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DESIGN OF ALTERNATING GRADIENT QUADRUPOLE LENSES

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A graphical method is described by means of which it is possible to design quadrupole lenses once locations of object and image are given. It is shown that similar procedures would be valid in the para-axial (Gauss) approximation, when applied to electron optical devices of any kind.

1. Introduction

Strong focusing[1]) of ion and electron beams has become widely used in many laboratories. A common assembly is that composed of two quadrupoles. Although this system is intrinsically astigmatic, so are generally the "objects" and the over-all result is a real, well defined image.

Different authors[2–4]) have treated the problem of designing strong focusing magnets, i.e. the problem of asigning currents and dimensions to both quadrupoles once the object position and the required location of the image are known. It seems worth while, however, to work out a method with a minimum of simplifying assumptions on the geometrical and electrical parameters. These parameters are: k_1 and k_2, field gradients in each quadrupole; d_1 and d_2, thickness of each quadrupole, and L, distance between both quadrupoles. In this paper we are going to assume that $d_1 = d_2 = d$. As a matter of fact, this only reflects t he way in which quadrupole lenses are generally constructed.

2. Optical Analogy of Quadrupole Lenses

Magnetic and electric quadrupoles behave as lenses convergent in one plane and divergent in another perpendicular to the first one. This effect can be best described by the well known matrix relations:

$$\begin{pmatrix} x_a \\ \dot{x}_a \end{pmatrix} = \begin{pmatrix} \cos \omega d & \frac{1}{\omega} \sin \omega d \\ -\omega \sin \omega d & \cos \omega d \end{pmatrix} \begin{pmatrix} x_b \\ \dot{x}_b \end{pmatrix} \qquad (1)$$

for the convergent plane, and

$$\begin{pmatrix} x_a \\ \dot{x}_a \end{pmatrix} = \begin{pmatrix} \cosh \omega d & \frac{1}{\omega} \sinh \omega d \\ \omega \sinh \omega d & \cosh \omega d \end{pmatrix} \begin{pmatrix} x_b \\ \dot{x}_b \end{pmatrix} \qquad 2)$$

for the divergent plane.

In eqs. (1) and (2) x_b and x_a refer to the distance of the particle from the z axis (optical axis of the quadrupole) before entrance to and after exit from the quadrupole field, $\dot{x}_b = (\mathrm{d}x/\mathrm{d}z)_b$, $\omega^2 = k/\mathrm{BR}$ in the magnetic case and $\omega^2 = k/2T$ in the electric case; here BR stands for the magnetic rigidity of the particle and T for its kinetic energy. Similarly, for a field-free region of length L:

$$\begin{pmatrix} x_a \\ \dot{x}_a \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_b \\ \dot{x}_b \end{pmatrix}. \qquad (3)$$

The preceding equations apply in the para-axial approximation to square shaped fields, which obviously do not represent the real case. However, it has been pointed out[5]) that calculations with the more realistic "bell" shaped fields give results that differ from those obtained with the previous equations in quantities of the order of 1%.

Let us consider now the usual two quadrupole lens. We have a CD plane (first convergent and

[1]) E. D. Courant, M. St. Livingston and H. S. Snyder, Phys. Rev. 88 (1952) 1190.

[2]) A.V. Crewe *et al.*, Use of strong focus magnet, internal report (The Enrico Fermi Institute for Advanced Studies).

[3]) D. A. Bromley and J. A. Bruner, AEC/NYO-3823 (1954).

[4]) H. Schneider, Nuclear Instruments 1 (1957) 268.

[5]) P. Grivet, A. Septier and J. Hue, CERN Symposium 1 (1956) 192.

then divergent) and a DC plane (first divergent and then convergent). The corresponding matrices are obtained by multiplication of those appearing in eqs. (1), (2) and (3) in the same order as the particle actually "sees" the respective fields. We call ω_1 and ω_2 the "strengths" of the first and second quadrupoles, respectively. The mean strength is defined as

$$\omega = (\omega_1\omega_2)^{\frac{1}{2}} \tag{4}$$

and the asymmetry parameter as

$$\gamma = \frac{\omega_1}{\omega} = \frac{\omega}{\omega_2}. \tag{5}$$

If we further take the quadrupole thickness d as unit of length, we have for the matrix elements

$$M_{11} = \cosh\frac{\omega}{\gamma}\cos\omega\gamma - L\omega\gamma\cosh\frac{\omega}{\gamma}\sin\omega\gamma - \gamma^2\sinh\frac{\omega}{\gamma}\sin\omega\gamma \tag{6a}$$

$$M_{12} = \frac{1}{\omega\gamma}\cosh\frac{\omega}{\gamma}\sin\omega\gamma + L\cosh\frac{\omega}{\gamma}\cos\omega\gamma + \frac{\gamma}{\omega}\sinh\frac{\omega}{\gamma}\cos\omega\gamma \tag{6b}$$

$$M_{21} = \frac{\omega}{\gamma}\sinh\frac{\omega}{\gamma}\cos\omega\gamma - L\omega^2\sinh\frac{\omega}{\gamma}\sin\omega\gamma - \omega\gamma\cosh\frac{\omega}{\gamma}\sin\omega\gamma \tag{6c}$$

$$M_{22} = \frac{1}{\gamma^2}\sinh\frac{\omega}{\gamma}\sin\omega\gamma + L\frac{\omega}{\gamma}\sinh\frac{\omega}{\gamma}\cos\omega\gamma + \cosh\frac{\omega}{\gamma}\cos\omega\gamma \tag{6d}$$

(CD plane) and

$$N_{11} = \cosh\omega\gamma\cos\frac{\omega}{\gamma} + L\omega\gamma\cos\frac{\omega}{\gamma}\sinh\omega\gamma + \gamma^2\sin\frac{\omega}{\gamma}\sinh\omega\gamma \tag{7a}$$

$$N_{12} = \frac{1}{\omega\gamma}\cos\frac{\omega}{\gamma}\cosh\omega\gamma + L\cos\frac{\omega}{\gamma}\cosh\omega\gamma + \frac{\gamma}{\omega}\sin\frac{\omega}{\gamma}\cosh\omega\gamma \tag{7b}$$

$$N_{21} = \omega\gamma\cos\frac{\omega}{\gamma}\sinh\omega\gamma - L\omega^2\sin\frac{\omega}{\gamma}\sinh\omega\gamma - \frac{\omega}{\gamma}\sin\frac{\omega}{\gamma}\cosh\omega\gamma \tag{7c}$$

$$N_{22} = \cos\frac{\omega}{\gamma}\cosh\omega\gamma - L\frac{\omega}{\gamma}\sin\frac{\omega}{\gamma}\cosh\omega\gamma - \frac{1}{\gamma^2}\sin\frac{\omega}{\gamma}\sinh\omega\gamma \tag{7d}$$

(DC plane). Inspection shows immediately the following relations to hold

$$M_{11}\left(\omega, \frac{1}{\gamma}\right) = N_{22}(\omega, \gamma) \tag{8a}$$

$$M_{12}\left(\omega, \frac{1}{\gamma}\right) = M_{12}(\omega, \gamma) \tag{8b}$$

$$M_{21}\left(\omega, \frac{1}{\gamma}\right) = N_{21}(\omega, \gamma) \tag{8c}$$

$$M_{22}\left(\omega, \frac{1}{\gamma}\right) = N_{11}(\omega, \gamma) \tag{8d}$$

$$M_{11}M_{22} - M_{12}M_{21} = N_{11}N_{22} - N_{12}N_{21} = 1\,. \tag{9}$$

We are thus left with three independent relations, from where in principle the three important parameters ω, γ and L could be solved. This appears as an impossible task from the analytical point of view. We are then going to look for a graphical method, based on the following well known optical relations:

$$f = -\frac{1}{M_{21}} \tag{10a}$$

$$z\pi_0 = z_1 + \frac{M_{22} - 1}{M_{21}} \tag{10b}$$

$$z\pi_1 = z_2 - \frac{M_{11} - 1}{M_{21}} \tag{10c}$$

$$\frac{1}{p_0} + \frac{1}{p_1} = \frac{1}{f}\,. \tag{10d}$$

Eqs. (10a) to (10d) are valid for the CD plane; the corresponding equations for the DC plane are obtained just replacing the M's by the N's. The meaning of the distances p_0, p_1 and f, and ordinates z_1, z_2, $z_{\pi 0}$ and $z_{\pi 1}$ is shown in the optical construction of fig. 1; π_0 and π_1 refer to the object and image principal planes. It is easy to see from fig. 1 and eqs. (10) that

$$s_0 = p_0 - \frac{M_{22} - 1}{M_{21}} \tag{11a}$$

$$s_1 = p_1 - \frac{M_{11} - 1}{M_{21}} \tag{11b}$$

for the CD plane and

$$s'_0 = p'_0 - \frac{N_{22} - 1}{N_{21}} \tag{12a}$$

$$s'_1 = p'_1 - \frac{N_{11} - 1}{N_{21}} \tag{12b}$$

for the DC plane.

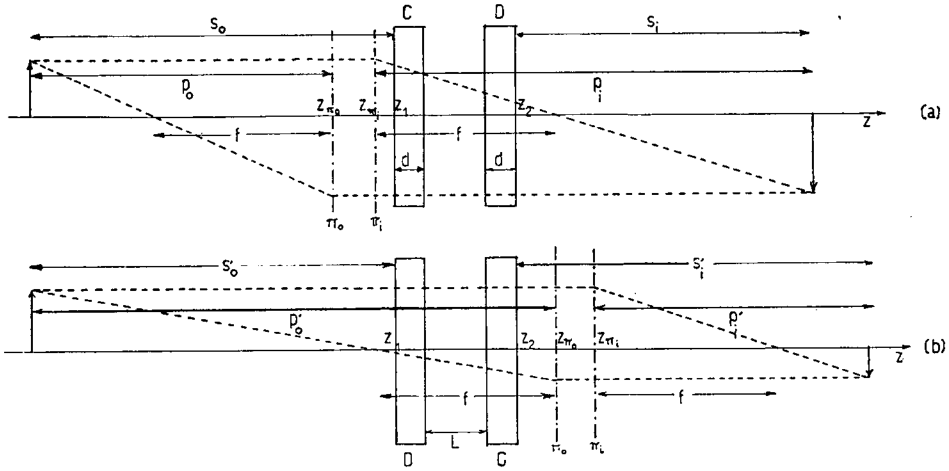

Fig. 1. Optical construction that relates object and image; (a) CD plane, (b) DC plane.

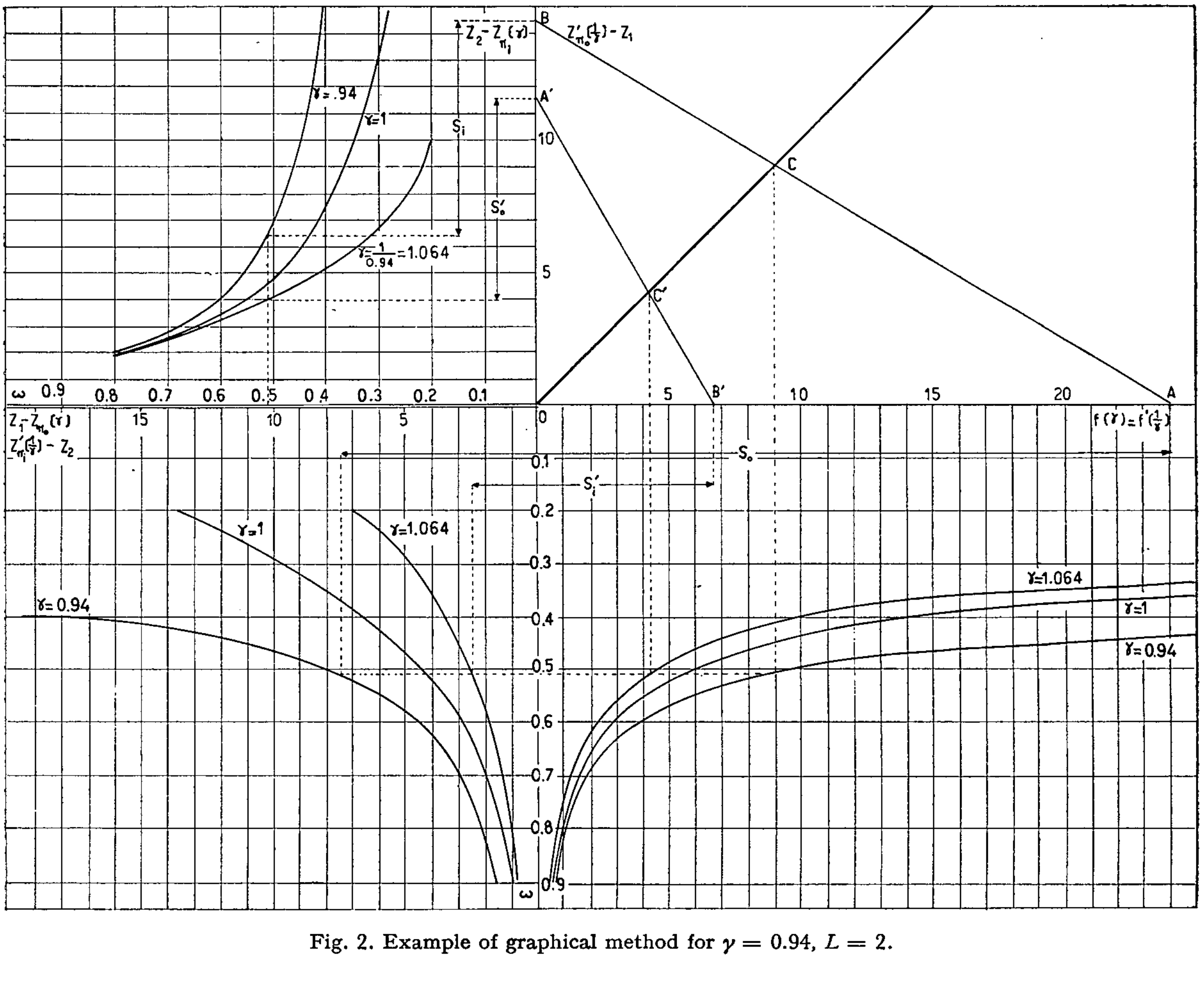

Fig. 2. Example of graphical method for $\gamma = 0.94$, $L = 2$.

3. Trial-and-Error Graphical Method

The whole preceding discussion is condensed in fig. 2, drawn for the particular value $L = 2$. In the first quadrant we have the geometrical construction resulting from (10d): if OA and OB stand for p_0 and p_1, (10d) implies that A, B and C must be aligned. The second quadrant contains the curves corresponding to

$$\frac{M_{11}(\gamma)-1}{M_{21}(\gamma)} = \frac{N_{22}\left(\frac{1}{\gamma}\right)-1}{N_{21}\left(\frac{1}{\gamma}\right)}$$

as functions of ω, for three different values of the parameter $\gamma: \gamma = 0.94$, $\gamma = 1.0$ and $\gamma = 1/0.94 = 1.0638$. As (10b) and (10c) show, the ordinates are the distances (in units of d) $z_2 - z_{\pi 1}(\gamma)$ for the CD plane and $z'_{\pi 0}(1/\gamma) - z_1$ for the DC plane.

In the third and fourth quadrants the downward vertical axis is used for abscissae, ω. In the third quadrant the functions

$$-\frac{M_{22}(\gamma)-1}{M_{21}(\gamma)} = -\frac{N_{11}\left(\frac{1}{\gamma}\right)-1}{N_{21}\left(\frac{1}{\gamma}\right)}$$

[$z_1 - z_{\pi 0}(\gamma)$ in the CD plane, $z'_{\pi 1}(1/\gamma) - z_2$ in the DC plane] are plotted. We finally have the focal distance

$$f = -\frac{1}{M_{21}(\gamma)} = -\frac{1}{N_{21}\left(\frac{1}{\gamma}\right)}$$

plotted against ω for the same aforementioned values of γ in the fourth quadrant.

Suppose now we are given the values of s_0, s_1, s'_0, s'_1 (in units of d; generally $s_1 = s'_1$, because anastigmatic images are sought). We then choose an arbitrary value of ω (a good rule of thumb is that the horizontal distance through ω between corresponding curves in the third and fourth quadrants be less than s_0) and $\gamma = 1$. Fig. 2 shows how to determine points A, B and C and correspondingly s_1, which of course will not generally coincide with that previously given; but information is obtained from the graph about where to look for a more accurate value of ω, which after two or three more trials is finally found. Repeating the same process for the DC plane (in this case s'_1 [s'_0] is read in the same axis as previously s_0 [s_1]). Attainment of a different value of ω shows the need of $\gamma \neq 1$. The same trial-and-error process must be accomplished with $\gamma \neq 1$, taking now into account that because of eqs. (8), when γ curves are used for the CD plane, $1/\gamma$ curves are required for the DC plane. Of course, curves for other values of γ and graphs for other values of L could be drawn, thus covering all possibilities.

4. Discussion and Final Remarks

The method just outlined allows the determination of the required parameters with far greater accuracy than that either implied by the approximations used or needed for the mechanical and electrical design of quadrupole lenses. Final adjustments are to be done experimentally[6]). The trial and error process is not very long (after some practice an hour is enough), but it requires long and tedious calculations for the plotting of curves. Fortunately, as a rough guide one graph as the one shown here suffices.

The whole discussion in this paper applies equally well to magnetic and electric quadrupoles. Moreover, it can be shown[7]) that the matrix formulation is generally valid for electron or ion optical instruments in the paraaxial approximation. Eqs. (10), (11) and (12) are then also valid, and so is the whole procedure described on the basis of these equations; naturally, different curves would appear in each case as a result of the different dependence of the matrix elements on the fundamental parameters.

Acknowledgement

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[6]) F. B. Schull, C. E. McFarland and M. M. Bretscher, Rev. Sci. Instr. 25 (1954) 365.

[7]) P. A. Sturrock, Static and dynamic electron optics (Cambridge University Press, 1955).