procedure quite simple for calculating the separation constants. When expressing the temperature field in its final form, it is considerably more expedient to use the Fourier–Bessel expansion [2]

$$T(\zeta,\bar\zeta,t) \simeq \sum_{n=1}^{N} B_n J_0[\alpha_n(\zeta\bar\zeta)^{1/2}]\,\exp(-\beta_n^2/c_\mathrm{p}\rho)t\,, \tag{17}$$

where J_0 is the Bessel function of the first kind and

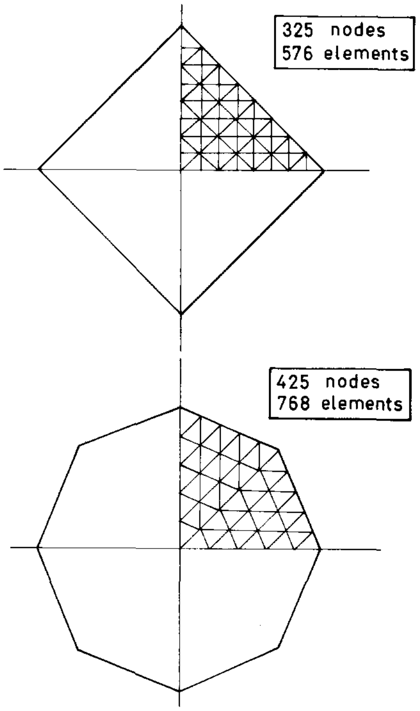

Fig. 3. Finite element net (schematic).

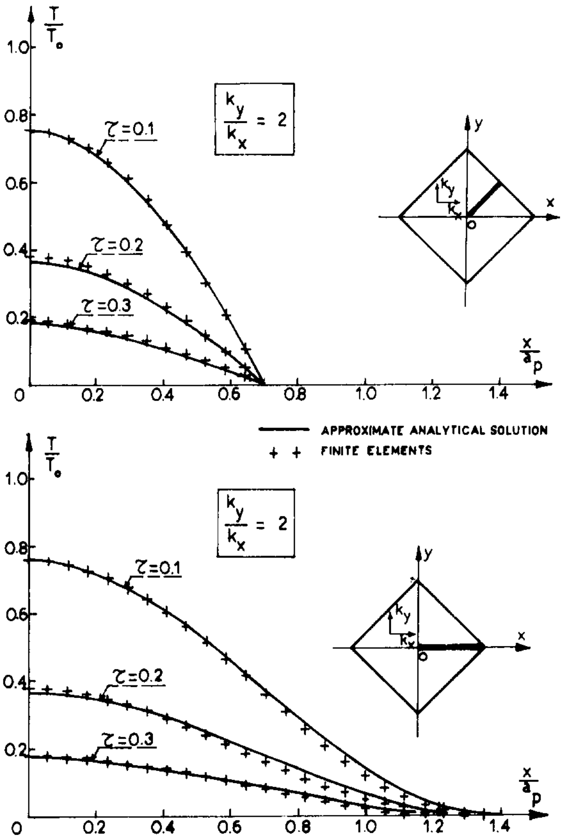

Fig. 4. Temperature distribution for a thermally orthotropic square plate ($k_y/k_x = 2$), (a) along the apothem; (b) along the diagonal.

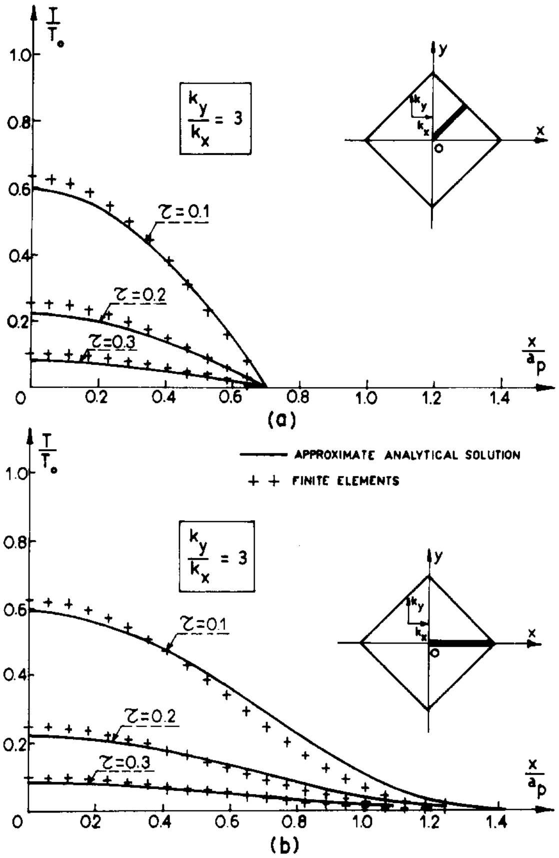

Fig. 5. Temperature distribution for a thermally orthotropic square plate ($k_y/k_x = 3$), (a) along the apothem; (b) along the diagonal.

order zero; the α_n's are the roots of $J_0(x)$ and the β_n's are given by the Fourier–Bessel expansion

$$\beta_n = \frac{2T_0}{\alpha_n J_1(\alpha_n)} . \tag{18}$$

The approach presented herein allows for an independent verification of the relative accuracy of the finite element method in a rather complex, diffusion-type problem.

3. The finite elements solution

Triangular elements are used and a linear variation of the temperature field is assumed inside the element (fig. 3).

Because of symmetry considerations, and taking into account the thermal orthotropy characteristics of the polygonal plates, it is necessary to consider, for each particular case, the subdomains shown in fig. 3.

4. Numerical results

From the point of view of the approximate analytical solution, it is quite remarkable that the functional relation (17) and the eigenvalues β_n's, are the same as those determined in ref. [2].

The rotation of the plate geometry with respect to the material axes is introduced by changing the positive sign of the term ζ^s in eq. (13) of ref. [2], to a minus sign [see eq. (13) of the present study]. This change in sign does not affect the determined eigen-

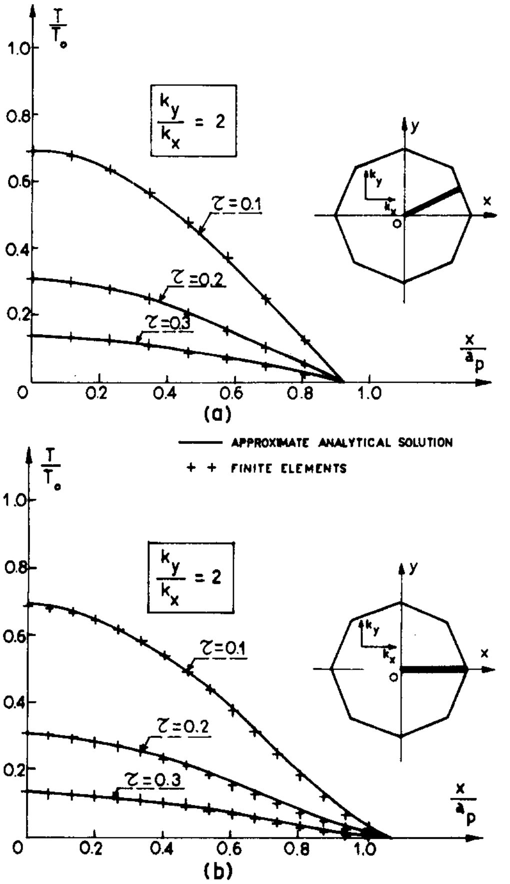

Fig. 6. Temperature distribution for a thermally orthotropic octogonal plate ($k_y/k_x = 2$), (a) along the apothem; (b) along the diagonal.

values [this fact is demonstrated easily performing the integrals in eq. (12)].

It must be emphasized now, that the actual temperature distribution in the z-plane is certainly different to that calculated in ref. [2], since the geometrical axes of the plate have rotated with respect to the material axes. This rotation is accomplished introducing the change in sign already mentioned.

Assume now that the angle of rotation is arbitrary. It will be necessary in this case to determine the mapping function using the Schwarz–Christoffel approach; the β_n's will be altered now but the functional relation (17) will be the same.

When making use of the finite element approach it will be necessary to construct a different net for each rotation.

Figs. 4 and 5 compare the dimensionless temperature, T/T_0 (a) along the apothem and (b) along the diagonal, for several values of the parameter $\tau = k_x t/a_p^2 c_p \rho$ for a square plate with $k_y/k_x = 2$ and 3, respectively.

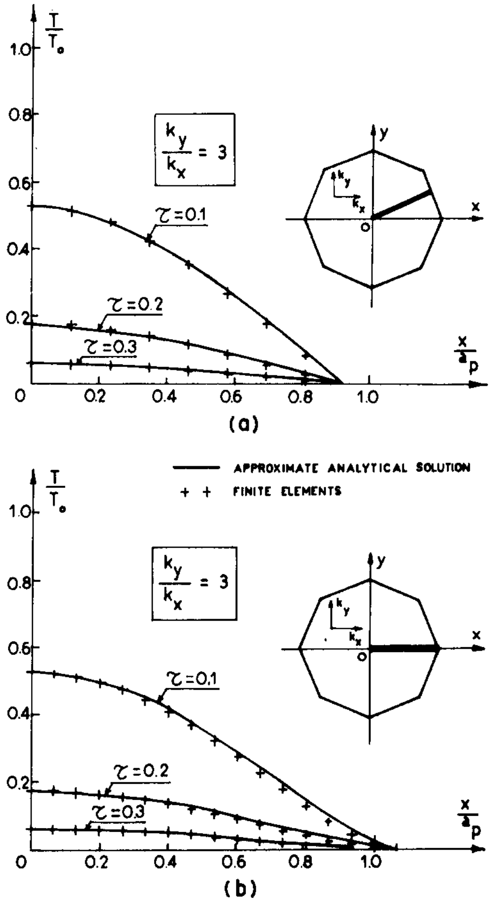

Fig. 7. Temperature distribution for a thermally orthotropic octogonal plate ($k_y/k_x = 3$), (a) along the apothem; (b) along the diagonal.

Figs. 6 and 7 deal with an octagonal configuration.

In general, reasonably good agreement is obtained between the approximate analytical solution and the finitie element solution, especially considering the simplicity of the coordinate functions employed in the analysis.

5. Conclusions

A unified approach is presented which allows for a simple solution of an important type of thermomechanical problem. Admittedly, the approach is not as general as the finite element method.

On the other hand, this study constitutes one of the few independent verifications of the relative accuracy of a finite element algrotihm in a rather complex diffusion-type problem.

Acknowledgements

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