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Finite Energy Sum Rules with $\pi\pi$ Veneziano Amplitudes

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Veneziano amplitudes for $\pi\pi$ scattering are analyzed with regards to finite energy sum rules. It is shown that significant discrepancies may appear in the evaluation of FESR for real parts of non-forward scattering amplitudes. On the contrary, failure of FESR for the imaginary part of the amplitude only appears when the second daughter Regge pole is included. Possible phenomenological implications are discussed.

§ 1. Introduction

Veneziano model[1] is mathematically dual[2] since it can be written alternatively as a sum of resonance-pole contributions in the direct or in the crossed channel showing Regge asymptotic behaviour when the corresponding Mandelstam variable is taken out of the real axis.

On the other hand, because of its characteristics, it is not automatic that Veneziano model satisfies duality in the original sense[3] of finite energy sum rules (FESR). Our purpose is to analyze thoroughly the duality of the model in this second sense. In fact, since the asymptotic Veneziano amplitude on the real axis contains, besides the Regge-poles contribution, a part which can be taken as Regge background integral,[4]~[7] FESR may not be satisfied.[8]

The original work[1] shows that the simple model with real trajectories satisfies FESR with integer moment for imaginary part of the amplitude up to the leading Regge pole contribution, and Yellin[9] has proved that the satisfaction is also obtained for the first Regge daughter when taking the energy cutoff half-way between two neighbouring resonances. We complete the analysis looking at the situation when lower daughters are included and considering also FESR for the real part of the amplitude.

The use of a naive Veneziano model with real trajectories for phenomenological applications is certainly questionable. However, it is reasonable to think that in order to give a meaning to a narrow-width model one must somehow

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perform an energy average of the amplitude. Therefore, our results could have implications to the broadly used fits of FESR with experimental amplitudes, particularly in connection with the difficulties in analyzing the non-forward scattering. On the other hand a possibility is sketched to determine the weight of the satellites to be added to the fundamental Veneziano block by requiring the exact fulfilment of FESR. This procedure tends to improve the agreement of the model with the widths of the most prominent $\pi\pi$ resonances.

§ 2. Finite energy sum rules

We use the Lovelace[10] amplitudes for $\pi\pi$-elastic scattering with defined isospin values in t-channel

$$A_1^t = V(s,t) - V(u,t), \qquad A_2^t = V(s,u), \tag{1}$$

with

$$V(x,y) = \frac{\Gamma(1-\alpha_\rho(x))\,\Gamma(1-\alpha_\rho(y))}{\Gamma(1-\alpha_\rho(x)-\alpha_\rho(y))}, \qquad \alpha_\rho(x) = \alpha_0 + x = \alpha_x. \tag{2}$$

It is particularly useful for our purpose to consider A_1^t, since its Regge-plane analysis in t-channel (or Khuri-plane analysis, which we shall prefer) shows an infinite set of Regge poles without fixed poles.[11]–[15] On the other hand A_2^t contains only wrong-signature fixed poles at -1, -3, ….[7] The features of A_0^t are a combination of those of the other two amplitudes.

The contribution of Khuri poles to A_1^t through a Sommerfeld-Watson transform is, with $\nu = (s-u)/4\mu$ and μ = pion mass,

$$A_1^{t*} = -\sum_{r=0}(-)^r \beta_r (-\nu)^{\alpha_t - r} \frac{1-(-)^r e^{-i\pi\alpha_t}}{\sin \pi\alpha_t} + \text{Background}, \tag{3}$$

where the explicit value of residues up to second daughter is

$$\beta_0 = -\pi \frac{(2\mu)^{\alpha_t}}{\Gamma(\alpha_t)}, \tag{4a}$$

$$\beta_1 = -\pi \frac{(2\mu)^{\alpha_t - 1}}{\Gamma(\alpha_t)} \frac{\alpha_t}{2}(c-1), \tag{4b}$$

$$\beta_2 = -\pi \frac{(2\mu)^{\alpha_t - 2}}{\Gamma(\alpha_t)} \frac{\alpha_t}{24}(\alpha_t - 1)(2 + 3c^2 - 6c - \alpha_t), \tag{4c}$$

where $c = 3\alpha_0 + 4\mu^2$.

We begin considering FESR for Im A_1^t. Since we have no fixed poles, both right and wrong-signature FESR are on an equal footing, and in fact they appear to be qualitatively equivalent. Considering the first right-signature FESR with some detail, expression (1) together with the usual $i\varepsilon$ prescription for resonances gives

$$I_1=\int_0^{\nu_c}\mathrm{Im}\,A_1{}^t d\nu=-\frac{\pi}{2\mu}\,\frac{1}{(\alpha_t+1)\Gamma(\alpha_t)}\,\frac{\Gamma(\alpha_t+N+2)}{\Gamma(N+1)}\,, \tag{5}$$

where N is the last resonance included in the energy range defined by ν_c.

For large N the use of Stirling expansion allows us to write $I_1=\sum_{i=0}I_1^{(i)}$, with the explicit form up to second non-leading order:

$$I_1^{(0)}=-\frac{\pi}{2\mu}\,\frac{1}{\Gamma(\alpha_t)}\,\frac{N^{\alpha_t+1}}{\alpha_t+1}\,, \tag{6a}$$

$$I_1^{(1)}=-\frac{\pi}{4\mu}\,\frac{\alpha_t+2}{\Gamma(\alpha_t)}N^{\alpha_t}, \tag{6b}$$

$$I_1^{(2)}=-\frac{\pi}{48\mu}\,\frac{\alpha_t}{\Gamma(\alpha_t)}(3\alpha_t^2+11\alpha_t+10)N^{\alpha_t-1}. \tag{6c}$$

We compare now expansion (6) with the contribution to the FESR of Khuri poles which, from Eq. (3), is $\sum_{r=0}\beta_r\nu_c{}^{\alpha_t+1-r}/(\alpha_t+1-r)$. For this purpose we take the cut-off as

$$\nu_c=\left(N+\sigma+1+\frac{\alpha_t}{2}-\frac{c}{2}\right)\Big/2\mu\,, \tag{7}$$

where $0<\sigma<1$, to place ν_c between the N-th and the $(N+1)$-th resonance. The introduction of the residues (4) shows that:

i) the leading term in N coincides with (6a) for any value of σ;

ii) to get agreement with first non-leading term (6b) one has to choose $\sigma=1/2$, i.e., to put ν_c half-way between two resonances;

iii) for the second non-leading contribution one cannot reproduce (6c) for any choice of σ. In fact, with $\sigma=1/2$ one obtains 11 instead of 10 for the last term in parenthesis.

Analogous discrepancies starting at second non-leading term are readily found for higher-moment right and wrong-signature FESR (for details of this type of calculations see Ref. 17).

Turning now our attention to FESR for $\mathrm{Re}A_1{}^t$, inserting again expression (1), one obtains for the first wrong-signature sum rule

$$R_1=\int_0^{\nu_c}\mathrm{Re}\,A_1{}^t d\nu=\frac{1}{2\mu}\sum_{n=0}\frac{1}{B(\alpha_t,n+1)}\ln\left|\frac{\nu_c^2}{\nu_n^2}-1\right|, \tag{8}$$

where $\nu_n=(n+1+\alpha_t/2-c/2)/2\mu$ gives the resonances position, which should be confronted to the Khuri poles contribution

$$R_K=\sum_{r=0}\beta_r tg\frac{\pi}{2}(\alpha_t-r)\,\nu_c{}^{(\alpha_t+1-r)}/(\alpha_t+1-r)\,. \tag{9}$$

It is difficult to compare (8) and (9) analytically, but it is anyhow apparent that

Table I.
FESR for Re $A_1{}^t$.

t[GeV^2]	R_1	R_K	R_a	R_s
0.	−17.4	−18.2		
−1.4	−14.8	−72.5	−15.0	−17.7

R_1: Resonances contribution (8), (ν_c[GeV]); R_K: Khuri poles contribution (9);
R_a: A_a contribution; R_s: Contribution of Khuri poles considered as powers of s.

both expressions cannot be consistent for all values of t, e.g., for $\alpha_t=-1$ (8) is finite whereas the parent contribution to (9) diverges. We have evaluated both (8) and (9) numerically assuming $\alpha_0=.5$ and taking the cutoff $s_c=3\,\text{GeV}^2$ half-way between 3-rd and 4-th resonances. The computation reported in Table I was done for two values of t, i.e., $t=0$ and $t=-1.4$. We obtain a severe disagreement for $t=-1.4$, as could be expected, the two first daughters being negligible in comparison with the parent Khuri pole contribution to R_K. It is interesting to study the behaviour with s_c of the discrepancy between R_1 and R_K. For the forward amplitude there is good agreement (within 10%) even for $s_c=2\,\text{GeV}^2$. On the contrary for the non-forward amplitude ($t=-1.4$), though the disagreement tends to decrease with increasing energy, even for a high cutoff as $s_c=15\,\text{GeV}^2$ the value of R_1 is still less than one half the corresponding one to R_K.

We may see that it is not simple to isolate the Regge (or Khuri) background contribution which is at the basis of the discrepancy of FESR. In fact, one could rewrite $A_1{}^t$ as

$$A_1{}^t=A_a+A_b\,, \tag{10}$$

with

$$A_a=\frac{\pi}{\Gamma(\alpha_t)}\left[\frac{\Gamma(\alpha_s+\alpha_t)}{\Gamma(\alpha_s)}\,\frac{e^{-i\pi\alpha_t}}{\sin\pi\alpha_t}-\frac{\Gamma(\alpha_s+\alpha_t+1-c)}{\Gamma(\alpha_s+1-c)}\,\frac{1}{\sin\pi\alpha_s}\right], \tag{11a}$$

$$A_b=\frac{\pi}{\Gamma(\alpha_t)}\,\frac{\Gamma(\alpha_s+\alpha_t)}{\Gamma(\alpha_s)}\,\frac{e^{i\pi\alpha_s}}{\sin\pi\alpha_s}, \tag{11b}$$

where A_a has a smooth Regge asymptotic behaviour on positive real s-axis (but clearly not in all s-plane), the oscillations due to s-channel resonances being caused by A_b. Moreover, a partial wave analysis of both parts, followed by a Sommerfeld-Watson transform, shows that Regge poles only appear in A_a. However, Regge background is not included totally in A_b, since the evaluation of $R_a=\int_0^{\nu_c}\text{Re }A_a\,d\nu$ at $t=-1.4$ (see Table I) shows almost the same result as with the total $A_1{}^t$. It is clear that formally the discrepancy shown by R_K is due to the divergent behaviour of the series of powers of ν near the origin. As an indication, if we consider the expansion of the parent and first two daughters in powers of s, instead of ν, the resulting contribution to the FESR (8), reported as R, in Table I, shows a smaller discrepancy since the integral is now less singular in its lower limit. In other words the discrepancy in FESR mainly appears because A_a coincides with the Regge poles contribution only at high energy, where Stirling expansion is valid. At low energy there is a difference which corresponds to a background content of A_a. Therefore, since on the other hand A_b is shown to be approximately superconvergent, we expect the fulfilment of the FESR for large t only at very high energy.

Considering now briefly the amplitude A_2^t, which has no Khuri poles, the zero-moment FESR for the imaginary part is, from (1),

$$I_2 = \int_0^{\nu_c} \mathrm{Im}\, A_2^t d\nu = \frac{\pi}{2\mu} \sum_{n=0}^{N} \frac{(-)^n}{B(t, n+1)} \xrightarrow[N\to\infty]{} \frac{\pi}{2\mu} t\, 2^{-(t+1)}, \quad t<0, \tag{12}$$

where we have taken for simplicity $\alpha_0 = .5$ and $\mu^2 = 0$. Therefore, as expected, the wrong-signature sum rule I_2 tends to the residue of the Khuri fixed pole at -1. Similar results appear for higher-moment FESR.

Finally, evaluating the first right-signature sum rule for $\mathrm{Re}\, A_2^t$ we obtain

$$R_2 = \int_0^{\nu_c} \mathrm{Re}\, A_2^t d\nu = -\frac{1}{2\mu} \sum_{n=0}^{\infty} \frac{(-)^n}{B(t, n+1)} \ln \left| \frac{\nu_c - \nu_n}{\nu_c + \nu_n} \right|. \tag{13}$$

We see that, for $t=0$, $R_2=0$ as it should be because of the absence of Regge poles, but for $t<0$ disagreement appears. We reproduce in Table II the result of R_2 for different values of t taking $s_c = 3\,\mathrm{GeV}^2$. It is seen that R_2 oscillates and increases in absolute value with $|t|$.

Table II.
FESR for $\mathrm{Re}\, A_2^t$.

$t[\mathrm{GeV}^2]$	0.	$-.5$	-1.0	-2.0
R_2	0.	$-.02$	$.12$	-2.65

R_2: Resonances contribution (13), (ν_c[GeV]).

Looking at the results of this section one could be surprised by the fact that background effects are relatively important even at high energy and for the imaginary part of the amplitude where affect the second daughter contribution to the FESR. The reason is that on real s-axis Veneziano formula has not a smooth Regge behaviour but possesses superimposed oscillations due to resonances at any energy as seen in Eq. (11). This oscillating background part, peculiar to a narrow-width model, is approximately, but not totally, superconvergent when it is averaged through a FESR. On the other hand the background content of A_a, which tends to zero when $s \to \infty$, is mainly responsible for the discrepancies in the FESR R appearing for a not too high cutoff.

At this regard it is interesting to see how s-channel resonances appear in the explicit evaluation of the background integral corresponding to Veneziano model. Taking A_3^t as the simplest example, and performing the integral along the imaginary axis $m=-L+ib$ (with positive L) of the Khuri plane between limits $-B$ and B, we obtain

$$\text{Background}=-\frac{1}{4\mu}\sum_n\frac{(-)^n}{B(\alpha_t+1-c,n+1)}\frac{1}{\nu_n}\left(\frac{\nu}{\nu_n}\right)^{-L}$$
$$\times\int_{-B}^{B}dbe^{ib\ln\nu/\nu_n}\left(\operatorname{ctg}\frac{\pi}{2}m+i\right). \tag{14}$$

Choosing an integer L, and being $\sin Bx/x\xrightarrow[B\to\infty]{}\pi\delta(x)$, we find

$$\operatorname{Im}\text{Background}=-\frac{\pi}{2\mu}\sum_{n=0}^{\infty}\frac{(-)^n}{B(\alpha_t+1-c,n+1)}\delta(\nu-\nu_n), \tag{15}$$

which corresponds to the s-channel resonances contribution to the amplitude. The divergent behaviour of the integral in Eq. (14) for $\nu=\nu_n$ makes disappear the apparent ν^{-L} behaviour for any value of L. One can evaluate similarly the background of A_1^t, but with complications due to the presence of Regge poles. In fact, for any fixed value of L there will be a corresponding value of t for which Regge poles will begin to cross the integration line, whose contributions must be subtracted from that obtained from the previous calculation.

§ 3. Discussion and phenomenological outlook

We have analyzed to which extent the "mathematically dual" Veneziano model satisfies FESR for both imaginary and real parts of the amplitude, which corresponds to the criteria of "global duality".

The general result is that Veneziano model satisfies FESR approximately, despite the fact that it has not pure asymptotic Regge behaviour on the real s-axis. However, going into details, one finds that FESR for the imaginary part show a discrepancy starting at the second Regge-pole daughter contribution. Numerically, this difference is clearly not too significant when the energy cutoff is reasonably high. On the other hand we found that FESR for real part of the amplitude model are numerically well satisfied, for the generally used cutoff energies, only for the forward scattering, whereas for non-forward direction severe discrepancies may appear in both A_1^t and A_3^t. Therefore, the global conclusion is that Veneziano model satisfies FESR better for imaginary than for real part, and better for forward than for non-forward direction.

Clearly, a possibility is that these features of Veneziano model, as other peculiarities, cannot be related to phenomenology.

On the other hand, a reason which can suggest to compare these results with experimental fits is to consider that the best way to attribute a phenomenological meaning to a narrow-width model is to average the amplitude through, e.g., a FESR. Moreover, one cannot say that the background part which is mainly responsible for the most relevant discrepancies, i.e., those of the sum rule for the real part R_1, is only peculiar to narrow-width models. Therefore one could think that the previously obtained features of Veneziano model have something to do with the fits of FESR with experimental amplitudes. In this way one can argue that the use of sum rules with both imaginary and real parts of the type of continuous moment sum rules (CMSR) is suitable for forward scattering. This is in agreement with the satisfactory results that such fits have provided in the past years. On the contrary our Veneziano results would imply that the use of sum rules which involve the real part of the amplitude for non-forward scattering is not convenient, because of background effects, to obtain good fits of Regge poles. This could perhaps explain the. difficulty of fitting CMSR for πN non-forward scattering (see e.g. Ref. 18). Of course, a quantitative comparison with the πN results cannot be done because of the lack of a convincing πN Veneziano model. We may point out that the use of naive πN Veneziano models implies that discrepancy in FESR for imaginary part begins at first daughter Regge pole because of the presence of at least two baryon trajectories, i.e., N and Δ [see Ref. 19) for details].

Another attitude can be to think that "global duality", in FESR sense, is a property that a model amplitude must have, at least within a certain approximation. This implies to impose the constraints of FESR on the Veneziano model, the satisfaction of which could allow to determine the weight of the satellite terms tentatively added to the fundamental Veneziano block.

A reason for such an attempt is that the simple Lovelace amplitude Eq. (1), though predicts correctly the gross phenomenological features of $\pi\pi$ scattering, fails in reproducing accurately the most prominent resonance elastic widths. In fact, scaling the model (1) to agree with the ρ width, the predicted width for f^0 is ~.085 GeV and for g ~.040 GeV, to be confronted to the experimental values .150 ± .025 and .167 ± .030 respectively, the last one not being completely elastic. [For a general discussion see Ref. 20) and references therein.] To obtain agreement with these widths, and with the daughters ones, several satellite terms must be added to Eq. (1).[21)]

The attempt to introduce a satellite with the FESR constraint has been done using, instead of (1), the amplitude

$$A_1' = -C^-(1,1) + \gamma D^-(2,2), \tag{16}$$

where C^- is the previous Lovelace amplitude and

$$D^-(2,2) = \frac{\Gamma(2-\alpha_s)\Gamma(2-\alpha_t)}{\Gamma(2-\alpha_s-\alpha_t)} - s \to u\,. \tag{17}$$

$D^-(2,2)$ is the first relevant satellite for FESR considerations, since it has leading Regge behaviour.

We found that, calculating the FESR R_1 analogous to (8), a value of $\gamma\approx.5$ gives fair agreement between both sides of this sum rule for Re A_1^t even for, e.g., $s_c=3\,\mathrm{GeV}^2$ and $t=-1.4\,\mathrm{GeV}^2$, and avoids the divergence for $\alpha_t=-1$. At the same time the discrepancy in the second non-leading term of the FESR for Im A_1^t corresponding to Eq. (5) is strongly reduced for $-2\lesssim t\lesssim 0$ with this value of γ. It is remarkable that the model with $\gamma\approx.5$ corresponds to elastic widths $\Gamma_{f_0}\approx.130\,\mathrm{GeV}$ and $\Gamma_g\approx.090\,\mathrm{GeV}$ close to the experimental ones. This value of γ is in agreement with the model of Ref. 21).

It is clear that a more complete treatment would require the introduction of a larger number of satellites, including non-leading ones, together with several constraints based on different orders of approximation of FESR and considering the Adler condition. Such a program is in progress, together with a generalization to N-point amplitudes. However, the results shown above already indicate the consistency between general fulfilment of FESR and improvement of predictions for elastic widths in the frame of Veneziano model.

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