Journal of Physics A:
Mathematical and Theoretical

PAPER

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To cite this article: Manuel O Cáceres 2020 *J. Phys. A: Math. Theor.* **53** 405002

IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **53** (2020) 405002 (22pp) https://doi.org/10.1088/1751-8121/aba655

Stochastic PDEs, random fields and exact mean-values

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Received 26 April 2020, revised 25 June 2020
Accepted for publication 15 July 2020
Published 1 September 2020

Abstract

Introducing projector-operator technique and algebra of Terwiel's cumulants we study stochastic linear partial differential equations with global and local disorder. We present the evolution equation for the mean-value of the field as a series in terms of Terwiel's cumulant operators. Then, we prove that if we use binary disorder with time exponential-correlated structure, as source of the stochastic perturbation, this series cuts leading to a treatable evolution equation. We apply this approach to find the exact mean-value solution of electromagnetic waves with stochastic absorption of energy in conducting media. This model shows the occurrence of novel time-scale separation phenomena. Local disorder in telegrapher's equation is also presented. Thus we show that strong disorder leads to anomalous behavior at short and long time regimes. In addition, other physical systems with global disorder are worked out to find exact mean-value solutions: finite-velocity diffusion in the presence of a deterministic force (Smoluchoswki-like process generalizing, in this way, Feynman–Kac's formula for its numerical solution); Lorentz' force on a fluctuating charge model (we calculate the diffusion coefficient transverse to the applied magnetic field); and a generalized non-Maxwellian velocity distribution (Ornstein–Uhlenbeck like process showing a noise-induced transition in the stationary distribution).

Keywords: finite-velocity diffusion, Smoluchoswki, Feynman–Kac, Cattaneo–Fick, Lorentz' force, disorder, random media

1. Introduction

The interest in solving stochastic partial differential equations (PDE) has been a subject with long history [1], in particular in physics the knowledge of the mean-value field contributes to understand complex transport phenomena with huge applications [2–4]. In addition, the mentioned problem has many implementations in several branches of biology, social sciences and technology [5–7].

Macroscopic concern with propagation in random media has been a subject of interest since the late 50 and the apply of formal perturbation, functional probabilistic methods, and pure algebraic method are some of the many techniques used to approach the problem [1]. It is also well known the failure of standard perturbations expansions [1, 4], so in view of these difficulties exact solutions are always welcome. On the other hand, discrete models of transport and effective medium approximations, have also been introduced to tackle different problems of propagation in random media [7–12].

Anomalous diffusion shows up in the transport of particles in heterogeneous media, this is the case when the governing PDE is perturbed with a random field $\xi(x)$ [6–8]. Relaxation to thermal equilibrium distributions shows up when the governing (Liouville) PDE is perturbed with a stochastic thermal process $\xi(t)$ [4, 7]. Interestingly all these problems can be presented unified studying stochastic PDEs. The theory of stochastic PDEs can be divided into two main classes. If the stochastic perturbation, in the evolution equation, is a time-dependent stochastic process $\xi(t)$ we can call it a 'global' disorder model; while if the stochastic perturbation is a space-dependent stochastic process $\xi(x)$, it is called 'local' disorder. More general cases with a space-time stochastic perturbations $\xi(x, t)$ are also very important, for example, in turbulence and fluids [4, 13]. In this work we will study these problem from a unified perspective to find *exact* solutions for the mean-value field averaged over *global* disorder. In particular, exact solutions for the mean-value field will be presented in the context of anomalous telegrapher's equation, which is nowadays of very much interest in physics, engineers and biological problems [5, 14, 15].

An important model of disorder that shares the crucial structure of 'Markov process' is the binary disorder with exponential correlation, called dichotomic noise [7, 16–18]. We will show that for (time) exponential-correlated binary disorder the perturbation expansion cuts the series (when it is written in terms of Terwiel's cumulant operators [19]) leading to a compact *exact* evolution equation for the mean-value of the random field. While an exponential correlation is not of general application, this binary disorder model allows to find exact results, which indeed are of very much interest. Dichotomic process is a class of non-Gaussian noise that is also of interest in modeling non-equilibrium dynamics, see [18] and references therein. Environments switching between two well differentiated states may also induce bimodatily [21] or multimodality under two-time scaled binary noise [22]. In general in the study of complexity, the dichotomous noise has been used for many purposes, considering non-Gaussian and non-white effects [16, 18, 20, 21, 23].

Other non-Markovian binary disorder models, like the one emulating intermittence [24], are very interesting but the algebra of their Terwiel's cumulants does not cancel to any order; so the mentioned series expansion does not cut and inevitable a perturbation analysis is demanded. In the present work we will focus on *exact* results in continuous media using algebra of Terwiel's cumulant, for the particular family of time exponential-correlated binary disorder.

Using our exact evolution equation for binary disorder, several physical applications will be presented. In particular the main example of *global* disorder will be in the context of electromagnetic waves with stochastic absorption of energy in conducting media. This case corresponds to study the telegrapher equation when the absorption of energy has temporal fluctuations. This model shows the occurrence of novel time-scale separation phenomena. As an example of *local* disorder we will work out the mean-value solution for the random telegrapher's equation; that is, when the finite-velocity of propagation has fluctuations coming from the heterogeneous media. This model shows the occurrence of short-time anomalous wave behavior, and long-time anomalous diffusion. These subjects are of interest in general finite-speed diffusion problems [14, 25, 26].

Appendices are left to present (in the context of global disorder) other results for different physical problems. First, we start obtaining telegrapher's equation [27–29] from the continuity equation of an incompressible fluid perturbed by dichotomic noise. We end this analysis generalizing Smoluchoswki's equation; that is, a finite-velocity diffusion process in the presence of a deterministic force. Also the solution of this exact evolution equation will be presented generalizing Feynman–Kac's formula [30]. As a second example of global disorder, we have worked out the diffusion of a particle in the presence of Lorentz' force when the charge of the particle may fluctuate in time. We have found the exact mean-value solution and we have calculated the diffusion coefficient which is of interest in plasma physics [31]. As third example of global disorder we have presented a Langevin model in presence of Stoke's velocity damping and dichotomous stochastic force. The exact mean-value evolution equation for this generalized Ornstein–Uhlenbeck process and its stationary solution have been recovered from the present Terwiel's cumulant approach [21, 23]. Last appendix E is left to present the algebra concerning local disorder in telegrapher's equation.

2. Stochastic perturbations

Consider the general evolution equation:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \left[\mathbf{L}_0 + \theta \tilde{\mathbf{L}}_I\right] \rho, \tag{1}$$

where $\mathbf{L}_0$ is an arbitrary operator and

$$\tilde{\mathbf{L}}_I = \xi\,(t)\,\mathbf{A}, \tag{2}$$

with $\xi\,(t)$ a zero mean-value stochastic process, here we assume that $\mathbf{L}_0$ and $\mathbf{A}$ commute with $\xi\,(t)$. Both linear operators $\mathbf{L}_0, \mathbf{A}$ (bounded or not) act on the abstract operator ρ. Here we are interested in calculating the average over all realizations of the process $\xi\,(t)$; that is $\langle\rho\rangle$. We use θ as an intensity parameter that help to follow perturbation units. In general it is convenient to work-out the problem in real-time, thus in order to simplify the analysis we transform (1) to the interaction representation, that is:

$$\sigma\,(t) = \mathrm{e}^{-\mathbf{L}_0 t}\rho\,(t)\,.$$

Therefore, the evolution equation for $\sigma\,(t)$ is reduced to:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \theta \mathbf{L}_I\,(t)\,\sigma, \tag{3}$$

with $\sigma\,(0)$ the initial condition and $\mathbf{L}_I\,(t)$ the *interacting* operator:

$$\mathbf{L}_I\,(t) = \mathrm{e}^{-\mathbf{L}_0 t}\tilde{\mathbf{L}}_I\,\mathrm{e}^{+\mathbf{L}_0 t} \tag{4}$$

Now we introduce a projection operator $\mathcal{P} = \mathcal{P}^2$, and its complement $\mathcal{Q} = \mathbf{I} - \mathcal{P}$, acting on any functional $f = f\,[\xi\,(\bullet)]$, to denote the mean-value over the 'disorder'; that is, we take the average over any functional of the stochastic process $\xi\,(t)$:

$$\mathcal{P}f = \langle f \rangle\,.$$

Applying $\mathcal{P}$ and $\mathcal{Q}$ to (3) we get respectively:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}\sigma\,(t) = \theta\mathcal{P}\mathbf{L}_I\,(t)\,\mathcal{P}\sigma\,(t) + \theta\mathcal{P}\mathbf{L}_I\,(t)\,\mathcal{Q}\sigma\,(t) \tag{5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Q}\sigma\,(t) = \theta\mathcal{Q}\mathbf{L}_I\,(t)\,\mathcal{P}\sigma\,(t) + \theta\mathcal{Q}\mathbf{L}_I\,(t)\,\mathcal{Q}\sigma\,(t)\,. \tag{6}$$

Equation (6) can formally be solved with the help of the auxiliary Green operator $\mathbf{T}\left(t\,|t'\right)$, then we get:

$$\mathcal{Q}\sigma\,(t) = \int_0^\infty H(t-t')\mathbf{T}\left(t\,|t'\right)\theta\mathcal{Q}\mathbf{L}_I\left(t'\right)\mathcal{P}\sigma\left(t'\right)\mathrm{d}t' + \mathbf{T}\left(t\,|0\right)\mathcal{Q}\sigma\,(0)\,, \tag{7}$$

where $H(t-t')$ is the step function, and the Green operator fulfills:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{T}\left(t\,|t'\right) = \theta\mathcal{Q}\mathbf{L}_I\,(t)\,\mathbf{T}\left(t\,|t'\right), \quad \mathbf{T}\left(t\,|t\right) = I. \tag{8}$$

Introducing solution (7) into (5) we get an *exact* evolution equation for the mean-value:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\sigma\,(t)\rangle = \theta^2\int_0^t \mathcal{P}\mathbf{L}_I\,(t)\,\mathbf{T}\left(t\,|t'\right)\mathcal{Q}\mathbf{L}_I\left(t'\right)\left\langle\sigma\left(t'\right)\right\rangle\mathrm{d}t' + \theta\mathcal{P}\mathbf{L}_I\,(t)\,\mathbf{T}\left(t\,|0\right)\mathcal{Q}\sigma\,(0)\,. \tag{9}$$

Here we have used $\mathcal{P}\mathbf{L}_I\,(t) = 0$ and the notation $\mathcal{P}\sigma\,(t) \equiv \langle\sigma\,(t)\rangle$. Assuming that the initial condition $\sigma\,(0)$ is not correlated with the stochastic process $\xi\,(t)$, the last term cancels. Therefore we can write the evolution equation (9) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\sigma\,(t)\rangle = \theta^2\int_0^t \mathbf{K}\left(t\,|t'\right)\left\langle\sigma\left(t'\right)\right\rangle\mathrm{d}t', \tag{10}$$

where

$$\mathbf{K}\left(t\,|t'\right) = \sum_{n=0}^{\infty}\mathbf{k}_n\left(t\,|t'\right) \tag{11}$$

$$\mathbf{k}_n\left(t\,|t'\right) = \theta^n\int_{t'}^t \mathrm{d}s_1\int_{t'}^{s_1}\mathrm{d}s_2\cdots\int_{t'}^{s_{n-1}}\mathrm{d}s_n\left\langle\mathbf{L}_I\,(t)\,\mathcal{Q}\mathbf{L}_I\,(s_1)\cdots\mathcal{Q}\mathbf{L}_I\,(s_n)\,\mathcal{Q}\mathbf{L}_I\left(t'\right)\right\rangle\,. \tag{12}$$

The integrand that appears in the n-fold integral (12) involves a Terwiel cumulant operator [7, 19] of order $n+2$. Due to the fact that $t \geqslant s_1 \geqslant \cdots \geqslant s_n \geqslant t'$ it is possible to argue some conclusions concerning each term appearing in sum (11). Thus, in principle a perturbation expansion can be applied to carry out the analysis of $\langle\sigma\,(t)\rangle$. The important point in expression (12) is that it involves a time-ordered expansion which facilitates the perturbation theory. In order to complete the description of the mean-value of the field we need to go back to the original representation, that is in terms of operator $\rho\,(t) = \mathrm{e}^{+\mathbf{L}_0 t}\sigma\,(t)$. It is simple to see that the mean-value of $\rho\,(t)$ evolves governed by the evolution equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\rho\,(t)\rangle = \mathbf{L}_0\,\langle\rho\,(t)\rangle + \theta^2\int_0^t \mathrm{e}^{+\mathbf{L}_0 t}\mathbf{K}\left(t\,|t'\right)\mathrm{e}^{-\mathbf{L}_0 t'}\left\langle\rho\left(t'\right)\right\rangle\mathrm{d}t'. \tag{13}$$

Generally any operator $\mathbf{L}_I\left(t_j\right)$ (appearing in $\mathbf{k}_n\left(t\left|t'\right.\right)$) involves an arbitrary random number $\xi\left(t_j\right)$. Therefore equation (13) is the starting point to study any perturbation series. The dominant term is of order $\theta^2\tau_c$ (coming from $\mathbf{k}_0\left(t\left|t'\right.\right)$), where τ_c is a characteristic time-scale measuring the time-domain where the integrand is non null. The next correction (coming from $\mathbf{k}_1\left(t\left|t'\right.\right)$) would be of order $\theta^3\tau_c^2$ in the Kubo number [2, 3].

In the present paper we are not interested in any perturbation approach, which have been extensively studies in the past, for a comprehensive review see [1, 4]. Here we will focus in presenting exact results that can be obtained from a particular structure of the stochastic process $\xi(t)$.

An evolution equation for the *local* disorder case, that is, when $\tilde{\mathbf{L}}_I$ depends on $\xi(x)$, will be presented in section 4.

2.1. Exact evolution equations

An important model of random perturbation $\xi(t)$ is the (Markovian) dichotomic stochastic process. That is, a time exponential-correlated binary process. In this particular case it is possible to see that all Terwiel's cumulants cancel except the first one: $\left\langle\mathbf{L}_I(t)\,\mathcal{Q}\mathbf{L}_I\left(t'\right)\right\rangle$. In fact, from (12) we get

$$\begin{aligned}\mathbf{k}_n\left(t\left|t'\right.\right) &= \theta^n\int_{t'}^{t}\mathrm{d}s_1\cdots\int_{t'}^{s_{n-1}}\mathrm{d}s_n\left\langle \mathrm{e}^{-\mathbf{L}_0 t}\tilde{\mathbf{L}}_I\mathrm{e}^{+\mathbf{L}_0 t}\mathcal{Q}\mathrm{e}^{-\mathbf{L}_0 s_1}\tilde{\mathbf{L}}_I\mathrm{e}^{+\mathbf{L}_0 s_1}\cdots\mathcal{Q}\mathrm{e}^{-\mathbf{L}_0 s_n}\right.\\ &\quad\left.\times\,\tilde{\mathbf{L}}_I\mathrm{e}^{+\mathbf{L}_0 s_n}\mathcal{Q}\mathrm{e}^{-\mathbf{L}_0 t'}\tilde{\mathbf{L}}_I\mathrm{e}^{+\mathbf{L}_0 t'}\right\rangle\\ &= \theta^n\int_{t'}^{t}\mathrm{d}s_1\cdots\int_{t'}^{s_{n-1}}\mathrm{d}s_n\left\langle\xi(t)\,\mathcal{Q}\xi(s_1)\cdots\mathcal{Q}\xi(s_n)\,\mathcal{Q}\xi\left(t'\right)\right\rangle\\ &\quad\times\,\mathrm{e}^{-\mathbf{L}_0 t}\mathbf{A}\mathrm{e}^{+\mathbf{L}_0(t-s_1)}\mathbf{A}\mathrm{e}^{+\mathbf{L}_0(s_1-s_2)}\cdots\mathbf{A}\mathrm{e}^{+\mathbf{L}_0(s_n-t')}\mathbf{A}\mathrm{e}^{+\mathbf{L}_0 t'},\end{aligned}\tag{14}$$

where the *naked* Terwiel cumulant (of order $n+2$):

$$\left\langle\xi(t)\,\mathcal{Q}\xi(s_1)\cdots\mathcal{Q}\xi(s_n)\,\mathcal{Q}\xi\left(t'\right)\right\rangle,\tag{15}$$

can be seen to cancel for all $n\geqslant 1$. To prove this fact, first note that for a stationary Markov binary process, any moment breaks in the form, see [32]:

$$\left\langle\xi(t)\,\xi(s_1)\cdots\xi(s_n)\,\xi\left(t'\right)\right\rangle=\left\langle\xi(t)\,\xi(s_1)\right\rangle\left\langle\xi(s_1)\,\xi(s_2)\right\rangle\cdots\left\langle\xi(s_n)\,\xi\left(t'\right)\right\rangle,\;\left\{t\geqslant s_1\geqslant\cdots\geqslant s_n\geqslant t'\right\}.\tag{16}$$

Now, using that the process is symmetric and stationary, all odd moments cancel out. In addition, Terwiel's cumulants adopt a simpler expression, see appendix A. Thus, Terwiel's cumulants (15), by the property (16), exactly cancel if $n\geqslant 1$ [7]. In order to exemplify this result, let me show a particular non-trivial Terwiel's cumulant of order $m=6$. Using (A2) and (16) we prove that:

$$\begin{aligned}\left\langle\xi(1)\,\mathcal{Q}\xi(2)\,\mathcal{Q}\xi(3)\,\mathcal{Q}\xi(4)\,\mathcal{Q}\xi(5)\,\mathcal{Q}\xi(6)\right\rangle &= \left\langle\xi(1)\,\xi(2)\,\xi(3)\,\xi(4)\,\xi(5)\,\xi(6)\right\rangle\\ &\quad-\left\langle\xi(1)\,\xi(2)\right\rangle\left\langle\xi(3)\,\xi(4)\,\xi(5)\,\xi(6)\right\rangle\\ &\quad-\left\langle\xi(1)\,\xi(2)\,\xi(3)\,\xi(4)\right\rangle\left\langle\xi(5)\,\xi(6)\right\rangle\\ &\quad+\left\langle\xi(1)\,\xi(2)\right\rangle\left\langle\xi(3)\,\xi(4)\right\rangle\left\langle\xi(5)\,\xi(6)\right\rangle\\ &= 0.\end{aligned}$$

The same follows for any even mth naked Terwiel's cumulant of order larger than $m \geqslant 4$. Therefore, for Markovian dichotomic noise the exact evolution equation (13) is written *only* in terms of the first non-null Terwiel cumulant operator. Note that we have used $\mathcal{P}\mathbf{L}_I(t) = 0$, nevertheless this fact does not impose any restriction in the general theory, but simplifies the algebra.

As mentioned before in the particular case when $\tilde{\mathbf{L}}_I$ involves a time exponential-correlated binary process, the only non-null contribution comes from $\mathbf{k}_0\left(t\,|t'\right)$. The Terwiel cumulant operator $\mathbf{k}_0\left(t\,|t'\right)$ can be written in terms of the two-points stationary correlation function:

$$\langle \xi(t_1)\,\xi(t_2)\rangle = \exp\left(-\left|t_1 - t_2\right|/T\right), \tag{17}$$

where T measures the time-scale of the noise. Thus, the *exact* evolution for the mean-value of the field gets:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle \rho(t)\rangle &= \mathbf{L}_0\langle \rho(t)\rangle + \theta^2 \int_0^t \mathrm{e}^{+\mathbf{L}_0 t}\mathbf{k}_0\left(t\,|t'\right)\mathrm{e}^{-\mathbf{L}_0 t'}\left\langle \rho\left(t'\right)\right\rangle \mathrm{d}t' \\
&= \mathbf{L}_0\langle \rho(t)\rangle + \theta^2 \int_0^t \mathrm{e}^{+\mathbf{L}_0 t}\left\langle \mathbf{L}_I(t)\,\mathcal{Q}\mathbf{L}_I\left(t'\right)\right\rangle \mathrm{e}^{-\mathbf{L}_0 t'}\left\langle \rho\left(t'\right)\right\rangle \mathrm{d}t' \\
&= \mathbf{L}_0\langle \rho(t)\rangle + \theta^2 \int_0^t \left\langle \xi(t)\,\xi\left(t'\right)\right\rangle \mathbf{A}\mathrm{e}^{+\mathbf{L}_0\left(t-t'\right)}\mathbf{A}\left\langle \rho\left(t'\right)\right\rangle \mathrm{d}t'.
\end{aligned} \tag{18}$$

Equation (18) is the first goal of the present paper, in the following section we will analyze a concrete example. That is, in the case when the random perturbation $\tilde{\mathbf{L}}_I$ in (1) involves a symmetric stochastic Markov dichotomic process $\xi(t)$. In appendices B–D we will present applications of formula (18) to different systems, leading to important exact results in theoretical and applied physics: (1) finite-velocity diffusion in the presence of a deterministic force, (2) Lorentz' force on a fluctuating charge model, (3) Langevin equation with a non-Gaussian time-correlated stochastic force.

As a second goal of the paper we will present, in a similar way, how to get the mean-value solution in the context of a local stochastic perturbation. In particular we will work out telegrapher's equation in heterogeneous media, leading to a finite-velocity (anomalous) diffusion process. General situations when a PDE is perturbed by a random field $\xi(x)$ can be called stochastic PDEs with 'local' disorder. Then, treatable mean-value evolution equations, for this type of systems, can be found when the process $\xi(x)$ is a space exponential-correlated binary disorder.

3. Global disorder

3.1. Electromagnetic waves with stochastic absorption of energy

The simplest kind of absorptive (electromagnetic) wave model is telegrapher's equation, where τ^{-1} is a measure of the wave attenuation and v is a characteristic velocity, both parameters given in terms of Maxwell's equations and Ohm's law, that is:

$$\tau = \varepsilon/4\pi\sigma \quad \text{and} \quad v^2 = c^2/\mu\varepsilon,$$

here ε is the dielectric and μ the permeability constants, c the light velocity, while σ is the conductivity (responsible of Joule effects in a waveguides).

Consider the situation when the attenuation factor is stochastic: $1/\tau \to \tau^{-1} + \theta\xi(t)$. Here the perturbation $\xi(t)$ is chosen to be a Markov dichotomic noise (with random values ± 1), note that in order to preserve the non-negativity of the energy attenuation factor, we adopt: $\theta \leqslant \tau^{-1}$. In this situation telegrapher's equation turns to be:

$$\left[\partial_t^2 + \frac{1}{\tau}\partial_t - v^2\partial_x^2\right]\psi = -\theta\xi(t)\,\partial_t\psi, \tag{19}$$

with noise correlation $\left\langle \xi(t)\,\xi(t')\right\rangle = \exp\left(-|t-t'|/T\right)$. In order to apply our Terwiel's approach to this evolution equation, it is necessary to transform (19) to the form:

$$\partial_t\psi = \phi \tag{20}$$

$$\left[\partial_t\phi + \frac{1}{\tau}\phi - v^2\partial_x^2\psi\right] = -\theta\xi(t)\,\phi. \tag{21}$$

This set of equations can be written in the form (1) for the vector field $\rho \equiv (\psi, \phi)$. In Fourier's representation expression for operators: $\mathbf{L}_0, \tilde{\mathbf{L}}_I = \xi(t)\,\mathbf{A}$ are given by:

$$\mathbf{L}_0 = \begin{pmatrix} 0 & 1 \\ -v^2k^2 & -1/\tau \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}. \tag{22}$$

Now, the exact evolution equation (18) adopts the form:

$$\partial_t \begin{pmatrix} \psi(k,t) \\ \phi(k,t) \end{pmatrix} = \mathbf{L}_0 \begin{pmatrix} \psi(k,t) \\ \phi(k,t) \end{pmatrix} + \theta^2 \int_0^t \mathrm{e}^{-(t-t')/T}\mathbf{A}\ \mathrm{e}^{+\mathbf{L}_0(t-t')}\mathbf{A} \begin{pmatrix} \psi(k,t') \\ \phi(k,t') \end{pmatrix} \mathrm{d}t'. \tag{23}$$

In order to apply this formula one has to compute:

$$\exp\left[t\begin{pmatrix} 0 & 1 \\ -v^2k^2 & -1/\tau \end{pmatrix}\right] = \frac{1}{\Delta}\begin{pmatrix} \lambda_+\mathrm{e}^{t\lambda_-} - \lambda_-\mathrm{e}^{t\lambda_+} & \mathrm{e}^{t\lambda_+} - \mathrm{e}^{t\lambda_-} \\ v^2k^2\left(\mathrm{e}^{t\lambda_-} - \mathrm{e}^{t\lambda_+}\right) & \lambda_+\mathrm{e}^{t\lambda_+} - \lambda_-\mathrm{e}^{t\lambda_-} \end{pmatrix}, \tag{24}$$

where:

$$\Delta = \sqrt{\frac{1}{\tau^2} - 4v^2k^2}, \quad \lambda_\pm = -\frac{1}{2}\left(\frac{1}{\tau} \mp \Delta\right), \tag{25}$$

the character of telegrapher's solutions is controlled by Δ. Substitution of (24) in (23) gives an exact equation for the average of the vector field (ψ, ϕ). Then we can write:

$$\partial_t \left\langle \psi(k,t)\right\rangle = \left\langle \phi(k,t)\right\rangle \tag{26}$$

$$\partial_t \left\langle \phi(k,t)\right\rangle + \frac{1}{\tau}\left\langle \phi(k,t)\right\rangle + v^2k^2\left\langle \psi(k,t)\right\rangle = -\theta^2\int_0^t \Lambda\left(k, t-t'\right)\left\langle \phi\left(k,t'\right)\right\rangle \mathrm{d}t', \tag{27}$$

with

$$\Lambda(k,t) = \left(\lambda_-\mathrm{e}^{-t(1/T-\lambda_-)} - \lambda_+\ \mathrm{e}^{-t(1/T-\lambda_+)}\right)/\Delta. \tag{28}$$

In order to compare with the unperturbed telegrapher's equation we take ∂_t in (26) and get:

$$\left[\partial_t^2 + \frac{1}{\tau}\partial_t + v^2k^2\right]\left\langle \psi(k,t)\right\rangle = -\theta^2\int_0^t \Lambda\left(k,t-t'\right)\partial_{t'}\left\langle \psi\left(k,t'\right)\right\rangle \mathrm{d}t'. \tag{29}$$

This expression teaches that, a stochastic absorption of energy in telegrapher's equation introduces in the evolution of the mean-value field, a non-local dispersive first order time derivative. Conversely if we take the inverse Fourier transform in (29), the non-local in time contribution introduces a space convolution. This means that a new time-scale for the separation between the wave-regime and the diffusive-regime will appear as a consequence of the stochastic absorption of energy (this time-scale separation is governed, in a non-trivial way, by kernel $\Lambda(k,t)$).

The analysis in Laplace's representation helps to study the features of the mean-value field. Taking the Laplace transform in (29) and assuming initial conditions: $\psi(x,t)|_{t=0} = \delta(x)$, $\partial_t\psi(x,t)|_{t=0} = 0$ we get:

$$\left[s^2 + s/\tau + v^2k^2\right]\langle\psi(k,s)\rangle = (s+1/\tau) - \theta^2\Lambda(k,s)\left(s\langle\psi(k,s)\rangle - 1\right), \quad (30)$$

solving for $\langle\psi(k,s)\rangle$ it finally gets the exact result:

$$\langle\psi(k,s)\rangle = \frac{\theta^2\Lambda(k,s)}{s^2 + s(1/\tau + \theta^2\Lambda(k,s)) + v^2k^2} + \frac{1}{s + (s\theta^2\Lambda(k,s) + v^2k^2)/(s+1/\tau)}, \quad (31)$$

with

$$\Lambda(k,s) = \frac{1}{\Delta}\left(\frac{\lambda_-}{s+1/T-\lambda_-} - \frac{\lambda_+}{s+1/T-\lambda_+}\right).$$

From (31) we see that both terms have dispersive features in the evolution of $\langle\psi(k,t)\rangle$, and so destroying the simple separation of scale, between wave-regime (for $t \ll \tau$) versus diffusion-regime (for $t \gg \tau$) that occurs for the conventional telegrapher's equation. Now the time-scale controlling this separation regimes strongly depends on the competition between disorder parameters θ, T and telegrapher's parameters v, τ.

To get a better understanding of the effects of an stochastic absorption of energy in a telegrapher equation, here we calculate the second moment; that is, the dispersion associated to the mean-value solution $\langle\psi(x,t)\rangle$. From (31) we get in Laplace's representation:

$$\begin{aligned}\left\langle x(s)^2\right\rangle &= -\left[\partial_k^2\langle\psi(k,s)\rangle\big|_{k=0}\right] \\ &= \left[\frac{2\tau v^2\left(1+sT+T/\tau\right)}{s^2\left[(s\tau+1)(sT+1) + \left(s\tau+1-\tau^2\theta^2\right)T/\tau\right]}\right].\end{aligned} \quad (32)$$

The case $\theta = 0$ (there is not stochastic fluctuations) gives: $\left\langle x(s)^2\right\rangle = 2\tau v^2/(s^2+s^3\tau)$ showing for $s\tau \gg 1$ and $s\tau \ll 1$ a clear separation of scale between ballistic against diffusion regimes, respectively.

A particular interesting case is when $\theta = \tau^{-1}$, this would correspond to strong global disorder (there are stochastic realizations with total absorption factor zero). In this case

$$\left\langle x(s)^2\right\rangle = \left[\frac{2\tau v^2\left(1+sT+T/\tau\right)}{s^2\left[(s\tau+1)(sT+1)+sT\right]}\right], \quad \theta = \tau^{-1}. \quad (33)$$

Thus, using Laplace inversion for $t \to 0$ we get $\left\langle x(t)^2\right\rangle \sim v^2t^2$, and at very long time $t \to \infty$ we get $\left\langle x(t)^2\right\rangle \sim 2(T+\tau)v^2t$, note that for global disorder there is not anomalous behavior (compare with the next section). On the other hand, it is possible to see that in the regime $T/\tau \gg 1$ the transient of the second moment is characterized by

$$\left\langle x(s)^2\right\rangle \simeq \frac{2Tv^2\left(s+1/\tau\right)}{s^2\left[1/\tau + Ts^2 + 2Ts/\tau\right]}, \quad \theta = \tau^{-1}. \quad (34)$$

This expression has an analytic inverse Laplace transform in terms of exponential functions with two characteristic time scales. From (34) we see that in the regime $T/\tau \gg 1$ the two time-scales characterizing the behavior of $\left\langle x(t)^2\right\rangle$ are given by (Laplace's poles):

Table 1. Time-dependent dispersion of the mean-value telegrapher's field perturbed by a stochastic absorption of energy, as a function of time-scale correlation of the noise, T, and telegrapher's parameters: finite-velocity v and mean rate of absorption of energy τ^{-1}.

Dispersion of $\langle\psi(x,t)\rangle$	$s\tau \gg 1$	$s\tau \ll 1$
$T/\tau \ll 1/2$	$\langle x(t)^2\rangle \simeq v^2\ t^2$	$\langle x(t)^2\rangle \simeq 2v^2\tau\ t$ if $\{2sT \ll 1; s^2 \ll 1/T\tau\}$
$T/\tau \gg 1$	$\langle x(t)^2\rangle \simeq v^2\ t^2$	$\langle x(t)^2\rangle \simeq 2v^2T\ t$ if $2Ts \ll 1$ $\langle x(t)^2\rangle \simeq v^2\ t^2/2$ if $2Ts \gg 1$

$$s_\pm \simeq \begin{cases} -1/2T \\ -2/\tau + 1/2T \end{cases}. \tag{35}$$

While in the opposite regime $T/\tau \ll 1$ we get:

$$\left\langle x(s)^2\right\rangle \simeq \frac{2Tv^2\left(s+1/T\right)}{s^2\left[1/\tau + Ts^2 + s\right]}, \quad \theta = \tau^{-1}, \tag{36}$$

In this case the two time-scales are characterized by poles:

$$s_\pm \simeq \begin{cases} -1/\tau \\ -1/T + 1/\tau \end{cases}. \tag{37}$$

These results end the proof that telegrapher's equation with stochastic absorption of energy leads to a novel separation of time-scales between the wave and diffusion regimes. This fact gives rise to new time-scale scenarios in the profile of the field, reminiscent of intermittent issues. In general, these time-scales are function of global disorder parameters θ, T and telegrapher's parameters τ, v.

In table 1 we present the different parameter regimes and different behaviors of the time-dependent dispersion of $\langle\psi(x,t)\rangle$, which can be obtained from the second moment in Laplace's representation (33).

From this table it is possible to see that the occurrence of a new wave vs diffusion time-scale separation happens when $T/\tau \gg 1$ and $s\tau \ll 1$ if $2Ts \gg 1$ (or $2Ts \ll 1$) respectively.

In addition the analysis of the velocity correlation function follows from Kubo's formula by studying function: $s^2\left\langle x(s)^2\right\rangle$. This velocity-correlation can fully be understood from exact result (31). It is interesting to note that the structure of the exact solution (31) is quite different to the one when the velocity of propagation in telegrapher's equation is heterogeneous due to the presence of local disorder. In this case, the local disorder introduces a time-dependent effective velocity in the mean-value field, leading, for strong disorder, to anomalous behavior. Therefore, the mean-value field is characterized by the length of the space-correlation of the local disorder $\xi(x)$. This analysis will be presented in the next section.

4. Local disorder

4.1. Telegrapher's equation in random media

Telegrapher's equation in heterogeneous media can be worked out using the effective medium approximation [33]. In the present section we analyze, in detail, results for the case when $\xi(x)$ is a symmetric space exponential-correlated binary disorder.

Consider telegrapher's equation in a random media. In this case, if the velocity of propagation has space fluctuations (due to the heterogeneous media) we can replace: $v^2 \to v^2 + \xi(x)$ in the generalized Cattaneo–Fick law [29]. Then, telegrapher's equation turns to be [33]:

$$\left[\partial_t^2 + \frac{1}{\tau}\partial_t - v^2\partial_x^2\right] P(x,t) = \partial_x \xi(x)\,\partial_x P(x,t) \equiv \mathbf{\Theta}_\xi P(x,t)\,. \tag{38}$$

From now on we use initial conditions: $P(x, t=0) = \delta(x)$, $\partial_t P(x,t)|_{t=0} = 0$.

Comparing with (1) (here $\theta = 1$) the operator $\tilde{\mathbf{L}}_I$ in the basis $|x\rangle$ has the form: $\tilde{\mathbf{L}}_I = \partial_x \xi(x)\,\partial_x \equiv \mathbf{\Theta}_\xi$. In Laplace's representation, equation (38) is:

$$\left(s^2 + \frac{s}{\tau} - v^2\partial_x^2\right) P(x,s) = \left(s + \frac{1}{\tau}\right)\delta(x) + \mathbf{\Theta}_\xi P(x,s)\,. \tag{39}$$

If we apply projectors operator to (39) we get the equivalent to (13), see appendix E:

$$\mathbf{\Lambda}\mathcal{P}P(x,s) = \left(s + \frac{1}{\tau}\right)\delta(x) + \mathcal{P}\mathbf{\Theta}_\xi \sum_{m=0}^{\infty}\left[\mathbf{\Lambda}^{-1}\mathcal{Q}\mathbf{\Theta}_\xi\right]^m \mathcal{P}P(x,s)\,, \tag{40}$$

where we have defined the differential operator: $\mathbf{\Lambda} = \left(s^2 + \frac{s}{\tau} - v^2\partial_x^2\right)$, and its integral representation $\mathbf{\Lambda}^{-1}$ with kernel:

$$g(x,s) = \exp\left(-\left|\frac{x}{v}\right|\sqrt{s\left(s+1/\tau\right)}\right) \Big/ 2\sqrt{v^2 s\left(s+1/\tau\right)}. \tag{41}$$

Sum terms in (40) take care of the average over the disorder. After integrating by parts and reordering Terwiel's cumulant operators, the sum in (40) can be represented in the integral form:

$$\mathcal{P}\partial_x \sum_{m=1}^{\infty}\int_{-\infty}^{+\infty} \mathrm{d}x_1 \cdots \int_{-\infty}^{+\infty} \mathrm{d}x_m\, \langle \xi(x)\,\xi(x_1)\cdots\xi(x_m)\rangle_{\mathrm{T}} \tag{42}$$
$$\times \left[\partial_{x_1}^2 g(x-x_1,s)\right]\cdots\left[\partial_{x_m}^2 g(x_{m-1}-x_m,s)\right]\partial_{x_m}\langle P(x_m,s)\rangle\,,$$

where $\langle \xi(x)\,\xi(x_1)\cdots\xi(x_m)\rangle_{\mathrm{T}}$ is a naked Terwiel's cumulant of order $m+1$. Using (42) we can take the Fourier transform in (40), and we get the solution:

$$\mathcal{P}P(k,s) = \frac{\left(s+1/\tau\right)}{s\left(s+1/\tau\right) + \left(v^2 + \gamma(k,s)\right)k^2}, \quad P(k,t=0) = 1,\ \partial_t P(k,t)|_{t=0} = 0, \tag{43}$$

where the effective velocity $\gamma(k,s)$ is given by:

$$\gamma(k,s) = \sum_{m=1}^{\infty}\int_{-\infty}^{+\infty}\mathrm{d}y_1 \cdots \int_{-\infty}^{+\infty}\mathrm{d}y_m\, \mathrm{e}^{-\mathrm{i}ky_m}\, \langle \xi(0)\,\xi(y_1)\cdots\xi(y_m)\rangle_{\mathrm{T}} \tag{44}$$
$$\times \left[\partial_{y_1}^2 g(0-y_1,s)\right]\cdots\left[\partial_{y_m}^2 g(y_{m-1}-y_m,s)\right].$$

As we mention before, in the particular case when the random field $\xi(x)$ is an exponential-correlated symmetric binary disordered, $\xi(x) = \pm\Delta$, the series (44) can be cut and we get the main contribution, see (E7) in appendix E:

$$\gamma_{\rm E}(k,s) = \int_{-\infty}^{+\infty} {\rm d}y_1\ {\rm e}^{-{\rm i}ky_1}\ \langle \xi(0)\,\xi(y_1)\rangle_{\rm T}\left[\partial_{y_1}^2 g(0-y_1,s)\right] \tag{45}$$
$$= \frac{\Delta^2\sqrt{s\left(s+1/\tau\right)/v^2}}{v^2}\frac{\left(\frac{1}{\lambda}+\sqrt{\frac{s(s+1/\tau)}{v^2}}\right)}{\left(\left(\frac{1}{\lambda}+\sqrt{\frac{s(s+1/\tau)}{v^2}}\right)^2+k^2\right)} - \frac{\Delta^2}{v^2}.$$

To do the integral we have explicitly used the two-point correlation of the binary disorder:

$$\langle \xi(x_1)\,\xi(x_2)\rangle_{\rm T} = \Delta^2\ \exp\left(-\left|x_1-x_2\right|/\lambda\right), \tag{46}$$

where λ is the length-scale of correlation and Δ the intensity (in units of square velocity). The solution for the average of telegrapher's field can be read form (43) using (45), and we get:

$$\langle P(k,s)\rangle = \left(s+\frac{v^2+\gamma_{\rm E}(k,s)}{\left(s+1/\tau\right)}k^2\right)^{-1} \equiv \left(s+\mathcal{D}_{\rm E}(k,s)\,k^2\right)^{-1}. \tag{47}$$

Then, we can call $\mathcal{D}_{\rm E}(k,s)$ the dispersive diffusive function, which can be analytically study in the limit $\tau \gg 1/s$ (wave regime), and in the limit $\tau \ll 1/s$ (diffusive regime).

In fact it is possible to see from function $\mathcal{D}_{\rm E}(k=0,s)$ that in the wave regime if the traveled distance $\left(v/s\right) \gg \lambda$, the effective-velocity is modified by the disorder. While in the opposite case $\left(v/s\right) \ll \lambda$, there is not a net effect from the action of the random media. On the other hand, in the limit $\tau \ll 1/s$ if the diffusive distance $\sqrt{\tau v^2/s} \gg \lambda$, there is some effect from the disorder; while in the opposite case $\sqrt{\tau v^2/s} \ll \lambda$, there is not a net effect form the random media. In the Laplace representation, the second moment $\left\langle x(s)^2\right\rangle$ can be studied using the Tauberian theorem [7]. Then, from (47), at short times ($s \to \infty$) we get:

$$\left\langle x(t\to 0)^2\right\rangle \simeq \frac{v^2}{2!}t^2 - \frac{\Delta^2}{3!\left|v\right|\lambda}t^3 + \cdots. \tag{48}$$

While at long times ($s \to 0$) we get:

$$\left\langle x(t\to\infty)^2\right\rangle \simeq \tau\left(v^2-\frac{\Delta^2}{v^2}\right)t + \frac{\tau^{1/2}\Delta^2\lambda}{\Gamma\left(3/2\right)\left|v\right|^3}t^{1/2} + \cdots. \tag{49}$$

Thus if the disorder is weak ($\Delta < v^2$) at very short times the heterogeneous media does not modified the ballistic behavior, nevertheless the value of the correlation length λ is important in modifying the whole wave regime. At very long time the behavior is diffusive, but from the *asymptotic* formula (49) we also see that the effect from disorder is to reduce the diffusion coefficient.

4.2. Strong-disorder in the telegrapher equation

In the particular case when the disorder produces a zero *static* total effective-velocity, it can be called strong disorder. For the binary disorder model this situation can be achieved when

Table 2. Time-dependent dispersion of the mean-value telegrapher's field in a random media, as a function of the correlation length of the disorder, λ, and telegrapher's parameters: velocity v and energy absorption time-scale τ.

Dispersion of $\langle P(x,t)\rangle$	$\lambda \ll v/s$	$\lambda \gg v/s$	$\lambda \ll \sqrt{v^2\tau/s}$	$\lambda \gg \sqrt{v^2\tau/s}$
$\tau \gg 1/s$ (short times)	$\left\langle x(t)^2\right\rangle \simeq \lvert v\rvert\,\lambda\, t$	$\left\langle x(t)^2\right\rangle \simeq \frac{v^2}{2}\, t^2$		
$\tau \ll 1/s$ (long times)			$\left\langle x(t)^2\right\rangle \simeq \frac{\lvert v\rvert\,\lambda\sqrt{\tau}}{\Gamma\left(3/2\right)}\, t^{1/2}$	$\left\langle x(t)^2\right\rangle \simeq \lvert v\rvert\,\lambda\, t$

the intensity of the random field $\xi(x)$ takes random value: $\pm\Delta = \pm v^2$. From current result it is possible to analyze this situation introducing the value $\Delta = v^2$ in (47).

In the asymptotic regime $s \to \infty$ we get for the effective-velocity (short-times):

$$v^2 + \gamma_{\mathrm{E}}\left(k=0, s\to\infty\right) \simeq v^2 \left(1 - \frac{1}{\lambda}\sqrt{\frac{v^2}{s^2}}\right)\cdots. \tag{50}$$

While in the asymptotic regime $s \to 0$ we get for the effective-velocity (long-times):

$$v^2 + \gamma_{\mathrm{E}}\left(k=0, s\to 0\right) \simeq \lambda\sqrt{\frac{sv^2}{\tau}}\left(1 - \lambda\sqrt{\frac{s}{v^2\tau}}\right) + \cdots. \tag{51}$$

These are genuine anomalous behaviors for the total effective-velocity $v^2 + \gamma_{\mathrm{E}}\left(k=0,s\right)$.

In addition, from function $\mathcal{D}_{\mathrm{E}}\left(k,s\right)$ evaluated at $\Delta = v^2$, we can study the behavior of mean-value telegrapher's field in the presence of strong disorder. For example from (47), using the Tauberian theorem it is possible to see that at short-times the second moment behaves 'anomalous ballistic'; while at long-times the behavior is subdiffusive.

In table 2 we present the different parameter regimes and different behaviors of the time-dependent dispersion of $\langle P(k,s)\rangle$, obtained from Laplace's representation of the second moment:

$$\left\langle x(s)^2\right\rangle = \frac{v^2 + \gamma_{\mathrm{E}}\left(k=0,s\right)}{s^2\left(s + 1/\tau\right)}, \quad \Delta = v^2. \tag{52}$$

For the case of strong local disorder the dispersion $\left\langle x(t)^2\right\rangle$ as a function of correlation-length of the disorder, λ, finite-velocity of propagation v and rate of absorption of energy τ^{-1} is shown in table 2.

For more than three decades, the so-called 'anomalous transport' has been the object of intense research with countless applications [5–8]. Anomalous diffusion arises in extremely disordered systems, and its most distinctive characteristic is that the second moment follows an asymptotic non-linear law. In the case of wave motion, we can consider anomalous behavior with respect to the quadratic (ballistic) asymptotic law: $\left\langle x(t)^2\right\rangle \propto t^2$. The interesting point in considering anomalous transport in telegrapher's equation, is the ability to show the transition from anomalous ballistic to anomalous diffusion. This issue has been studied from related fractional calculus. However, the so called fractional telegrapher's equation is set in an ad hoc fashion by replacing the ordinary derivatives by fractional derivatives in telegrapher's equation. Or by deriving fractional telegrapher's equation from the concept of the continuous-time random-walk (CTRW formalism), or by using a fractional generalization of the persistent random walk, see [15] and references therein. In the present work we justify on physical grounds (random media) the occurrence of anomalous behavior in telegrapher's equation, predicting from first principles anomalous finite-velocity diffusion. These results are of interest in the general analysis of finite-velocity diffusion process in heterogeneous systems [14, 25, 26].

5. Conclusions

5.1. Concerning global disorder

Introducing, if necessary, time derivatives of the random field as new unknowns, the class of stochastic propagation equation that we have worked out can be written as:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \left[\mathbf{L}_0 + \theta\tilde{\mathbf{L}}_I\right]\rho, \tag{53}$$

where $\tilde{\mathbf{L}}_I = \xi(t)\,\mathbf{A}$ and the unknown function ρ (scalar or vector-valued) is called the random field, here $\mathbf{L}_0$ and $\mathbf{A}$ are nonrandom linear operators and $\xi(t)$ is a stochastic process.

Introducing projectors and Terwiel's cumulant operators we have presented a series expansion describing the evolution equation for the mean-value of the random field ρ. We have proved using the algebra of naked Terwiel's cumulants, that if $\xi(t)$ is a dichotomic process the series cuts giving rise to an exact and treatable evolution equation for the mean-value of the field.

We have applied our formalisms to several physical problems. In particular, we have studied electromagnetic waves with stochastic absorption of energy, predicting a non-trivial time-scale separation between wave and diffusion in the mean-value telegrapher field, see table 1. In appendices we have also presented the application of our Terwiel's cumulant approach to different physical systems: in appendix B we have worked out a finite-velocity diffusion process in the presence of a deterministic force, thus we have generalized Feynman–Kac's formula for its numerical solution. In appendix C we have studied a fluctuating charge particle in the presence of Lorentz' force, and we have calculated the diffusion coefficient transverse to the magnetic field. In appendix D a Langevin particle in the presence of a non-Gaussian correlated stochastic force is solved, showing a noise-induced transition in its stationary distribution.

5.2. Concerning local disorder

In the case when the stochastic perturbation in (53) depends on a static random field $\xi(x)$ we can call it a 'local' disorder model. An important case is when $\tilde{\mathbf{L}}_I = f\left[\xi(x), \mathbf{A}\right]$, as presented in section 4. In general $\xi(x)$ does not commute with $\mathbf{A}$ nor $\mathbf{L}_0$. Thus, in the series expansion (40) Terwiel's cumulant operators must be reordered explicitly in each particular situation. After arrangement of these Terwiel cumulant operators we can obtain naked Terwiel's cumulants of the form: $\langle\xi(x)\,\mathcal{Q}\xi(x_1)\cdots\mathcal{Q}\xi(x_m)\rangle$. If the random field $\xi(x)$ is a space exponential-correlated symmetric binary disorder we can conclude that all naked Terwiel's cumulants cancel for $m \geqslant 2$ when x_j-points are ordered in a sequence. This general situation can be used to find treatable evolution equations in many physical systems, as we have presented in the study of finite-velocity diffusion in random media, see appendix E. In this case we have shown that due to the intensity and correlation length of the disorder, the mean-value of telegrapher's field can behave anomalous at short and long times, table 2.

Another important local disorder case corresponds when $\tilde{\mathbf{L}}_I = f\left[\xi(x,t), \mathbf{A}\right]$ in (53), assuming that $\xi(x,t)$ is a stationary binary process in space and time. Like in the previously case, the problem must be worked out in each particular situation considering the non-commuting algebra with operators $\mathbf{A}$ and $\mathbf{L}_0$, to be able to arrive to naked Terwiel's cumulants in the series expansion. From this rearrangement it is possible to show that the series cuts if the binary disorder has space-time ordered naked Terwiel's cumulants, thus treatable evolution equations for the mean-value field can be worked out in a similar way. Research in that direction will be presented in future contributions.

Acknowledgments

MOC thanks funding provided by CONICET (Grant No. PIP 112-201501-00216, CO), and Grant from Secretaría de Ciencia, Técnica y Postgrado, Num. 06/C565 Univ. Nac. Cuyo (2020).

Appendix A. Naked Terwiel's cumulants

Terwiel's cumulant operators $\left\langle \mathbf{L}_I(t)\,\mathcal{Q}\mathbf{L}_I(s_1)\cdots\mathcal{Q}\mathbf{L}_I(s_n)\,\mathcal{Q}\mathbf{L}_I(t')\right\rangle$ appear in (12). Then, after some arrangement we define naked Terwiel's cumulant: $\left\langle \xi(t)\,\mathcal{Q}\xi(s_1)\cdots\mathcal{Q}\xi(s_n)\,\mathcal{Q}\xi(t')\right\rangle$. In general, these non-commuting cumulants can be written in terms of common moments by formula:

$$\begin{aligned}&\langle \xi(1)\,\mathcal{Q}\xi(2)\cdots\mathcal{Q}\xi(m-1)\,\mathcal{Q}\xi(m)\rangle\\&\quad=\sum_{i=0}^{m-1}(-1)^i\sum_{1\leqslant l_1<\cdots<l_i<m}\langle\xi_1\cdots\xi_{l_1}\rangle\left\langle\xi_{l_1+1}\cdots\xi_{l_2}\right\rangle\cdots\left\langle\xi_{l_i+1}\cdots\xi_m\right\rangle.\end{aligned}\tag{A1}$$

Some examples of naked Terwiel cumulants $\langle \xi(1)\,\mathcal{Q}\xi(2)\cdots\mathcal{Q}\xi(m-1)\,\mathcal{Q}\xi(m)\rangle \equiv \langle\xi_1\xi_2\cdots\xi_{m-1}\xi_m\rangle_{\mathrm{T}}$ are:

$$\begin{aligned}\langle\xi_1\rangle_{\mathrm{T}}&=\langle\xi_1\rangle\\\langle\xi_1\xi_2\rangle_{\mathrm{T}}&=\langle\xi_1\xi_2\rangle-\langle\xi_1\rangle\langle\xi_2\rangle\\\langle\xi_1\xi_2\xi_3\rangle_{\mathrm{T}}&=\langle\xi_1\xi_2\xi_3\rangle-\langle\xi_1\rangle\langle\xi_2\xi_3\rangle-\langle\xi_1\xi_2\rangle\langle\xi_3\rangle+\langle\xi_1\rangle\langle\xi_2\rangle\langle\xi_3\rangle\\\langle\xi_1\xi_2\xi_3\xi_4\rangle_{\mathrm{T}}&=\langle\xi_1\xi_2\xi_3\xi_4\rangle-\langle\xi_1\rangle\langle\xi_2\xi_3\xi_4\rangle-\langle\xi_1\xi_2\rangle\langle\xi_3\xi_4\rangle-\langle\xi_1\xi_2\xi_3\rangle\langle\xi_4\rangle\\&\quad+\langle\xi_1\rangle\langle\xi_2\rangle\langle\xi_3\xi_4\rangle+\langle\xi_1\rangle\langle\xi_2\xi_3\rangle\langle\xi_4\rangle+\langle\xi_1\xi_2\rangle\langle\xi_3\rangle\langle\xi_4\rangle\\&\quad-\langle\xi_1\rangle\langle\xi_2\rangle\langle\xi_3\rangle\langle\xi_4\rangle.\end{aligned}$$

If random variables are symmetric all odd Terwiel's cumulants are zero, and from (A1) it follows that even cumulants simplify drastically, which facilitate the algebra. For example for $m=6$ it gets:

$$\begin{aligned}\langle\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6\rangle_{\mathrm{T}}&=\langle\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6\rangle-\langle\xi_1\xi_2\rangle\langle\xi_3\xi_4\xi_5\xi_6\rangle\\&\quad-\langle\xi_1\xi_2\xi_3\xi_4\rangle\langle\xi_5\xi_6\rangle+\langle\xi_1\xi_2\rangle\langle\xi_3\xi_4\rangle\langle\xi_5\xi_6\rangle.\end{aligned}\tag{A2}$$

Appendix B. Telegrapher's equation and its generalized Smoluchoswki version

The (stochastic) equation of continuity for an incompressible fluid, in one dimension, provides a physical example of global disorder. Taking

$$\mathbf{L}_0=0\quad\text{and}\quad\mathbf{A}=-\partial_x$$

in (1) it gets:

$$\partial_t \rho(x,t) = -\theta \xi(t) \partial_x \rho(x,t). \tag{B1}$$

Interestingly this continuity equation leads to telegrapher's equation when the stochastic perturbation $\xi(t)$ is chosen to be the Markov dichotomic noise. That is, (B1) represents a stochastic Liouville equation dealing with a finite-velocity diffusion approach.

The exact evolution for $\langle \rho(x,t) \rangle$ can be read from (B1) and (18):

$$\begin{aligned}\partial_t \langle \rho(x,t) \rangle &= \theta^2 \int_0^t \left\langle \xi(t) \xi\left(t'\right) \right\rangle \mathbf{A}\mathbf{A} \left\langle \rho\left(t'\right) \right\rangle \, \mathrm{d}t'. \\ &= \theta^2 \partial_x^2 \int_0^t \exp\left(-\left|t-t'\right|/T\right) \left\langle \rho\left(t'\right) \right\rangle \, \mathrm{d}t'.\end{aligned} \tag{B2}$$

Applying the operator ∂_t to this equation it is simple to get telegrapher's equation:

$$\left[\partial_t^2 + \frac{1}{T}\partial_t - \theta^2 \partial_x^2\right] \langle \rho(x,t) \rangle = 0, \quad \partial_t \langle \rho(x,t) \rangle|_{t=0} = 0. \tag{B3}$$

We noted that here the parameter θ has dimension of velocity and the dichotomic process $\xi(t)$ is dimensionless.

Conversely any stochastic realizations of telegrapher's equation can be seen as the generalization of Feynman–Kac formula for the Brownian motion. In fact, for a particular realization of process $\xi(t)$, see (B1), the function $\rho(x,t)$ can be written in the form:

$$\rho(x,t) = \exp\left[-\theta \int_0^t \mathrm{d}t' \xi\left(t'\right) \partial_x\right] \rho(x, t=0). \tag{B4}$$

Thus, in Fourier representation we can write the mean-value $\langle \mathcal{F}_k [\rho(x,t)] \rangle$ over all realizations of the (Markov) symmetric dichotomic noise, as:

$$\begin{aligned}\langle \rho(k,t) \rangle &= \left\langle \exp\left[+\mathrm{i}\theta k \int_0^t \mathrm{d}t' \xi\left(t'\right)\right] \right\rangle_{P[\xi(\bullet)]} \\ &= \frac{1}{2}\left\{ \left\langle \exp\left[+\mathrm{i}\theta k \int_0^t (-1)^{N(t')} \mathrm{d}t'\right] \right\rangle_{P[N(t)]} \right. \\ &\quad \left. + \left\langle \exp\left[-\mathrm{i}\theta k \int_0^t (-1)^{N(t')} \mathrm{d}t'\right] \right\rangle_{P[N(t)]} \right\},\end{aligned} \tag{B5}$$

here we have taken the initial condition $\rho(k,t)|_{t=0} = 1$. This is just Kac's result [30] for the solution of telegrapher's equation, written in terms of all realizations of the Poisson process $N(t)$ with:

$$P[N(t) = q] = \mathrm{e}^{-(t-t_0)/2T} \frac{\left((t-t_0)/2T\right)^q}{q!}, \quad q = 0, 1, 2, \cdots.$$

That is, for any $t > t_0$, the probability that process $N(t)$ is equal to q at elapsed time $t - t_0$ is $P[N(t) = q]$. For any finite sequence $t_0 < t_1 < \cdots < t_{n-1} < t_n$ any increment $\Delta N_{i+1,i} \equiv N\left(t_{i+1}\right) - N(t_i)$ is independent and generalizes Wiener's increments in the Brownian motion. Thus, solution (B5) is the counterpart of Feynman–Kac's formula for the diffusion equation.

B.1. Generalized Kac's formula

Another interesting case that can be worked out within our approach is the simile of the Smoluchoswki process but with finite-velocity diffusion. That is, consider the stochastic differential equation: $\dot{x} = f(x) + \theta\xi(t)$ with $\xi(t)$ chosen to be the Markov dichotomic noise. The continuity equation in the presence of a force $f(x)$ is:

$$\partial_t \rho(x,t) = -\partial_x \left(f(x) + \theta\xi(t)\right) \rho(x,t).$$

In this case, comparing with (1), we get

$$\mathbf{L}_0 = -\partial_x f(x) \quad \text{and} \quad \mathbf{A} = -\partial_x.$$

Thus, we can work out this model in a similar way as before. Now, any stochastic realization of the field $\rho(x,t)$ can be written as:

$$\rho(x,t) = \exp\left[-\partial_x \int_0^t \mathrm{d}t' \left\{f(x) + \theta\xi\left(t'\right)\right\}\right] \rho(x,t=0).$$

The mean-value of the field $\langle\rho(x,t)\rangle$ is governed by the *exact* evolution equation (18), which in the present case adopts the form:

$$\partial_t \langle\rho(x,t)\rangle = \mathbf{L}_0 \langle\rho(x,t)\rangle + \theta^2 \int_0^t \left\langle \xi(t)\xi\left(t'\right)\right\rangle \partial_x\, \mathrm{e}^{+\mathbf{L}_0(t-t')} \partial_x \left\langle\rho\left(x,t'\right)\right\rangle \mathrm{d}t', \tag{B6}$$

with

$$\partial_t \langle\rho(x,t)\rangle|_{t=0} = \mathbf{L}_0 \langle\rho(x,t)\rangle|_{t=0}. \tag{B7}$$

This equation can be written in an alternative way resembling the telegrapher process, but in the presence of a deterministic force $f(x)$. Applying ∂_t to equation (B6) we get

$$\left[\partial_t^2 + \left(\frac{1}{T} - \mathbf{L}_0\right)\partial_t - \frac{\mathbf{L}_0}{T} - \theta^2\partial_x^2\right] \langle\rho(x,t)\rangle = \theta^2 \int_0^t \partial_x \mathbf{L}_0\, \mathrm{e}^{(\mathbf{L}_0 - 1/T)(t-t')} \partial_x \left\langle\rho\left(x,t'\right)\right\rangle \mathrm{d}t' \tag{B8}$$

This equation clearly manifests a finite-velocity diffusion evolution, in addition we get a non-local in time contribution coming from the interaction between force operator: $\mathbf{L}_0 = -\partial_x f(x)$ and the finite-velocity diffusion restriction. As expected in the limit $T \to 0$ and $\theta \to \infty$ such that $T\theta^2 \to D$ (Gaussian limit) the Smoluchoswki equation is recover.

As commented before, see (B5), the solution of (B8) can be written in terms of realizations of dichotomic's process:

$$\begin{aligned}
\langle\rho(x,t)\rangle &= \left\langle \exp\left[-\partial_x \int_0^t \mathrm{d}t' \left\{f(x) + \theta\xi\left(t'\right)\right\}\right] \rho(x,t=0)\right\rangle_{P[\xi(\bullet)]} \\
&= \frac{1}{2}\left\langle \exp\left[-\partial_x \int_0^t \left\{f(x) - \theta(-1)^{N(t')}\right\} \mathrm{d}t'\right] \rho(x,t=0)\right\rangle_{P[N(t)]} \\
&\quad + \frac{1}{2}\left\langle \exp\left[-\partial_x \int_0^t \left\{f(x) + \theta(-1)^{N(t')}\right\} \mathrm{d}t'\right] \rho(x,t=0)\right\rangle_{P[N(t)]}.
\end{aligned} \tag{B9}$$

This formula is the generalization of Kac's formula for a telegrapher process in the presence of a deterministic force. What is interesting in this very elegant result, is to write the solution of a finite-velocity Smoluchoswki process in terms of the Poisson process. This representation could be used to analyze relativistic quantum processes in the presence of an external electromagnetic field [34].

Appendix C. Lorentz' force on a fluctuating charge model

Another example of global disorder is provided by the dynamics associated to a *fluctuating charge* model. Consider a particle with velocity $\vec{V}$ in a magnetic field in the $\hat{z}$ direction. Using Lorentz' force the perpendicular velocity satisfies the stochastic evolution equation (in dimensionless units)

$$\dot{V}_x(t) = q\,(t)\,V_y\,(t)\,, \quad \dot{V}_y(t) = -q\,(t)\,V_x\,(t)\,. \tag{C1}$$

Here we assume that the random charge can be written as $q\,(t) = q_0 + \theta\xi\,(t)$ where $\xi\,(t)$ is a Markov dichotomic process with mean-value zero. The set of equations (C1) can easily be handled in complex notation:

$$\dot{u} = -\mathrm{i}\,\{\omega_0 + \theta\xi\,(t)\}\,u \tag{C2}$$

where $u = \left(V_x + \mathrm{i}V_y\right)$, Larmor's frequency is ω_0 and $\xi(t) \in \mathcal{R}_e$ takes the stochastic values ± 1. To calculate $\langle u\rangle$ and $\langle u^*(t)u(0)\rangle$ we use the *exact* evolution equation (18), with the identification:

$$\mathbf{L}_0 = -\mathrm{i}\omega_0 \quad \text{and} \quad \mathbf{A} = -\mathrm{i}.$$

Note that now $\mathbf{L}_0$ and $\tilde{\mathbf{L}}_I$ are not operators so ρ is a complex scalar. From this model we can study the space spreading of a packet-distribution in a plasma diffusion model.

The mean-value $\langle u\rangle$ follows from (18) as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\langle u\,(t)\rangle = -\mathrm{i}\omega_0\,\langle u\,(t)\rangle - \theta^2\int_0^t\,\exp\left(-\left|t-t'\right|/T\right)\;\mathrm{e}^{-\mathrm{i}\omega_0\left(t-t'\right)}\left\langle u\left(t'\right)\right\rangle\,\mathrm{d}t',$$

this equation can be solved analytically. With initial condition $\langle u\,(t=0)\rangle = u_0$ it gets:

$$\langle u\,(t)\rangle_{u_0} = \mathrm{e}^{-(\mathrm{i}\omega_0+1/2T)t}\left[\cosh\left(t\sqrt{\left(\frac{1}{2T}\right)^2-\theta^2}\right) + \frac{1/2T}{\sqrt{\left(\frac{1}{2T}\right)^2-\theta^2}}\;\sinh\right.$$
$$\left.\times\left(t\sqrt{\left(\frac{1}{2T}\right)^2-\theta^2}\right)\right]u_0. \tag{C3}$$

This is just Kubo's result, that he obtained when studying 'motional narrowing' [4]. Here we have found this mean-value in a quite different context using our Terwiel theoretical approach.

From this results we can calculate the space-diffusion coefficient associated to the dynamics of a fluctuating charge model. First we have to calculate de velocity correlation function: $\langle u^*(t)u(0)\rangle$, which can immediately be obtained from (C3) taking the average over the thermal initial condition; that is:

$$\langle u^*(t)u(0)\rangle = \left\langle \langle u(t)\rangle_{u_0} u_0^* \right\rangle_{P(u_0,0)}$$
$$= \mathrm{e}^{-(\mathrm{i}\omega_0+1/2T)t}\left[\cos\left(t\sqrt{\theta^2-\left(\frac{1}{2T}\right)^2}\right)\right.$$
$$\left. + \frac{1/2T}{\sqrt{\theta^2-\left(\frac{1}{2T}\right)^2}}\sin\left(t\sqrt{\theta^2-\left(\frac{1}{2T}\right)^2}\right)\right]\left\langle |u(0)|^2\right\rangle_{\mathrm{Th}}. \quad \text{(C4)}$$

Another important quantity to be calculated is the Fourier transform of this correlation function:

$$K(\omega) \equiv \mathcal{F}_\omega\left[\langle u(t)\, u^*(0)\rangle\right] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\omega t}\,\langle u(t)\, u^*(0)\rangle\,\mathrm{d}t \quad \text{(C5)}$$
$$= \frac{1}{\pi T}\frac{\theta^2}{\left[\theta^2-\Delta\omega^2\right]^2+\left(\frac{1}{T}\right)^2\Delta\omega^2}\left\langle |u(0)|^2\right\rangle_{\mathrm{Th}},\quad \Delta\omega\equiv\omega-\omega_0.$$

Here $\left\langle |u(0)|^2\right\rangle_{\mathrm{Th}}$ is a (thermal) second moment, and $\Delta\omega$ the difference in frequency with respect to the (non-stochastic) Larmor frequency. The space-diffusion coefficient in a transversal direction to the magnetic field follows from Einstein relation [7]. Noting that in the stationary state $\left\langle V_\perp(t)\, V_\perp(t')\right\rangle = \left\langle u(t)\, u^*(t')\right\rangle/2$ we conclude:

$$D_\perp = \int_0^\infty \left\langle V_\perp(t')\, V_\perp(0)\right\rangle_{\mathrm{st}}\mathrm{d}t' = \frac{1}{2}\int_0^{+\infty}\langle u(t)\, u^*(0)\rangle\,\mathrm{d}t = \frac{\pi}{2}K(\omega)\big|_{\omega=0}$$
$$= \frac{\left\langle |u(0)|^2\right\rangle_{\mathrm{Th}}}{2T}\frac{\theta^2}{\left[\theta^2-\Delta\omega^2\right]^2+\left(\frac{1}{T}\right)^2\Delta\omega^2}. \quad \text{(C6)}$$

This diffusion coefficient characterizes the spread of a packet diffusing, due to charge changes, in a plasma model [31].

Appendix D. Generalized Langevin equation

Another important global disorder example, is the generalized Langevin equation for a Brownian motion with a dichotomic stochastic force. In dimensionless units of mass we write the differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}V = -\gamma V + \theta\xi(t),\quad V(0) = V_0, \quad \text{(D1)}$$

here the term $-\gamma V$ represents Stoke's velocity damping, and $\xi(t)$ is the stochastic force (dimensionless units). The stochastic Liouville equation will be in this case

$$\partial_t P\left(V,t\,|V_0,t_0\right) = -\partial_V\left(-\gamma V+\theta\xi(t)\right)P\left(V,t\,|V_0,t_0\right), \quad \text{(D2)}$$

with initial condition:

$$\lim_{t\to t_0} P\left(V,t\,|V_0,t_0\right)\to\delta\left(V-V_0\right).$$

From this equation, and comparing with (1) and (2) we take

$$\mathbf{L}_0 = \gamma \frac{\partial}{\partial V} V, \quad \tilde{\mathbf{L}}_I = \xi(t)\, \mathbf{A} \tag{D3}$$

$$\mathbf{A} = -\frac{\partial}{\partial V}. \tag{D4}$$

We find it convenient to use bracket and operator notation:

$$\begin{aligned}
P\left(V, t \,|V', 0\right) &= \langle V| \rho(t) |V'\rangle \\
\langle V| \mathbf{L}_0 |V'\rangle &= \delta\left(V - V'\right) \gamma \frac{\partial}{\partial V} V \\
\langle V| \tilde{\mathbf{L}}_I |V'\rangle &= -\delta\left(V - V'\right) \xi(t) \frac{\partial}{\partial V} \\
\mathbf{I} &= \int \mathrm{d}V'' \,|V''\rangle \langle V''|\,, \\
\langle V|V'\rangle &= \delta\left(V - V'\right) \\
\overline{\rho(t)} &\equiv \langle \rho(t)\rangle = \mathcal{P}\rho(t)\,.
\end{aligned}$$

From (D3) and (D4) the evolution equation (18) can be written in the form:

$$\begin{aligned}
\partial_t \langle V| \overline{\rho(t)} |0\rangle = \langle V| \mathbf{L}_0 \overline{\rho(t)} |0\rangle \\
+ \theta^2 \int_0^t \mathrm{e}^{-(t-t')/T} \langle V| \mathbf{A} \int \mathrm{d}V'' \,|V''\rangle \langle V''| \,\mathrm{e}^{+\mathbf{L}_0(t-t')} \\
\times \int \mathrm{d}V' \,|V'\rangle \langle V'| \mathbf{A}\overline{\rho(t)} |0\rangle \,\mathrm{d}t',
\end{aligned} \tag{D5}$$

note that in this example θ has dimensions of velocity/time. Now using the explicit Green function:

$$\begin{aligned}
G\left(V, V' \,|t - t'\right) &\equiv \langle V| \,\mathrm{e}^{+\mathbf{L}_0(t-t')} \,|V'\rangle \\
&= \mathrm{e}^{\gamma(t-t')} \delta\left[V \mathrm{e}^{\gamma(t-t')} - V'\right] H(t - t'),
\end{aligned} \tag{D6}$$

where $H(t - t')$ is the step function, and denoting $\overline{P(V,t)} \equiv \langle V| \overline{\rho(t)} |0\rangle$ we arrive to:

$$\partial_t \overline{P(V,t)} = \gamma \frac{\partial}{\partial V} V \overline{P(V,t)} + \theta^2 \frac{\partial}{\partial V} \int \mathrm{d}V' \int_0^t \mathrm{d}t' G\left(V, V' \,|t - t'\right) \mathrm{e}^{-(t-t')/T} \frac{\partial}{\partial V'} \overline{P(V,t)}. \tag{D7}$$

This mean-value evolution equation was also found using van Kampen's lema in reference [21]. Introducing the operator identity: $\mathrm{e}^{\gamma\tau V \partial_V} f(V, t-\tau) = f(V\mathrm{e}^{\gamma\tau}, t-\tau)$, and using that:

$$\partial_V^2 (V\partial_V)^n [\bullet] = (V\partial_V + 2)^n \partial_V^2 [\bullet]\,,$$

we finally get:

$$\frac{\mathrm{d}\overline{P(V,t)}}{\mathrm{d}t} - \gamma \frac{\partial}{\partial V} V \overline{P(V,t)} = \theta^2 \int_0^t \mathrm{e}^{-\tau/T} \exp\left[\gamma\tau\left(V \frac{\partial}{\partial V} + 2\right)\right] \frac{\partial^2 \overline{P(V, t-\tau)}}{\partial V^2} \mathrm{d}\tau, \tag{D8}$$

from which the evolution of $\overline{P(V,t)}$ can be studied. In particular, the stationary state $\overline{P(V,t\to\infty)}$ showing a noise-induced phase transition from $\cup$-shape to $\cap$-shape distributions depending on the parameter $2T\gamma \lesseqgtr 1$ [21, 23].

Other generalized Brownian motions, but in the presence of non-harmonic potentials will show similar noise-induced phase transitions [18]. Here what it is interesting to point out is that by using our generalized Kac's formula (B9) we can study, numerically, the mean-value field in a very suitable and elegant way.

Appendix E. Local disorder in telegrapher's equation

Using initial conditions: $P(x,t=0)=\delta(x)$ and $\partial_t P(x,t)|_{t=0}=0$, Laplace's representation of (38) is:

$$\Lambda P(x,s)=\left(s+\frac{1}{\tau}\right)\delta(x)+\boldsymbol{\Theta}_\xi P(x,s), \tag{E1}$$

where Λ is the differential operator $\Lambda\equiv\left[s^2+\frac{s}{\tau}-v^2\partial_x^2\right]$. If we apply projector operator $\mathcal{P}$ to this equation we get:

$$\begin{aligned}\Lambda\mathcal{P}P(x,s)&=\left(s+\frac{1}{\tau}\right)\delta(x)+\mathcal{P}\boldsymbol{\Theta}_\xi P(x,s)\\&=\left(s+\frac{1}{\tau}\right)\delta(x)+\mathcal{P}\boldsymbol{\Theta}_\xi\mathcal{P}P(x,s)+\mathcal{P}\boldsymbol{\Theta}_\xi\mathcal{Q}P(x,s),\end{aligned} \tag{E2}$$

here we have used that $\mathcal{P}+\mathcal{Q}=\mathbf{I}$. Applying $\mathcal{Q}$ to (E1) we get:

$$\Lambda\mathcal{Q}P(x,s)=\mathcal{Q}\boldsymbol{\Theta}_\xi\mathcal{P}P(x,s)+\mathcal{Q}\boldsymbol{\Theta}_\xi\mathcal{Q}P(x,s). \tag{E3}$$

This equation can formally be solved by Dyson's series:

$$\mathcal{Q}P(x,s)=\sum_{m=1}^{\infty}\left(\Lambda^{-1}\mathcal{Q}\boldsymbol{\Theta}_\xi\right)^m\mathcal{P}P(x,s), \tag{E4}$$

where Λ^{-1} is the integral representation of Λ with Green's function:

$$g(x,s)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}k}{s^2+\frac{s}{\tau}+v^2k^2}=\exp\left(-\left|\frac{x}{v}\right|\sqrt{s\left(s+1/\tau\right)}\right)\Big/2\sqrt{v^2s\left(s+1/\tau\right)}; \tag{E5}$$

that is:

$$\Lambda^{-1}\left[f(x,s)\right]\to\int_{-\infty}^{+\infty}g\left(x-x',s\right)f\left(x',s\right)\mathrm{d}x'.$$

Introducing (E4) into (E2) we can write:

$$\begin{aligned}\Lambda\mathcal{P}P(x,s)&=\left(s+\frac{1}{\tau}\right)\delta(x)+\mathcal{P}\boldsymbol{\Theta}_\xi\mathcal{P}P(x,s)+\mathcal{P}\boldsymbol{\Theta}_\xi\sum_{m=1}^{\infty}\left(\Lambda^{-1}\mathcal{Q}\boldsymbol{\Theta}_\xi\right)^m\mathcal{P}P(x,s)\\&=\left(s+\frac{1}{\tau}\right)\delta(x)+\mathcal{P}\boldsymbol{\Theta}_\xi\sum_{m=0}^{\infty}\left(\Lambda^{-1}\mathcal{Q}\boldsymbol{\Theta}_\xi\right)^m\mathcal{P}P(x,s),\end{aligned} \tag{E6}$$

which is (40).

Using the space-convolution structure of operator $\Lambda^{-1}[\bullet]$ and integrating by parts, equation (E6) can be transformed to the Fourier representation:

$$\frac{\mathcal{P}P(k,s)}{g(k,s)} = \left(s+\frac{1}{\tau}\right)$$
$$+\int_{-\infty}^{+\infty} \mathrm{d}x\ \mathrm{e}^{ikx}\ \mathcal{P}\partial_x \sum_{m=1}^{\infty} \int_{-\infty}^{+\infty} \mathrm{d}x_1 \cdots \int_{-\infty}^{+\infty} \mathrm{d}x_m\ \langle \xi(x)\xi(x_1)\cdots\xi(x_m)\rangle_{\mathrm{T}}$$
$$\times \left[\partial_{x_1}^2 g(x-x_1,s)\right]\cdots\left[\partial_{x_m}^2 g(x_{m-1}-x_m,s)\right]\partial_{x_m}\langle P(x_m,s)\rangle, \qquad \text{(E7)}$$

where $g(k,s)^{-1} = \left(s^2+\frac{s}{\tau}+v^2k^2\right)$ and $\mathcal{P}\mathbf{\Theta}_\xi = 0$. Therefore we can write:

$$\left(1+k^2 g(k,s)\gamma(k,s)\right)\mathcal{P}P(k,s) = \left(s+\frac{1}{\tau}\right)g(k,s),$$

where $\gamma(k,s) = \sum_{m=1}^{\infty}\gamma_m(k,s)$ can be read from:

$$-k^2\gamma(k,s)\langle P(k,s)\rangle \equiv \int_{-\infty}^{+\infty}\mathrm{d}x\ \mathrm{e}^{ikx}\ \mathcal{P}\mathbf{\Theta}_\xi \sum_{m=1}^{\infty}\left(\Lambda^{-1}\mathcal{Q}\mathbf{\Theta}_\xi\right)^m\langle P(x_m,s)\rangle,$$

from which (43) and (44) follow.

Result (E7) is the starting point to introduce an effective medium approximation to tackle the general problem for arbitrary disorder [33]. But, in the particular case when the random field $\xi(x)$ is an exponential-correlated symmetric binary disorder, Terwiel's cumulants $\langle\xi(x)\xi(x_1)\cdots\xi(x_m)\rangle_{\mathrm{T}}$ adopt the particular structure that they are zero for any $m \geqslant 2$, when x_j-*points* are ordered in a space sequence, see (15) replacing time t by space x. Thus, after reordering integral domains all Terwiel's cumulants with $m \geqslant 2$ cancel. The rest of contributions come from integrals with overlapping domains, which can be neglected.

In summary, even when the m-fold integral (E7), or (44), is not space ordered, for the present binary disorder we can neglect all Terwiell's cumulant with $m \geqslant 2$. Measuring the error on this assumption is still and open question, and will be addressed in a future work.

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