

## Perturbation theory for operational quantum non-Markovianity

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The definition of memory in operational approaches to quantum non-Markovianity depends on the statistical properties of different sets of outcomes related to successive measurement processes performed over the system of interest. Using projector techniques we develop a perturbation theory that enable to expressing both joint probabilities and outcome correlations in terms of the unperturbed system density-matrix propagator. This object defines the open system dynamics in the absence of measurement processes. Successive series terms, which are scaled by the system-environment interaction strength, consist in a convolution structure involving system propagators weighted by higher-order bath correlations. The formalism is corroborated by studying different dynamics that admit an exact description. Using the perturbative approach, unusual memory effects induced by the interplay between the system-environment interaction and measurement processes are found in finite temperature reservoirs.

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### I. INTRODUCTION

Most features characterizing an open quantum system dynamics can be recovered from a *perturbative* approach to the full system-environment dynamics. For example, the Born-Markov approximation (BMA) [1–3] is able to describe paradigmatic phenomena like decoherence and dissipation. Even more, *quantum Markovianity* was originally related to this “first-order” approximation [4,5]. This association relies on the local-in-time property of the density-matrix evolution. Higher-order contributions to the system quantum master equation, which may be obtained from projector techniques [6], were then related to quantum memory effects [2].

Over the past decade, the previous point of view was reviewed significantly. Instead of the BMA, the hallmark of quantum Markovianity becomes the theory of quantum dynamical semigroups [7]. In this alternative scenario [8,9], memory effects are determined from different departures the system *density-matrix propagator* may develop with respect to a “Markovian” Lindblad dynamics [10–26]. While this approach is consistent, departures from BMA lose their relation with quantum memory effects; neither compatibility with the classical notion of non-Markovianity is achieved.

Alternative *operational approaches* [27] to quantum non-Markovianity have been introduced recently [28–34]. Instead of focusing on mappings from density operators to density operators, the presence of memory is determined from the statistical properties of different outcomes obtained from successive *measurement processes* performed during the system evolution. In this alternative context, some remarkable features are achieved. For example, consistence with the classical definition of non-Markovianity is immediate. In fact, given a sequence of measurement outcomes, Markovianity can be

checked through the corresponding conditional probabilities [28]. In addition, and also in contrast to previous nonoperational approaches, any possible dynamical departure from BMA renders the dynamic non-Markovian [29,30]. This is a relevant property because memory effects may develop even when the system propagator satisfies a completely positive divisibility (semigroup) property [28,29] or even when it is defined by a time-independent Lindblad evolution [30,35].

The definition of quantum non-Markovianity from an operational perspective implies an intrinsic dependence of memory effects on measurement operators, which leads to a richer structure when compared to the classical (incoherent) case [31–33]. Interestingly, experimental setups for measuring quantum memory effects in an operational way were implemented recently in Refs. [33,34].

In spite of the previous properties and achievements, in contrast to the other approaches, (operational) memory effects cannot be characterized by knowing solely the system quantum master equation or propagator [34,36]. In fact, even when obtained in an *exact way*, it is insufficient to describe multitime (outcome) probability distributions and correlations. This feature has its physical origin in the modification or dependence of the environment state on system outcomes [29,30], which in consequence, due to departures from BMA, modifies the system dynamic between successive measurements. Thus, in this context, departures from Markovianity cannot be characterized through a unique system evolution. The main goal of this paper is to formulate a general theoretical tool for tackling this problem, which in turn allows us to study which underlying structure characterizes memory effects when defined in an operational way.

We formulate a perturbation theory that allows characterizing memory effects in operational approaches to quantum

non-Markovianity. It is written as a function of the *unperturbed system propagator*, which defines the open system dynamics in absence of measurement processes. The study is valid for arbitrary system-environment interactions and is performed with projector operator techniques [2,6].

The formalism is developed in the case where three system measurement processes are performed, being applied to both joint probabilities [28] and a conditional past-future (CPF) correlation [29]. The projector approach naturally leads to an expansion series in the system-environment coupling strength. We found that successive order contributions consist in a convolution term involving two system propagators weighed by higher-order bath correlations. This structure arises for both quantum and classical environmental fluctuations. These findings generalize the results found in [34], which were derived for specific system-bath interaction Hamiltonians. Generalization to an arbitrary number of measurement processes follows straightforwardly from the present results.

The validity of the formalism is confirmed by studying different dynamics that admit an exact treatment such as dephasing and decay in a bosonic bath at zero temperature. As an application, we study memory effects in the thermalization of a two-level system. We found that memory effects may be increased by raising the environment temperature. Since in general increasing the reservoir temperature leads to a decrease of memory, this unusual effect is also discussed in detail.

The paper is outlined as follows. In Sec. II we review how memory effects can be determined from joint probabilities and the CPF correlation. In Sec. III we develop the perturbation theory for both classical and quantum environment fluctuations. In Sec. IV we apply the perturbation theory to dynamics that admit an exact treatment. In addition, we study memory effects induced by thermal reservoirs. In Sec. V we provide the Conclusions. Auxiliary calculation details are presented in the Appendix.

## II. OPERATIONAL MEMORY WITNESSES

Memory effects in open quantum systems can be determined by subjecting the system to successive measurement processes and checking if the corresponding probability structure satisfies the usual Markovian definition [28]. As in a classical regime, it is simple to realize that a *minimal number* of three system observations is necessary to detect memory effects [29]. Labeling with  $x \rightarrow y \rightarrow z$  the successive measurement outcomes, their joint probability  $P(z, y, x)$  can be written as

$$P(z, y, x) = P(z|y, x)P(y|x)P(x), \quad (1)$$

where, in general,  $P(b|a)$  denotes the conditional probability of  $b$  given  $a$ . Markovianity is defined by the equality  $P(z, y, x) \stackrel{M}{=} P(z|y)P(y|x)P(x)$ , that is,  $P(z|y, x) \stackrel{M}{=} P(z|y)$ . By an explicit calculation, this property can easily be rewritten in terms of a *conditional* past-future independence, leading to the *equivalent condition*  $P(z, x|y) \stackrel{M}{=} P(z|y)P(x|y)$ . The vantage of this last formulation is that Markovianity can be easily checked with a single scalar function. In fact, defining the CPF

correlation [29],

$$C_{pf}(t, \tau)|_y = \sum_{zx} [P(z, x|y) - P(z|y)P(x|y)]O_z O_x, \quad (2)$$

Markovianity implies  $C_{pf}(t, \tau)|_y = 0$ , while  $C_{pf}(t, \tau)|_y \neq 0$  witnesses memory effects. In this equation, the sequence  $\{x\} \rightarrow y \rightarrow \{z\}$  labels the outcomes at each stage, while the *scalar* quantities  $\{O_z\}$  and  $\{O_x\}$  are the corresponding values of the observables for each outcome at the initial and final (past and future) observation times. The outcome  $y$  gives the conditional character of the correlation. The parameters  $t$  and  $\tau$  denote the time intervals between the first and second, and between the second and third measurements, respectively.

In a quantum regime, joint probabilities as well as the CPF correlation intrinsically depend on the chosen measurement processes. Each one is defined by a set of operators that determine the system state transformation upon observing an outcome (see Fig. 1 in Ref. [29]). They are denoted as  $\{\Omega_x\}$ ,  $\{\Omega_y\}$ , and  $\{\Omega_z\}$ , being normalized to the system identity matrix,  $\sum_z \Omega_z^\dagger \Omega_z = \sum_y \Omega_y^\dagger \Omega_y = \sum_x \Omega_x^\dagger \Omega_x = I$ . For simplicity, the intermediate measurement is assumed a pure projective one [28,29], that is,  $\Omega_y^\dagger \Omega_y = \delta_{y,y'} \Omega_y$ , where the number of one-dimensional projectors  $[\Omega_y \equiv |y\rangle\langle y|]$  is equal to the dimension of the system Hilbert space. Under this condition, we define quantum Markovianity by the validity of the previous statistical statements for arbitrary measurement processes [33].

Both operational memory witnesses, Eqs. (1) and (2), can be directly related. Using that  $P(z, x|y) = P(z, y, x)/P(y)$ ,  $P(z|y) = P(z, y)/P(y)$ , and  $P(x|y) = P(y, x)/P(y)$ , we get the equivalent expression

$$C_{pf}(t, \tau)|_y = \sum_{zx} \frac{O_z O_x}{P^2(y)} [P(z, y, x)P(y) - P(z, y)P(y, x)]. \quad (3)$$

Here, all statistical objects can be written in terms of the joint probability  $P(z, y, x)$  [Eq. (1)]. In fact,  $P(z, y) = \sum_x P(z, y, x)$ ,  $P(y, x) = \sum_z P(z, y, x)$ , and  $P(y) = \sum_{z,x} P(z, y, x)$ . These formulas are always valid in a classical regime, and also in the quantum case for the scheme presented here. In fact, we are considering the situation in which the three successive measurements are performed on the system and the partial probabilities are always obtained by marginating some of the outcome variables.

For the explicit calculation of  $P(z, y, x)$  or  $C_{pf}(t, \tau)|_y$  we must define the evolution of the system-environment arrangement between measurements. Both (total) unitary dynamics and stochastic Liouville dynamics are considered. In these two situations we introduce the usual partition of system and environment to distinguish between accessible and inaccessible degrees of freedom, respectively.

### A. Unitary system-environment dynamics

We assume that the system and the environment are described by a unitary evolution with Hamiltonian  $H_T(t)$ . The total density matrix  $\rho_t^{se}$  evolves as

$$\frac{d}{dt} \rho_t^{se} = \mathcal{L}_t^{se}[\rho_t^{se}], \quad \mathcal{L}_t^{se}[\bullet] = -i[H_T(t), \bullet]. \quad (4)$$

As usual, the total Hamiltonian is written as  $H_T(t) = H_s + H_e + H_I$ . Each contribution corresponds to the system, the environment, and their interaction Hamiltonian, respectively. The previous equation can be integrated as  $\rho_t^{se} = \mathcal{E}_{t,0}[\rho_0^{se}]$ , where the bipartite propagator between two arbitrary times  $t_a < t_b$  is

$$\mathcal{E}_{t_b,t_a} \equiv \left[ \exp \int_{t_a}^{t_b} dt' \mathcal{L}_{t'}^{se} \right]. \quad (5)$$

Here  $[\dots]$  denotes a time-ordering operation, which is necessary due to the dependence of  $\mathcal{L}_{t'}^{se}$  on time. This case arises, for example, when working in an interaction representation or when the system is submitted to an external time-dependent field.

The system density matrix follows from a partial trace over the environmental degrees of freedom,  $\rho_t = \text{Tr}_e(\rho_t^{se})$ . Thus

$$\rho_t = \Lambda_{t,t_0}[\rho_0] \equiv \text{Tr}_e(\mathcal{E}_{t,t_0}[\rho_0 \otimes \sigma_0]), \quad (6)$$

where  $\Lambda_{t,t_0}$  is the system density-matrix propagator. For simplicity, we assume  $t_0 = 0$  and separable initial conditions,  $\rho_0^{se} = \rho_0 \otimes \sigma_0$ , where  $\rho_0$  and  $\sigma_0$  are the initial states of the system and the bath, respectively. Accordingly, it means that the three successive measurements corresponding to the

outcomes  $\{x\}$ ,  $\{y\}$ , and  $\{z\}$  are performed at times 0,  $t$ , and  $t + \tau$ , respectively.

From standard quantum measurement theory, the expression for the three-joint probability is [37]

$$P(z, y, x) = \text{Tr}_{se}(E_z \mathcal{E}_{t+\tau,t}[\rho_y \otimes \text{Tr}_s(E_y \mathcal{E}_{t,0}[\tilde{\rho}_x \otimes \sigma_0])]), \quad (7)$$

where  $\tilde{\rho}_x \equiv \Omega_x \rho_0 \Omega_x^\dagger$  and  $E_i \equiv \Omega_i^\dagger \Omega_i$ . With the normalization  $\tilde{\rho}_x/P(x)$ , where  $P(x) = \text{Tr}_s[\tilde{\rho}_x] = \text{Tr}_s[E_x \rho_0]$ ,  $\tilde{\rho}_x$  defines the system state given that the outcome  $x$  was observed in the first measurement. Similarly,  $\rho_y = E_y = |y\rangle\langle y|$  ( $\text{Tr}_s[\rho_y] = 1$ ) is the collapsed (pure) system state after the second measurement. We notice that in Eq. (7) the evolution in the interval  $(0, t)$  can be written in terms of the unperturbed system propagator  $\Lambda_{t,0}$  defined in Eq. (6). Nevertheless, this object is insufficient to describe the dynamics in the interval  $(t, t + \tau)$  because the initial bath state does not remain unchanged,  $\sigma_0 \rightarrow \text{Tr}_s(E_y \mathcal{E}_{t,0}[\tilde{\rho}_x \otimes \sigma_0])$ . In fact, this feature is a witness of memory effects [29] whose description, for arbitrary system-environment interactions, is performed in the following section.

From Eq. (7), the CPF correlation [Eq. (3)] can be written as

$$C_{pf}(t, \tau)|_y = \frac{1}{P^2(y)} \sum_{zx} O_z O_x \text{Tr}_{se}(E_z \mathcal{E}_{t+\tau,t}[\rho_y \otimes \text{Tr}_s(E_y \mathcal{E}_{t,0}[\tilde{\varphi}_{yx} \otimes \sigma_0])]). \quad (8)$$

The *auxiliary* matrix  $\tilde{\varphi}_{yx}$  is defined in the system Hilbert space, and reads  $\tilde{\varphi}_{yx} \equiv \tilde{\rho}_x P(y) - \tilde{\rho} P(y, x)$ , where in addition  $\tilde{\rho} \equiv \sum_{x'} \tilde{\rho}_{x'}$ . Explicitly, it reads

$$\tilde{\varphi}_{yx} = \tilde{\rho}_x \text{Tr}_s(E_y \Lambda_{t,0}[\tilde{\rho}]) - \tilde{\rho} \text{Tr}_s(E_y \Lambda_{t,0}[\tilde{\rho}_x]). \quad (9)$$

Similarly, the probability  $P(y)$  is given by

$$P(y) = \sum_{x'} \text{Tr}_s(E_y \Lambda_{t,0}[\tilde{\rho}_{x'}]). \quad (10)$$

From the previous two expressions, we notice that both  $\tilde{\varphi}_{yx}$  and  $P(y)$  can be written in terms of the unperturbed system propagator  $\Lambda_{t,0}$ . On the other hand, it is simple to show that the matrix  $\tilde{\varphi}_{yx}$  never vanishes. In fact, imposing the condition  $\tilde{\varphi}_{yx} = 0$  in Eq. (9), after performing a sum over  $y$ , using that  $\sum_y E_y = \text{I}$ , and that  $\Lambda_{t,0}$  preserves trace, jointly with  $P(x) = \text{Tr}_s(E_x \rho_0)$ , it follows  $\tilde{\rho}_x/P(x) = \tilde{\rho}$ . As this equality must be valid for arbitrary  $x$ , it is an incongruence whenever the first measurement has more than one possible outcome.

We notice that Eq. (8), disregarding the sum operation and under the replacement  $\tilde{\varphi}_{yx} \rightarrow \tilde{\rho}_x$ , has the same structure as Eq. (7). This similitude allows us to formulate a perturbation theory that straightforwardly applies to both kinds of objects. The same relation is also valid for higher statistical objects (Sec. IV C).

### B. Stochastic Liouville dynamics

In addition, we deal with the case in which the open system evolution is defined by a stochastic Liouville dynamics,

$$\frac{d}{dt} \rho_t^{st} = \mathcal{L}_t^{st}[\rho_t^{st}], \quad \mathcal{L}_t^{st}[\bullet] = -i[H_{st}(t), \bullet], \quad (11)$$

where the effect of the environment is included as stochastic classical degrees of freedom in the Hamiltonian  $H_{st}(t)$ . The evolution can be integrated as  $\rho_t^{st} = \mathcal{E}_{t,0}^{st} \rho_0$ , where the stochastic propagator between two arbitrary times  $t_a < t_b$  is

$$\mathcal{E}_{t_b,t_a}^{st} \equiv \left[ \exp \int_{t_a}^{t_b} dt' \mathcal{L}_{t'}^{st} \right]. \quad (12)$$

As before,  $[\dots]$  denotes a time-ordering operation. The system density matrix  $\rho_t = \overline{\rho_t^{st}}$  follows after averaging over realizations (over bar symbol) of the stochastic Liouville superoperator  $\mathcal{L}_t^{st}$ . This operation is equivalent to tracing over the degrees of freedom of the environment in the case of total unitary evolution. Thus we write

$$\rho_t = \Lambda_{t,t_0}[\rho_0] \equiv \overline{\mathcal{E}_{t,t_0}^{st}[\rho_0]}, \quad (13)$$

where, as before,  $\Lambda_{t,t_0}$  ( $t_0 = 0$ ) is the system density-matrix propagator. For simplicity we assumed that the initial system state  $\rho_0$  is *uncorrelated* from the noise fluctuations.

From quantum measurement theory, it is possible to obtain [37]

$$P(z, y, x) = \overline{\text{Tr}_s(E_z \mathcal{E}_{t+\tau,t}^{st}[\rho_y]) \text{Tr}_s(E_y \mathcal{E}_{t,0}^{st}[\tilde{\rho}_x])}, \quad (14)$$

where, as before,  $\tilde{\rho}_x = \Omega_x \rho_0 \Omega_x^\dagger$  and  $E_i = \Omega_i^\dagger \Omega_i$ . Similar to the unitary case, here the (average) evolution in the interval  $(0, t)$  can be written in terms of the unperturbed propagator (13), but it is unable to describe the dynamics in the interval  $(t, t + \tau)$  because it is correlated with the previous environmental fluctuations.

From Eq. (14) it is possible to write the CPF correlation [Eq. (3)] as

$$C_{pf}(t, \tau)|_y = \sum_{zx} \frac{O_z O_x}{P^2(y)} \overline{\text{Tr}_s(E_z \mathcal{E}_{t+\tau, t}^s[\rho_y]) \text{Tr}_s(E_y \mathcal{E}_{t, 0}^s[\tilde{\varphi}_{yx}])}, \quad (15)$$

where  $\tilde{\varphi}_{yx}$  and  $P(y)$  can be read from Eqs. (9) and (10), respectively, with the propagator  $\Lambda_{t, 0}$  defined by Eq. (13).

Here, we can see that Eqs. (14) and (15) present a similar structure and both expressions can be related under the same mapping that connects Eqs. (7) and (8).

### III. PERTURBATION THEORY

The joint probabilities [Eqs. (7) and (14)] together with the associated CPF correlations [Eqs. (8) and (15)] may, in principle, be calculated in an exact way. Except for a few system-environment models [30,34], this is not the case. Thus, in general, a perturbative treatment must be introduced.

As usual in the study of quantum master equations, projector techniques [2,6] are one of the most powerful tools to formulate a perturbative treatment. The main idea behind the projector technique consists in introducing two complementary projection superoperators: one of them yields the information of the reduced system density matrix and is therefore said to give the relevant part, while the other one gives the irrelevant part [2], corresponding to the environment degrees of freedom. The formalism becomes useful because the irrelevant part can be obtained (integrated) in an exact way from the underlying system-environment evolution, providing in consequence an exact formal expression for the system dynamics.

Even when the environment state is different at the beginning of each time interval between consecutive measurements, we show that projector techniques can be applied to the present problem. Given that in nonoperational memory approaches memory effects are mainly determined from the unperturbed system density-matrix propagator, we use the projector technique considering this object as an input of the full approach.

For both unitary system-environment interactions as well as stochastic Liouville dynamics we find exact series expansions for the joint probabilities  $P(z, y, x)$ . This result straightforwardly leads to an expansion for the CPF correlation  $C_{pf}(t, \tau)|_y$ . While the complete set of functions associated to the joint probability  $P(z, y, x)$  allow one to characterize departures from a Markovian limit, we remark that  $C_{pf}(t, \tau)|_y$  allows one to witness and characterize memory effects with only one single function.

#### A. Unitary system-environment dynamics

Acting on an arbitrary bipartite state  $\rho^{se}$ , the relevant and irrelevant projector contributions are defined, respectively, by

the standard relations [6]

$$\mathcal{P}[\rho^{se}] = \text{Tr}_e(\rho^{se}) \otimes \sigma_0, \quad \mathcal{Q}[\rho^{se}] = \rho^{se} - \text{Tr}_e(\rho^{se}) \otimes \sigma_0. \quad (16)$$

Here,  $\sigma_0$  is a reference state of the bath. We assume that it coincides with the initial environment state. The case of more general initial states is dealt with in Sec. IV C.

The projectors satisfy  $\mathcal{P} + \mathcal{Q} = \text{I}_{se}$ , where  $\text{I}_{se}$  is the bipartite identity operator. Thus, introducing the operator  $\text{I}_{se}$  in front of each propagator  $\mathcal{E}$  in Eq. (7), we get

$$P(z, y, x) = \text{Tr}_{se}(E_z \mathcal{P} \mathcal{E}_{t+\tau, t}[\rho_y \otimes \text{Tr}_s(E_y \mathcal{P} \mathcal{E}_{t, 0}[\tilde{\rho}_x \otimes \sigma_0])]) \\ + \text{Tr}_{se}(E_z \mathcal{Q} \mathcal{E}_{t+\tau, t}[\rho_y \otimes \text{Tr}_s(E_y \mathcal{Q} \mathcal{E}_{t, 0}[\tilde{\rho}_x \otimes \sigma_0])]). \quad (17)$$

In deriving this expression we used that  $\text{Tr}_e(\mathcal{Q}[\rho^{se}]) = 0$ , where the equality is valid for arbitrary system-environment states  $\rho^{se}$ . Now, the goal is to write  $P(z, y, x)$  solely in terms of the unperturbed system propagator.

In terms of projectors, the system propagator [Eq. (6)] between two arbitrary times [ $t_a < t_b$ ] emerges from the relevant contribution as  $\mathcal{P} \mathcal{E}_{t_b, t_a}[\rho \otimes \sigma_0] = \Lambda_{t_b, t_a}[\rho] \otimes \sigma_0$ . Hence we notice that the first line in Eq. (17) can in fact be written in terms of two system propagators,  $\Lambda_{t_b, t_a}[\rho] = \text{Tr}_e(\mathcal{E}_{t_b, t_a}[\rho \otimes \sigma_0])$ . On the other hand, in the second line, as usual, we note that for separable initial conditions the irrelevant part  $\mathcal{Q} \mathcal{E}_{t_b, t_a}$  in the projector technique can be expressed as (see the Appendix)

$$\mathcal{Q} \mathcal{E}_{t_b, t_a} = \int_{t_a}^{t_b} dt' \mathcal{G}_{t_b, t'} \mathcal{Q} \mathcal{L}_{t'}^{se} \mathcal{P} \mathcal{E}_{t', t_a}, \quad (18)$$

where  $t_a < t_b$  are arbitrary times, and

$$\mathcal{G}_{t_b, t_a} \equiv \left[ \exp \int_{t_a}^{t_b} dt' \mathcal{Q} \mathcal{L}_{t'}^{se} \right]. \quad (19)$$

Therefore, the contribution proportional to  $\mathcal{Q} \mathcal{E}_{t_b, t_a}$  in Eq. (17) can also be written in terms of the unperturbed propagator  $\mathcal{P} \mathcal{E}_{t_b, t_a}[\rho \otimes \sigma_0]$  [Eq. (6)]. On the other hand, the relevant part (in the second line) can also be integrated as (see the Appendix)

$$\mathcal{P} \mathcal{E}_{t_b, t_a} = \mathcal{P} \mathcal{E}_{t_b, t_a} \mathcal{P} + \int_{t_a}^{t_b} dt' \mathcal{P} \mathcal{E}_{t_b, t'} \mathcal{P} \mathcal{L}_{t'}^{se} \mathcal{G}_{t', t_a} \mathcal{Q}. \quad (20)$$

This expression is of central importance for the development of the formalism because it enables one to characterize the projected system dynamics in terms of the unperturbed propagator even when considering *arbitrary initial environment states*.

Introducing explicitly Eqs. (18) and (20) in  $P(z, y, x)$ , using that  $\mathcal{P} \mathcal{Q} = 0$  and after some algebra, from Eq. (17) we get

$$P(z, y, x) = \text{Tr}_s(E_z \Lambda_{t+\tau, t}[\rho_y]) \text{Tr}_s(E_y \Lambda_{t, 0}[\tilde{\rho}_x]) + \int_0^\tau d\tau' \int_0^{t'} dt' \text{Tr}_s[E_z \Lambda_{t+\tau, t+\tau'} \text{Tr}_e(\tilde{\Phi}_{t+\tau', t}^{se}[\rho_y \otimes \text{Tr}_s(E_y \Phi_{t', t'}^{se} \Lambda_{t', 0}[\tilde{\rho}_x] \otimes \sigma_0)])], \quad (21)$$

where the system propagator  $\Lambda_{t_b,t_a}$  is defined by Eq. (6). For shortening the expression we introduced the system-environment superoperators

$$\tilde{\Phi}_{t_b,t_a}^{se} = \mathcal{L}_{t_b}^{se} \mathcal{G}_{t_b,t_a}, \quad \Phi_{t_b,t_a}^{se} \equiv \mathcal{G}_{t_b,t_a} \mathcal{Q} \mathcal{L}_{t_a}^{se}, \quad (22)$$

where  $t_a < t_b$  are arbitrary times.

Equation (21) is the main result of this section. It expresses  $P(z, y, x)$  in an *exact* way as a function of the unperturbed system propagator. We notice that the first contribution corresponds to a Markovian description, where  $P(z, y, x) \stackrel{M}{=} P(z|y)P(y|x)P(x)$  with  $P(z|y) \stackrel{M}{=} \text{Tr}_s(E_z \Lambda_{t+\tau,t}[\rho_y])$  and  $P(y|x)P(x) \stackrel{M}{=} \text{Tr}_s(E_y \Lambda_{t,0}[\tilde{\rho}_x])$  with  $P(x) = \text{Tr}_s(\tilde{\rho}_x) = \text{Tr}_s(E_x \rho_0)$ .

Consistently, the second (integral) contribution in Eq. (21) takes into account memory effects. It consists of a convolution structure involving two unperturbed system propagators [ $\Lambda_{t+\tau,t+\tau'}$  and  $\Lambda_{t',0}$ ] weighted by the ‘‘correlation’’ between the bipartite operators  $\tilde{\Phi}_{t+\tau',t}^{se}$  and  $\Phi_{t',0}^{se}$  [Eq. (22)]. We notice that these two contributions are the only ones in Eq. (21) that depend on the environment degrees of freedom. Their correlation is built from the environment trace operation,  $\text{Tr}_e(\tilde{\Phi}_{t+\tau',t}^{se} \cdots \Phi_{t',0}^{se} \cdots)$ . This object can be written as a series in the interaction strength. Similar to the case of quantum master equations [2,6], it follows naturally by introducing the expansion  $\mathcal{G}_{t_b,t_a} = \mathbb{I}_{se} + \int_{t_a}^{t_b} d\tau_1 \mathcal{Q} \mathcal{L}_{\tau_1}^{se} + \int_{t_a}^{t_b} d\tau_2 \int_{t_a}^{\tau_2} d\tau_1 \mathcal{Q} \mathcal{L}_{\tau_2}^{se} \mathcal{Q} \mathcal{L}_{\tau_1}^{se} + \cdots$  in their definition [Eq. (22)]. Thus the integral contribution in Eq. (21) can be written as a series in the interaction strength (proportional to  $\mathcal{L}_t^{se}$ ), each term involving two system propagators and bath correlations of increasing order. This final result provides us the basis for a perturbative study of memory effects defined in an operational way.

The previous series expansion straightforwardly can be extended to the CPF correlation. Using the similarity between Eqs. (7) and (8), it follows that the first (Markovian) term in Eq. (21) does not contribute to  $C_{pf}(t, \tau)|_y$ . In fact,  $\sum_x O_x \text{Tr}_s(E_y \Lambda_{t,0}[\tilde{\varphi}_{yx}]) = 0$ . Thus consistently the CPF correlation only depends on the second integral contribution, which in fact measures the memory effects. We get

$$C_{pf}(t, \tau)|_y = \sum_{z,x} \frac{O_z O_x}{P^2(y)} \int_0^t dt' \int_0^{\tau} d\tau' \Xi(z, x|y)[\tilde{\varphi}_{yx}], \quad (23)$$

where

$$\Xi(z, x|y)[\bullet] \equiv \text{Tr}_{se}(E_z \tilde{\Upsilon}_{t+\tau',t}^{se}[\rho_y \otimes \text{Tr}_s(E_y \Upsilon_{t',t'}^{se}[\bullet])]). \quad (24)$$

This term defines the integrand in Eq. (21). For shortening the expression, here we introduced the superoperator

$$\tilde{\Upsilon}_{t+\tau',t}^{se}[\bullet] \equiv \Lambda_{t+\tau,t+\tau'} \tilde{\Phi}_{t+\tau',t}^{se}[\bullet] \quad (25a)$$

and similarly

$$\Upsilon_{t',t'}^{se}[\bullet] \equiv \Phi_{t',0}^{se}[\Lambda_{t',0}[\bullet] \otimes \sigma_0]. \quad (25b)$$

The final expression (23) enables one to perform a perturbative theory for the CPF correlation developed as a series in terms of the system-environment interaction strength. In

fact, similar to the case of probabilities, expansion of the superoperators  $\tilde{\Phi}_{t_b,t_a}^{se}$  and  $\Phi_{t_b,t_a}^{se}$  [Eq. (22)] in powers of  $\mathcal{L}_t^{se}$  allows us to write

$$\Xi(z, x|y)[\bullet] = \sum_{n=1}^{\infty} \Xi^{(n)}(z, x|y)[\bullet]. \quad (26)$$

Each series contribution  $\Xi^{(n)}$  involves higher-order bath correlations. In fact, the index  $n$  labels the bath correlation order that appears in each term. In order to understand this relation we analyze the first-order case. Hence we can approximate  $\tilde{\Phi}_{t_b,t_a}^{se} = \mathcal{L}_{t_b}^{se} \mathcal{G}_{t_b,t_a} \simeq \mathcal{L}_{t_b}^{se}$ , and  $\Phi_{t_b,t_a}^{se} = \mathcal{G}_{t_b,t_a} \mathcal{Q} \mathcal{L}_{t_a}^{se} \simeq \mathcal{Q} \mathcal{L}_{t_a}^{se} = \mathcal{L}_{t_a}^{se}$ , where the last equality relies on the usual assumption  $\mathcal{P} \mathcal{L}_{t_a}^{se} \mathcal{P} = 0$  [2]. Using these approximations, from Eqs. (24) and (25) it follows that  $\Xi^{(1)}$  consists of the convolution of the two system propagators  $\Lambda_{t+\tau,t+\tau'}$  and  $\Lambda_{t',0}$  weighted by the ‘‘environment correlation’’  $\text{Tr}_e(\mathcal{L}_{t+\tau'}^{se} \cdots \mathcal{L}_{t'}^{se} \sigma_0)$ .

An explicit expression of  $\Xi^{(1)}$  in terms of ‘‘operator bath correlations’’ can only be obtained after knowing the specific form of the bipartite Liouville superoperator,  $\mathcal{L}_t^{se}[\bullet] = -i[H_T(t), \bullet]$ . As usual, we consider the bipartite Hamiltonian

$$H_T(t) = \sum_{\mu} S_t^{\mu} \otimes B_t^{\mu}. \quad (27)$$

Here, the dummy index  $\mu$  labels different contributions of system and bath operators,  $S_t^{\mu}$  and  $B_t^{\mu}$ , respectively. Equation (27) is written in an interaction representation with respect to the bare system and bath Hamiltonians. Assuming, as usual, that expectation values of the bath operators are null,  $\text{Tr}_e(B_t^{\mu} \sigma_0) = 0$ , and considering Hermitian operators, from Eqs. (24) and (25), based on the previous first-order approximations, we get

$$\begin{aligned} \Xi^{(1)}(z, x|y)[\bullet] &= \sum_{\mu,\nu} \{ \text{Tr}_s(E_z \Lambda_{t+\tau,t+\tau'}[\rho_y S_{t+\tau'}^{\mu}]) - \text{c.c.} \} \\ &\times \{ \chi_{\mu\nu}(\tau' + t') \text{Tr}_s(E_y S_{t-t'}^{\nu} \Lambda_{t',0}[\bullet]) - \text{c.c.} \}. \end{aligned} \quad (28)$$

In this explicit form, it becomes evident that  $\Xi^{(1)}$  is defined by the convolution of two system propagators weighted by the environment operator correlations. They are defined as  $\chi_{\mu\nu}(t, t') \equiv \text{Tr}_e(B_t^{\mu} B_{t'}^{\nu} \sigma_0)$  [2]. For simplicity they are assumed stationary:  $\chi_{\mu\nu}(t + \tau', t - t') = \chi_{\mu\nu}(\tau' + t')$ . The structure of (28) is similar to that found in Ref. [34] for models that admit an exact analytic calculation.

Higher-order terms  $\Xi^{(n)}$  include higher bath correlations that involve a higher number of bath operators. For bosonic environments,  $\Xi^{(n)}$  involves a product of  $n$  (two-time) correlations  $\chi_{\mu\nu}(t, t')$ . In this case, the Markovian limit can be checked straightforwardly assuming the approximation  $\chi_{\mu\nu}(\tau' + t') \approx \delta(\tau' + t')$ . Thus, in Eq. (23), the double time integral  $\int_0^t dt' \int_0^{\tau} d\tau'$  of the successive series terms vanishes, leading consistently to a vanishing CPF correlation.

We remark that the exact expressions (21) and (23) explicitly depend on the unperturbed propagator  $\Lambda_{t_b,t_a}$ . This object, when it is not available in an exact analytical way, using

standard tools [2,6], can be approximated to the same order as the joint probability or CPF correlation.

### B. Stochastic Liouville dynamics

The previous perturbation theory can also be developed for the case of stochastic Liouville dynamics, Eqs. (14) and (15). Instead of the projectors (16), here they are defined as

$$\mathcal{P}[f_{st}] = \overline{f_{st}}, \quad \mathcal{Q}[f_{st}] = f_{st} - \overline{f_{st}}, \quad (29)$$

$$P(z, y, x) = \text{Tr}_s(E_z \Lambda_{t+\tau, t}[\rho_y]) \text{Tr}_s(E_y \Lambda_{t, 0}[\tilde{\rho}_x]) + \int_0^\tau dt' \int_0^{t'} dt'' \overline{\text{Tr}_s(E_z \Lambda_{t+\tau, t+t'} \tilde{\Phi}_{t+\tau', t}^{st}[\rho_y]) \text{Tr}_s(E_y \Phi_{t', t'}^{st} \Lambda_{t', 0}[\tilde{\rho}_x])}, \quad (31)$$

where here

$$\tilde{\Phi}_{t_b, t_a}^{st} \equiv \mathcal{L}_{t_b, t_a}^{st} \mathcal{G}_{t_b, t_a}^{st}, \quad \Phi_{t_b, t_a}^{st} \equiv \mathcal{G}_{t_b, t_a}^{st} \mathcal{Q} \mathcal{L}_{t_a}^{st} \quad (32)$$

( $t_a < t_b$ ) and correspondingly

$$\mathcal{G}_{t_b, t_a}^{st} = \left[ \exp \int_{t_a}^{t_b} dt' \mathcal{Q} \mathcal{L}_{t'}^{st} \right]. \quad (33)$$

The propagator  $\Lambda_{t_b, t_a}$  is defined by Eq. (13). The CPF correlation, using the similitude of Eqs. (14) and (15), can be written from Eq. (31) as

$$C_{pf}(t, \tau)|_y = \sum_{z, x} \frac{O_z O_x}{P^2(y)} \int_0^t dt' \int_0^{t'} d\tau' \overline{\Xi(z, x|y)[\tilde{\varphi}_{yx}]}, \quad (34)$$

where

$$\overline{\Xi(z, x|y)[\bullet]} \equiv \overline{\text{Tr}_s(E_z \tilde{\Upsilon}_{t+\tau', t}^{st}[\rho_y]) \text{Tr}_s(E_y \Upsilon_{t', t'}^{st}[\bullet])}. \quad (35)$$

Similarly, we defined

$$\tilde{\Upsilon}_{t+\tau', t}^{st}[\bullet] \equiv \Lambda_{t+\tau, t+\tau'} \tilde{\Phi}_{t+\tau', t}^{st}[\bullet] \quad (36a)$$

and the stochastic superoperator

$$\Upsilon_{t', t'}^{st}[\bullet] \equiv \Phi_{t', t'}^{st}[\Lambda_{t', 0}[\bullet]]. \quad (36b)$$

Furthermore,  $\tilde{\varphi}_{yx}$  is the matrix defined by Eq. (9). From Eqs. (35) and (36) the perturbation theory follows as in the previous case, that is, after expanding  $\mathcal{G}_{t_b, t_a}^{st}$  [Eq. (33)] as a series in the stochastic Liouville operator  $\mathcal{L}_{t'}^{st}$ .

## IV. EXAMPLES AND APPLICATIONS

In this section we apply the perturbation theory for different dynamics of interest such as dephasing induced by a Gaussian nonwhite noise and dissipation induced by a non-Markovian bosonic thermal bath.

### A. Non-Markovian dephasing

We consider a two-level system driven by a dephasing stochastic Hamiltonian. The stochastic system state  $\rho_t^{st}$  evolves as

$$\frac{d}{dt} \rho_t^{st} = -i \xi(t) [\sigma_z, \rho_t^{st}], \quad (37)$$

where  $\sigma_z$  is the  $\hat{z}$ -Pauli matrix (eigenvectors  $|\pm\rangle$ ) and  $\xi(t)$  is a (real) stationary color Gaussian noise with vanishing average  $\overline{\xi(t)} = 0$  and stationary correlation  $\chi(t - t') =$

where  $f_{st}$  is an arbitrary functional of the noise fluctuations. After introducing the identity  $\mathcal{P} + \mathcal{Q} = 1$  in Eq. (14), we get

$$P(z, y, x) = \overline{\text{Tr}_s(E_z \mathcal{P} \mathcal{E}_{t+\tau, t}^{st}[\rho_y]) \text{Tr}_s(E_y \mathcal{P} \mathcal{E}_{t, 0}^{st}[\tilde{\rho}_x])} + \overline{\text{Tr}_s(E_z \mathcal{P} \mathcal{E}_{t+\tau, t}^{st}[\rho_y]) \text{Tr}_s(E_y \mathcal{Q} \mathcal{E}_{t, 0}^{st}[\tilde{\rho}_x])}. \quad (30)$$

Using similar transformations and solutions as in the previous section, for the joint probability we obtain

$\overline{\xi(t)\xi(t')} = (\gamma/2\tau_c) \exp[-|t - t'|/\tau_c]$ . We consider that the system begins in its upper state,  $\rho_0 = |+\rangle\langle +|$ . Furthermore, the three measurements are performed in the  $\hat{x}$  direction in the Bloch sphere,  $\{\Omega_x\} = \{\Omega_y\} = \{\Omega_z\} = |\hat{x}_\pm\rangle\langle \hat{x}_\pm|$ , where  $|\hat{x}_\pm\rangle = (|+\rangle \pm |-\rangle)/\sqrt{2}$  are the eigenvectors of  $\sigma_{\hat{x}}$ , the  $\hat{x}$ -Pauli matrix. Thus  $x = \pm 1$ ,  $y = \pm 1$ , and  $z = \pm 1$ . Similarly,  $O_x = x$  and  $O_z = z$ . Under the previous conditions, both the joint probabilities and CPF correlation can be obtained in an exact analytical way. Explicit expressions can be found in Ref. [30] [ $(\gamma/2\tau_c) \leftrightarrow g^2$ ]. Similarly, for this model it is possible to obtain explicit recursive relations and expressions for the successive series terms [Eq. (26)], which are of order  $(\gamma\tau_c)^n$ . Due to the symmetry of the problem, the first-order contribution vanishes.

In Fig. 1 we plot the CPF correlation  $C_{pf}(t, \tau)|_y$  at equal measurement time intervals  $t = \tau$ , for different noise correlation times. Both the exact expression and the perturbation theory estimation are shown. The unperturbed system propagator  $\Lambda_{t_b, t_a}$  [Eq. (13)] was taken as the exact one. Similar to the exact expression, the CPF correlation obtained by adding successive series terms is independent of the conditional  $y = \pm 1$ . We found that for smaller noise correlation times the convergence to the exact expression is increased, which shows the consistence of the perturbation theory. Furthermore, we checked that, to the same order, the set of eight joint probabilities  $P(z, y, x)$  [Eq. (31)] is definite positive. This feature also supports the developed formalism.

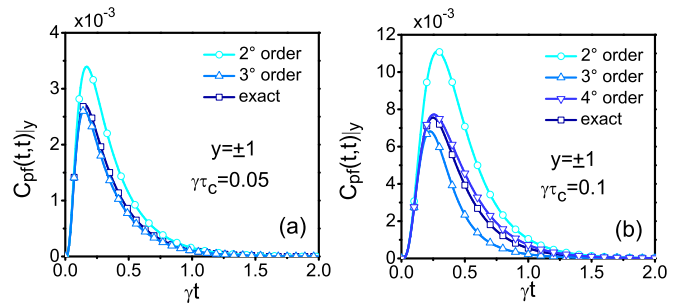


FIG. 1. CPF correlation  $C_{pf}(t, t)|_y$  ( $y = \pm 1$ ) obtained by adding successive contributions in the perturbation theory for the dephasing dynamics (37). The noise correlation is  $\chi(t) = (\gamma/2\tau_c) \exp[-|t|/\tau_c]$ . The three measurements are performed in the  $\hat{x}$ -Bloch direction, while the system begins in its upper state. The parameters in (a) and (b) respectively are  $\gamma\tau_c = 0.05$  and  $0.1$ .

### B. Non-Markovian bosonic bath

The decay of a two-level system in a bosonic environment is described by the total Hamiltonian [2]

$$H_{\text{tot}} = \frac{\omega_0}{2} \sigma_z + \sum_k \omega_k b_k^\dagger b_k + \sum_k (g_k \sigma_+ b_k + g_k^* \sigma_- b_k^\dagger), \quad (38)$$

where  $[b_k, b_k^\dagger] = 1$  are the creation-annihilation bosonic operators and  $\sigma_+ = |+\rangle\langle -|$ ,  $\sigma_- = |-\rangle\langle +|$  are the raising and lowering operators of the system. Memory effects in this dynamics can also be analyzed in an operational approach to quantum non-Markovianity.

We consider two different measurement schemes. In the first one, the three measurements are performed in  $\hat{z}$ -Bloch direction ( $\hat{z}$ - $\hat{z}$ - $\hat{z}$  scheme), while in the second one, the first and last measurements are performed in the  $\hat{x}$ -Bloch direction, with the intermediate one in the  $\hat{z}$  direction ( $\hat{x}$ - $\hat{z}$ - $\hat{x}$  scheme). These observables are defined in the representation interaction with respect to the system and bath free evolutions. In this frame, the total Hamiltonian reads

$$H_{\text{tot}}(t) = \sigma_+ B(t) + \sigma_- B^\dagger(t), \quad (39)$$

where  $B(t) = \sum_k g_k b_k \exp[+i(\omega_0 - \omega_k)t]$ . Furthermore, the initial bipartite state is taken as

$$\rho_0^{\text{se}} = |\psi_0\rangle\langle\psi_0| \otimes \sigma_0, \quad |\psi_0\rangle = (a|+\rangle + b|-\rangle), \quad (40)$$

with normalized coefficients  $a$  and  $b$ . The initial bath state  $\sigma_0$  is taken as a thermal one with respect to the environment bare Hamiltonian  $\sum_k \omega_k b_k^\dagger b_k$ . For both measurement schemes, the perturbation theory enables us to study the dependence of memory effects with temperature. Given the bosonic property of the bath, its complete set of (operator) correlations can be written in terms of only two:

$$\chi_\downarrow(t) \equiv \text{Tr}_e[B(t)B^\dagger \sigma_0], \quad \chi_\uparrow(t) \equiv \text{Tr}_e[B^\dagger(t)B \sigma_0]. \quad (41)$$

*Zero temperature.* For different physical arrangements, the environment temperature can be (effectively) taken as null. Thus  $\sigma_0 = |0\rangle\langle 0|$ , where  $|0\rangle \equiv \prod_k |0\rangle_k$ . Each state  $|0\rangle_k$  corresponds to the vacuum state of each bosonic mode. As is well known [2], in this case the full system-environment dynamic admits a simple analytical solution, given also an exact expression for the unperturbed system propagator  $\Lambda_{I_b, t_a}$  [Eq. (6)]. Furthermore, the CPF correlation and joint probabilities can also be calculated in an exact way. In fact, the open system dynamics and the CPF correlation have been implemented and measured in a photonic setup [34].

We consider a Lorentzian spectral bath density. Thus the environment correlations read  $\chi_\downarrow(t) = (\gamma/2\tau_c) \exp[-|t|/\tau_c]$ , while the zero temperature condition leads to  $\chi_\uparrow(t) = 0$ . In Fig. 2 we plot the CPF correlation  $C_{pf}(t, t)|_y$  at equal time intervals for both measurement schemes and the conditional  $y = -1$ . In the  $\hat{z}$ - $\hat{z}$ - $\hat{z}$  scheme [(a) and (b)], the first-order contribution vanishes. Similar to the previous case (at second order), a decrease in the bath correlation time leads to a higher convergence with the exact analytical result [34]. On the other hand, in the  $\hat{x}$ - $\hat{z}$ - $\hat{x}$  scheme [(c) and (d)], the first-order contribution coincides with the exact solution. Thus, while higher-order contributions do not vanish, their addition cancels out. These results also support the consistence of the perturbation theory. In addition, we found that, to the same

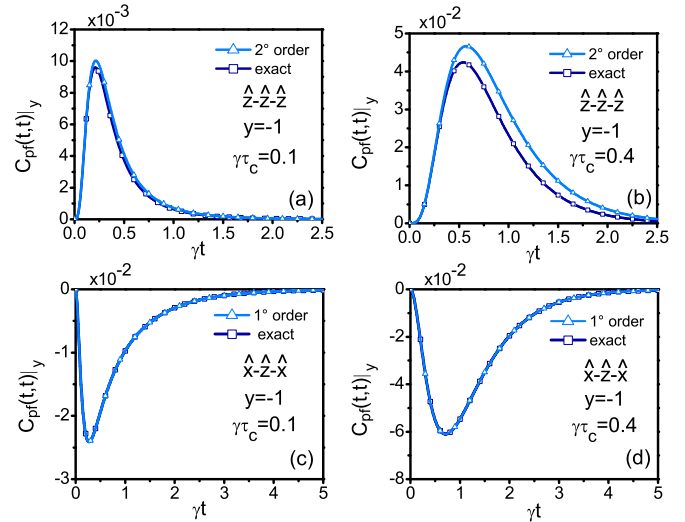


FIG. 2. CPF correlation  $C_{pf}(t, t)|_y$  ( $y = -1$ ) obtained in a perturbative way for the dissipative dynamics (39) with environment at zero temperature. Its correlations are  $\chi_\downarrow(t) = (\gamma/2\tau_c) \exp[-|t|/\tau_c]$  and  $\chi_\uparrow(t) = 0$ . Panels (a) and (b) correspond to the  $\hat{z}$ - $\hat{z}$ - $\hat{z}$  measurement scheme, while (c) and (d) to the  $\hat{x}$ - $\hat{z}$ - $\hat{x}$  scheme. The bath correlation time  $\tau_c$  is indicated in each plot. In all cases, the initial system state is  $|\psi_0\rangle = (\sqrt{p}|+\rangle + \sqrt{1-p}|-\rangle)$  with  $p = 0.8$ .

order, all joint probabilities  $P(z, y, x)$  [Eq. (21)] are definite positive.

For the conditional  $y = +1$ , the exact calculation of the CPF correlation leads to [34]

$$C_{pf}(t, \tau)|_{y=+1} \stackrel{\text{z-z-z}}{=} 0, \quad C_{pf}(t, \tau)|_{y=+1} \stackrel{\text{x-z-x}}{=} 0. \quad (42)$$

In this case, the joint probabilities  $P(z, y, x)$  can be written as [38]

$$P(z, +1, x) \stackrel{\text{z-z-z}}{=} P(z|+1)P(+1|x)P(x) \quad (y \leftrightarrow +1) \quad (43)$$

and for the second scheme as

$$P(z, +1, x) \stackrel{\text{x-z-x}}{=} P(z)P(+1)P(x) \quad (y \leftrightarrow +1). \quad (44)$$

These expressions correspond to the first contribution in Eq. (21), while the integral contribution vanishes. Thus, in this restricted case ( $y = +1$ ), the Markov property is fulfilled leading to a vanishing CPF correlation and, consequently, the perturbation theory loses its meaning.

*Finite temperature.* For finite temperature, a simple expression for the unperturbed system propagator is not available. In addition, neither the CPF correlation nor the joint probabilities can be obtained in an exact analytical way. Nevertheless, this case can be dealt with the developed perturbation theory.

At finite temperature, both bath correlations [Eq. (41)] must be considered. As a model, we take  $\chi_\downarrow(t) = (\bar{n} + 1)(\gamma/2\tau_c) \exp[-|t|/\tau_c]$  and  $\chi_\uparrow(t) = \bar{n}(\gamma/2\tau_c) \exp[-|t|/\tau_c]$ , where  $\bar{n}$  is the average number of bosonic bath excitations at the natural frequency of the system. When  $\bar{n} = 0$  the previous Lorentzian case at null temperature is recovered. This correlation model arises when the dependence on frequency of the number of thermal bath excitations is almost a flat function around the natural system frequency [3]. In this

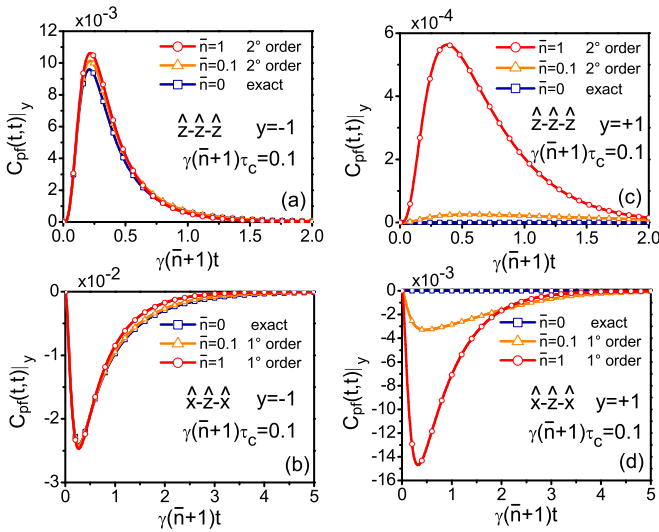


FIG. 3. CPF correlation  $C_{pf}(t, t)_y$  obtained in a perturbative way for the dissipative dynamics (39) with an environment at finite temperature. Its correlations are  $\chi_{\downarrow}(t) = (\bar{n} + 1)(\gamma/2\tau_c) \exp[-|t|/\tau_c]$  and  $\chi_{\uparrow}(t) = \bar{n}(\gamma/2\tau_c) \exp[-|t|/\tau_c]$ . Panels (a) and (b) correspond to  $y = -1$  for the  $\hat{z}-\hat{z}-\hat{z}$  and  $\hat{x}-\hat{z}-\hat{x}$  measurement schemes, respectively. In (c) and (d)  $y = +1$ . The bath correlation parameters are indicated in each plot. In all cases, the initial system state  $|\psi_0\rangle$  is the same as in Fig. 2.

approximation, temperature increases the “intensity” of the environment fluctuations, while their correlation time is independent of it. Consistently, the unperturbed density-matrix propagator is taken as the (exact) zero-temperature propagator [2], with an extra similar contribution that takes into account (thermally induced) transitions from the lower to the upper system state.

By using the previous assumptions, in Fig. 3 we plot the CPF correlation obtained from the perturbation theory [Eqs. (23) and (26)]. For both measurement schemes ( $\hat{z}-\hat{z}-\hat{z}$  and  $\hat{x}-\hat{z}-\hat{x}$ ), and for the conditional  $y = -1$  [Figs. 3(a) and 3(b)], the memory effects (amplitude of the CPF correlation) weakly depend on temperature. Small departures with respect to the vanishing temperature case (Fig. 2) are observed. Due to the normalization of the time axis [ $\gamma(\bar{n} + 1)t$ ], a natural change of time scale (shrinking due to the increasing of the effective system decay rates) is not observed.

On the other hand, for the conditional  $y = +1$  [Figs. 3(c) and 3(d)] an unusual and strong dependence on temperature is observed in both measurement schemes. In fact, in this case, in the limit of vanishing temperature [ $\bar{n} \rightarrow 0$ ] the CPF correlation vanishes; see Eq. (42). This property is a consequence of the restricted Markovian property defined by Eqs. (43) and (44). We find that by increasing temperature the CPF correlation becomes not null. In fact, its maximal amplitude increases with temperature.

The previous unusual effect, that is, an increase of the memory effects with temperature, does not rely on the specific environmental properties such as the proposed correlation model. It relies on the symmetries of the problem, which are defined by the system-environment interaction and the quantum measurement processes. For the conditional  $y = +1$ , an

increase of the bath temperature leads to an extra dissipation channel that breaks the conditional statistical independence of the first and last (past and future) measurement outcomes. From the point of view of joint probabilities, temperature leads to extra contributions that break the Markovian property expressed by Eqs. (43) and (44). In fact, for  $y = +1$  we found that the first integral series contributions in Eq. (21) are proportional to  $\bar{n}$ , while in the previous case ( $y = -1$ ) are proportional to  $(\bar{n} + 1)$ . We checked that by increasing temperature, the memory effects saturate. In addition, a usual vanishing of memory effects with temperature can be introduced or recovered through a temperature-dependent bath correlation time.

### C. Generalizations

The present approach is generalizable to different cases of interest. First, the formalism can be extended by considering arbitrary initial conditions,  $\rho_0^{se} \neq \rho_0 \otimes \sigma_0$ . For example, the bath state can be an arbitrary one, different from the reference state in the projector definition [Eq. (16)]. In addition, system and environment may be correlated at the initial time. These situations lead to an extra term in the perturbation theory [see Eq. (A2) in the Appendix].

It is important to notice that the Markovian property strictly requires the study of higher statistical objects. For example, one may consider higher joint probabilities  $P(x_m, \dots, x_1)$  involving  $m$ -measurement processes ( $m \geq 3$ ) [28]. While in the context of open quantum system this is not necessary in general [29], the present perturbation theory can be straightforwardly extended to this situation. In fact, while the explicit forms become more complex, the calculation steps remain almost the same. In the case of  $m$ -measurement processes,  $(m - 1)$  system propagators are involved in the memory convolution term, where the corresponding weight is also defined by the environment correlations. The same result applies to higher CPF correlations [29].

### V. SUMMARY AND CONCLUSIONS

Using projector techniques, we have developed a perturbation theory for describing memory effects in operational approaches to quantum non-Markovianity, where the system dynamics is explicitly observed at different times. The formalism leads to exact expressions of both joint probabilities and correlations, which are written in terms of the unperturbed system density-matrix propagator. We worked out the minimal case of three measurement processes. Memory contributions are defined by a convolution integral involving two system propagators, where the successive series terms are weighted by higher-order bath correlations. In a bosonic or Gaussian case they can be reduced to two-point correlations. This result clarifies which structure determines memory effects in operational approaches to quantum non-Markovianity.

As examples, we applied the theory to different open system dynamics that admit an exact treatment, such as dephasing induced by stochastic Hamiltonians and decay of a two-level system in a bosonic reservoir at zero temperature. The perturbative theory consistently approaches the exact solutions. We also studied non-Markovian effects that emerge when considering thermal baths. Unusual memory effects

arise due to the interplay between the measurement process, the bipartite dynamics, and the environment temperature. We found that, depending on the chosen measurement processes and conditionals, memory effects may grow with the environment temperature. This feature can be understood from a special interplay between the previous ingredients, where an extra dissipative channel induced by the environment temperature facilitates the development of memory effects.

To conclude, our theory provides a solid basis for analyzing memory effects in operational approaches to quantum non-Markovianity. Application to other physical arrangements and statistical objects can be tackled by using the developed formalism.

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#### APPENDIX: FORMAL SOLUTIONS FROM PROJECTOR TECHNIQUES

As usual [6], a bipartite system-environment evolution,  $(d/dt)\rho_i^{se} = \mathcal{L}_i^{se}\rho_i^{se}$ , can be split as

$$\frac{d}{dt}\mathcal{P}\rho_i^{se} = \mathcal{P}\mathcal{L}_i^{se}\mathcal{P}\rho_i^{se} + \mathcal{P}\mathcal{L}_i^{se}\mathcal{Q}\rho_i^{se}, \quad (\text{A1a})$$

$$\frac{d}{dt}\mathcal{Q}\rho_i^{se} = \mathcal{Q}\mathcal{L}_i^{se}\mathcal{P}\rho_i^{se} + \mathcal{Q}\mathcal{L}_i^{se}\mathcal{Q}\rho_i^{se}, \quad (\text{A1b})$$

where the projectors  $\mathcal{P}$  and  $\mathcal{Q}$  are given by Eq. (16). The irrelevant part  $\mathcal{Q}\rho_i^{se}$  can be integrated as

$$\mathcal{Q}\rho_i^{se} = \mathcal{G}_{t,t_0}\mathcal{Q}\rho_{t_0}^{se} + \int_{t_0}^t dt' \mathcal{G}_{t,t'} \mathcal{Q}\mathcal{L}_{t'}^{se}\mathcal{P}\rho_{t'}^{se}, \quad (\text{A2})$$

where the corresponding propagator is

$$\mathcal{G}_{t,t'} = \left[ \exp \int_{t'}^t dt' \mathcal{Q}\mathcal{L}_{t'}^{se} \right]. \quad (\text{A3})$$

By introducing the solution Eq. (A2) into Eq. (A1) the evolution of the relevant part  $\mathcal{P}\rho_i^{se}$  can be written as

$$\frac{d}{dt}\mathcal{P}\rho_i^{se} = \int_{t_0}^t dt' \mathbb{K}(t,t')\mathcal{P}\rho_{t'}^{se} + \mathbb{I}_{t,t_0}, \quad (\text{A4})$$

where the exact memory kernel is

$$\mathbb{K}(t,t') = \delta(t-t')\mathcal{P}\mathcal{L}_i^{se} + \mathcal{P}\mathcal{L}_i^{se}\mathcal{G}_{t,t'}\mathcal{Q}\mathcal{L}_{t'}^{se}, \quad (\text{A5})$$

while the inhomogeneous term is

$$\mathbb{I}_{t,t_0} = \mathcal{P}\mathcal{L}_i^{se}\mathcal{G}_{t,t_0}\mathcal{Q}\rho_{t_0}^{se}. \quad (\text{A6})$$

The evolution (A4) can be solved in a formal way by noticing that the solution of the homogeneous part can be written as  $\mathcal{P}\mathcal{E}_{t,t_0}\mathcal{P}\rho_{t_0}^{se}$ . Thus the full solution is given by

$$\mathcal{P}\rho_i^{se} = \mathcal{P}\mathcal{E}_{t,t_0}\mathcal{P}\rho_{t_0}^{se} + \int_{t_0}^t dt' \mathcal{P}\mathcal{E}_{t,t'}\mathcal{P}\mathcal{L}_{t'}^{se}\mathcal{G}_{t',t_0}\mathcal{Q}\rho_{t_0}^{se}, \quad (\text{A7})$$

where we have used the explicit expression for the inhomogeneous contribution, Eq. (A6). The general solution Eqs. (A2) and (A7) support Eqs. (18) and (20), respectively.

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- [1] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Atom-Photon Interactions, Basic Process and Applications* (John Wiley & Sons, New York, 1992).
- [2] H. P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002).
- [3] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, UK, 1995).
- [4] I. de Vega and D. Alonso, Dynamics of non-Markovian open quantum systems, *Rev. Mod. Phys.* **89**, 015001 (2017).
- [5] L. Li, M. J. W. Hall, and H. M. Wiseman, Concepts of quantum non-Markovianity: A hierarchy, *Phys. Rep.* **759**, 1 (2018).
- [6] F. Haake, *Statistical Treatment of Open Systems by Generalized Master Equations*, Springer Tracts in Modern Physics Vol. 66 (Springer, New York, 1973), p. 98; H. Grabert, *Projection Operator Techniques in Nonequilibrium Statistical Mechanics*, Springer Tracts in Modern Physics Vol. 95 (Springer, New York, 1982).
- [7] R. Alicki and K. Lendi, *Quantum Dynamical Semigroups and Applications*, Lecture Notes in Physics Vol. 717 (Springer, Berlin, Heidelberg, 1987).
- [8] H. P. Breuer, E. M. Laine, J. Piilo, and V. Vacchini, Colloquium: Non-Markovian dynamics in open quantum systems, *Rev. Mod. Phys.* **88**, 021002 (2016).
- [9] A. Rivas, S. F. Huelga, and M. B. Plenio, Quantum non-Markovianity: characterization, quantification and detection, *Rep. Prog. Phys.* **77**, 094001 (2014).
- [10] H. P. Breuer, E. M. Laine, and J. Piilo, Measure for the Degree of Non-Markovian Behavior of Quantum Processes in Open Systems, *Phys. Rev. Lett.* **103**, 210401 (2009).
- [11] M. M. Wolf, J. Eisert, T. S. Cubitt, and J. I. Cirac, Assessing Non-Markovian Quantum Dynamics, *Phys. Rev. Lett.* **101**, 150402 (2008).
- [12] A. Rivas, S. F. Huelga, and M. B. Plenio, Entanglement and Non-Markovianity of Quantum Evolutions, *Phys. Rev. Lett.* **105**, 050403 (2010).
- [13] E.-M. Laine, J. Piilo, and H.-P. Breuer, Measure for the non-Markovianity of quantum processes, *Phys. Rev. A* **81**, 062115 (2010).
- [14] X.-M. Lu, X. Wang, and C. P. Sun, Quantum Fisher information flow and non-Markovian processes of open systems, *Phys. Rev. A* **82**, 042103 (2010).
- [15] A. K. Rajagopal, A. R. Usha Devi, and R. W. Rendell, Kraus representation of quantum evolution and fidelity as manifestations of Markovian and non-Markovian forms, *Phys. Rev. A* **82**, 042107 (2010).
- [16] D. Chruściński, A. Kossakowski, and A. Rivas, Measures of non-Markovianity: Divisibility versus backflow of information, *Phys. Rev. A* **83**, 052128 (2011).
- [17] S. Luo, S. Fu, and H. Song, Quantifying non-Markovianity via correlations, *Phys. Rev. A* **86**, 044101 (2012).
- [18] S. Lorenzo, F. Plastina, and M. Paternostro, Geometrical characterization of non-Markovianity, *Phys. Rev. A* **88**, 020102(R) (2013).

- [19] D. Chruściński and S. Maniscalco, Degree of Non-Markovianity of Quantum Evolution, *Phys. Rev. Lett.* **112**, 120404 (2014).
- [20] F. F. Fanchini, G. Karpat, B. Çakmak, L. K. Castelano, G. H. Aguilar, O. Jiménez Farías, S. P. Walborn, P. H. Souto Ribeiro, and M. C. de Oliveira, Non-Markovianity through Accessible Information, *Phys. Rev. Lett.* **112**, 210402 (2014).
- [21] C. Addis, G. Brebner, P. Haikka, and S. Maniscalco, Coherence trapping and information backflow in dephasing qubits, *Phys. Rev. A* **89**, 024101 (2014).
- [22] M. J. W. Hall, J. D. Cresser, L. Li, and E. Andersson, Canonical form of master equations and characterization of non-Markovianity, *Phys. Rev. A* **89**, 042120 (2014).
- [23] P. Haikka, J. D. Cresser, and S. Maniscalco, Comparing different non-Markovianity measures in a driven qubit system, *Phys. Rev. A* **83**, 012112 (2011); C. Addis, B. Bylicka, D. Chruściński, and S. Maniscalco, Comparative study of non-Markovianity measures in exactly solvable one- and two-qubit models, *ibid.* **90**, 052103 (2014).
- [24] J. Trapani and M. G. A. Paris, Nondivisibility versus backflow of information in understanding revivals of quantum correlations for continuous-variable systems interacting with fluctuating environments, *Phys. Rev. A* **93**, 042119 (2016).
- [25] B. Bylicka, M. Johansson, and A. Acín, Constructive Method for Detecting the Information Backflow of Non-Markovian Dynamics, *Phys. Rev. Lett.* **118**, 120501 (2017).
- [26] S. Chakraborty, Generalized formalism for information backflow in assessing Markovianity and its equivalence to divisibility, *Phys. Rev. A* **97**, 032130 (2018); S. Chakraborty and D. Chruscinski, Information flow versus divisibility for qubit evolution, *ibid.* **99**, 042105 (2019).
- [27] S. Milz, F. A. Pollock, and K. Modi, An introduction to operational quantum dynamics, *Open Syst. Inf. Dyn.* **24**, 1740016 (2017); F. Costa and S. Shrapnel, Quantum causal modeling, *New J. Phys.* **18**, 063032 (2016).
- [28] F. A. Pollock, C. Rodríguez-Rosario, T. Frauenheim, M. Paternostro, and K. Modi, Operational Markov Condition for Quantum Processes, *Phys. Rev. Lett.* **120**, 040405 (2018); Non-Markovian quantum processes: Complete framework and efficient characterization, *Phys. Rev. A* **97**, 012127 (2018).
- [29] A. A. Budini, Quantum Non-Markovian Processes Break Conditional Past-Future Independence, *Phys. Rev. Lett.* **121**, 240401 (2018).
- [30] A. A. Budini, Conditional past-future correlation induced by non-Markovian dephasing reservoirs, *Phys. Rev. A* **99**, 052125 (2019).
- [31] P. Taranto, F. A. Pollock, S. Milz, M. Tomamichel, and K. Modi, Quantum Markov Order, *Phys. Rev. Lett.* **122**, 140401 (2019); P. Taranto, S. Milz, F. A. Pollock, and K. Modi, Structure of quantum stochastic processes with finite Markov order, *Phys. Rev. A* **99**, 042108 (2019).
- [32] M. R. Jørgensen and F. A. Pollock, Exploiting the Causal Tensor Network Structure of Quantum Processes to Efficiently Simulate Non-Markovian Path Integrals, *Phys. Rev. Lett.* **123**, 240602 (2019).
- [33] S. Yu, A. A. Budini, Y.-T. Wang, Z.-J. Ke, Y. Meng, W. Liu, Z.-P. Li, Q. Li, Z.-H. Liu, J.-S. Xu, J.-S. Tang, C.-F. Li, and G.-C. Guo, Experimental observation of conditional past-future correlations, *Phys. Rev. A* **100**, 050301(R) (2019).
- [34] T. de Lima Silva, S. P. Walborn, M. F. Santos, G. H. Aguilar, and A. A. Budini, Detection of quantum non-Markovianity close to the Born-Markov approximation, *Phys. Rev. A* **101**, 042120 (2020).
- [35] L. Accardi, A. Frigerio, and J. T. Lewis, Quantum stochastic processes, *Publ. RIMS, Kyoto Univ.* **18**, 97 (1982) (see Appendix section).
- [36] Y.-Y. Hsieh, Z.-Y. Su, and H.-S. Goan, Non-Markovianity, information backflow, and system-environment correlation for open-quantum-system processes, *Phys. Rev. A* **100**, 012120 (2019).
- [37] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevLett.121.240401> of Ref. [29] for a detailed derivation.
- [38] These results can straightforwardly be derived by using  $P(z, y, x) = P(z, x|y)P(y)$ , where  $P(y) = \sum_x P(y|x)P(x)$ , and using the exact probability expressions of Ref. [34].