TREATMENT OF THE SPURIOUS STATES IN NUCLEAR FIELD THEORY[☆]

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It is shown, in the framework of a schematic model, that the nuclear field theory (based on the particle-vibration coupling), is a systematic method to eliminate the spurious states occurring in the descriptions of a many-body system which are based on the concept of elementary modes of excitation.

In a previous note [1] the perturbation theory rules for evaluating all the coupling and anharmonicity effects involved when superposing particles and phonons were given. There are, however, situations where a full diagonalization is called for, as in the isolation of a spurious state. In the present note we study a generalization of the model considered in [1] in order to illustrate the application of the nuclear field theory to such situations.

The model considered consists of a number, 2Ω, of single particle levels in which particles interact pairwise by means of a "monopole" force

$$H = H_{sp} + H_{int}, \tag{1}$$

where

$$H_{sp} = \frac{1}{2}\sum_{m=1}^{\Omega} \epsilon_m (a^+_{m,1}a_{m,1} - a^+_{m,-1}a_{m,-1}), \tag{2}$$

and

$$H_{int} = -VA^+A, \tag{3}$$

with

$$A^+ = \sum_{m=1}^{\Omega} a^+_{m,1}a_{m,-1}. \tag{4}$$

The interaction strength is given by the constant V and the energy to excite a particle from the state $(m, -1)$ to its paired level $(m, 1)$ is ϵ_m. The vacuum state $|0\rangle$ is taken to be the Ω-particle configuration with all the $(m,-1)$ states filled and all the $(m, 1)$ states empty.

The energy of the i-th phonon is determined by the RPA dispersion relation (cf. rule (IV) of ref. [1])

$$\sum_{m=1}^{\Omega} \frac{1}{\epsilon_m - \omega_i} = \frac{1}{V} \qquad (i = 1, 2, \dots, \Omega). \tag{5}$$

The eigenfunction corresponding to the different modes is

$$|n_i = 1\rangle = \sum_m \frac{\Lambda_i}{\epsilon_m - \omega_i} a^+_{m,1}a_{m,-1}|0\rangle. \tag{6}$$

The particle-vibration coupling constant is given by

$$\Lambda_i = \langle n_i = 1|H_{int}|m,1;m',-1\rangle = \left[\sum_m \frac{1}{(\epsilon_m - \omega_i)^2}\right]^{-1/2} \delta(m,m'), \tag{7}$$

where $|n_i = 1\rangle$ denotes a state containing one phonon, while $|m, 1; m, -1\rangle$ is the eigenstate of a particle-hole excitation.

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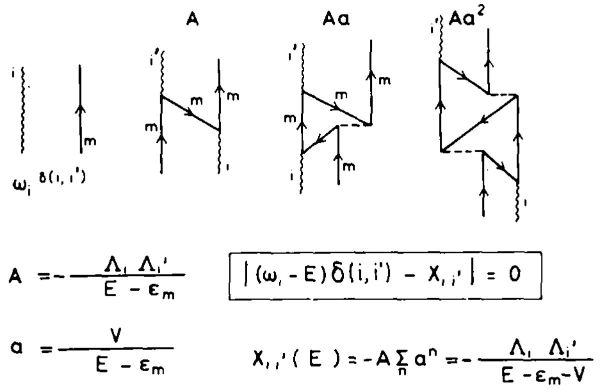

Fig. 1(a) Lower order contributions to the energy matrix element between the basis states $|n_i = 1; m, 1\rangle$. The dotted line stands for the model bare interaction (cf. eq. (8)). The quantity $X_{ii'}(E)$ is the matrix element iterated to all orders in $1/\Omega$. The framed equation is the secular equation of the problem, and is equivalent to the dispersion relation (11).

The other interaction to be included according to ref. [1] is the four-point vertex which has the value

$$\langle m, 1; m', -1 | H_{\text{int}} | m'', 1; m''', -1\rangle = -V\delta(m, m')\,\delta(m'', m''') . \tag{8}$$

The single-particle energies to be used in calculating the different graphs are $\frac{1}{2}\epsilon_m$, because the Hartree-Fock contribution of H_{int} is zero (rule (IV) of ref. [1]).

Similarly to H_{int}, the "inelastic operator" $a^+_{m,1}$, $a_{m,-1}$ has two different matrix elements, namely

$$\langle n_i = 1 | a^+_{m,1} a_{m,-1} | 0\rangle = \frac{\Lambda_i}{\epsilon_m - \omega_i} , \tag{9}$$

and

$$\langle m', 1; m'', -1 | a^+_{m,1} a_{m,-1} | 0\rangle = \delta(m, m')\delta(m', m'') . \tag{10}$$

In what follows we discuss the simplest system displaying spurious states, namely the system comprising an odd particle in the orbit $(m, 1)$, in addition to a single phonon excitation of the vacuum‡. If we displace the zero point energy to $\frac{1}{2}\epsilon_m$, the unperturbed energy of the basis state $|n_i = 1; m, 1\rangle$ is ω_i.

The lower order corrections to this energy which do not contain bubbles (cf. rule (III). ref. [1]) are drawn in fig. 1(a). Iterating these processes to infinite order we obtain the secular equation displayed in the same figure and which is equivalent to the dispersion relation

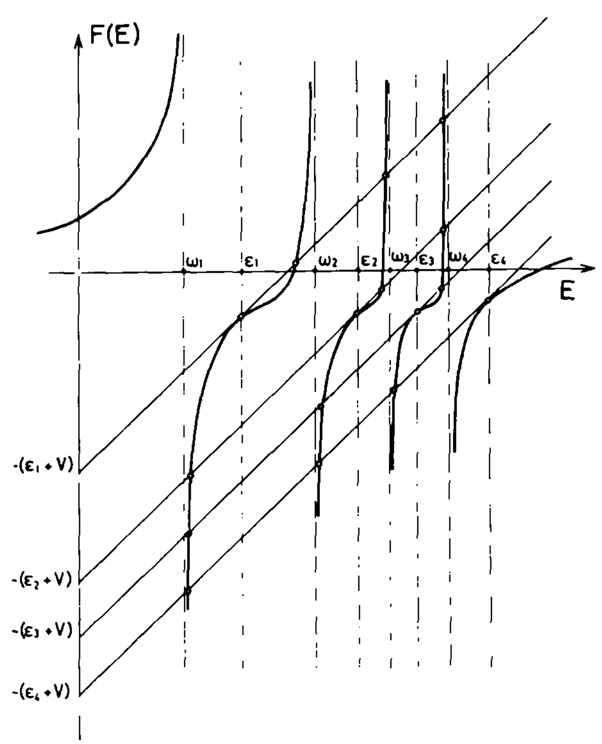

Fig. 1(b) Graphical solution of the dispersion relation (11), for the case $\Omega = 4$. The function $F(E) = \Sigma_i \Lambda_i^2/(\omega_i - E)$ is displayed as a continuous thick line while the parallel lines $E - \epsilon_m - V$ have been drawn as thin continuous lines intersecting the ordinate axis at $-(\epsilon_m + V)$. The intersections between the two functions give the eigenvalues of the secular equation. For each value of ϵ_m there are $\Omega + 1$ roots, the root at $E = \epsilon_m$ being double.

$$F(E) = \sum_{i=1}^{\Omega} \frac{\Lambda_i^2}{\omega_i - E} = E - \epsilon_m - V . \tag{11}$$

There is one equation for each single-particle level because the monopole force cannot change the m-state of the odd particle. The relation (11) can be solved

‡ According to rule (I) of ref. [1], initial and final states may involve both collective modes and particle modes, but not any particle configuration that can be replaced by a combination of collective modes. The exclusion of the states $|m, 1; m', 1; m', -1\rangle$ eliminates most of the double counting of two-particle, one-hole states. The Ω "proper" states of the form $|n_i = 1; m, 1\rangle$ are allowed. However, there are only Ω-1 two-particle, one-hole states in which the odd-particle is in the state $(m, 1)$. Therefore, a spurious state remains in the spectrum based on elementary modes of excitation.

graphically as shown in fig. 1(b). The energy $E = \epsilon_m$ is always a root of (11), in fact a double root since

$$\left[\frac{\mathrm{d}F(E)}{\mathrm{d}E}\right]_{E=\epsilon_m} = \sum_i \frac{\Lambda_i}{(\omega_i - \epsilon_m)^2} = 1 \qquad (12)$$

and the line $E - \epsilon_m - V$ is at 45°. The remaining intersections of this line and the function $F(E)$ give rise to $\Omega - 1$ additional roots whose energy agrees with the physical eigenvalues obtained from the exact solution of the model. We denote these roots (qm) and the amplitudes of the basis states in the diagonalized eigenvectors are given by

$$\xi_{iqm} \equiv \langle qm | n_i = 1; m, 1\rangle = -N_{qm} \frac{\Lambda_i}{\omega_i - E_{qm}}, \qquad (13)$$

where N_{qm} is the normalization of the state and is determined by [2]

$$1 = \langle qm | qm\rangle$$

$$= \sum_{i,i'} \left(\delta(i,i') - \left[\frac{\partial X_{ii'}(E)}{\partial E}\right]_{E=E_{qm}}\right) \xi_{i'qm}\xi^*_{iqm} \qquad (14)$$

$$= N^2_{qm}\left[\sum_i \frac{\Lambda_i}{(\omega_i - E_{qm})^2} - 1\right],$$

where $X_{ii'}$ is the matrix appearing in the secular equation and given in fig. 1(a). For $E_{qm} = \epsilon_m$ the factor multiplying N^2_{qm} is zero (cf. eq. (12)). There are thus only $\Omega - 1$ states (for given m) that have a finite normalization.

The above results can be illuminated from a somewhat different point of view, in terms of the non-orthogonality and over-completeness of the basis states $(n_i = 1; m, 1)$ when expanded in terms of two particle-one hole components. Thus we obtain the overlap matrix

$$Z_{ii'}(\epsilon_m) \equiv \langle n_i = 1; m, 1 | n_{i'} = 1; m, 1\rangle$$

$$= \sum_{m' \neq m} \frac{\Lambda_i \Lambda_{i'}}{(\epsilon_{m'} - \omega_i)(\epsilon_{m'} - \omega_{i'})} \qquad (15)$$

$$= \delta(i,i') - \frac{\Lambda_i \Lambda_{i'}}{(\epsilon_m - \omega_i)(\epsilon_m - \omega_{i'})}.$$

Because of the non-orthogonality of the basis, the eigenvalues of the system are determined by the relation $|Z(E)\,(H - E)| = 0$. This is fulfilled for $|H - E| = 0$ which yields the $\Omega - 1$ physical roots, as well as for $|Z(E)| = 0$ which occurs for the spurious root $E_{qm} = \epsilon_m$ of (11).

In order to illustrate the use of the eigenvectors obtained above by diagonalization, we shall calculate the transition matrix elements for single particle transfer and inelastic scattering leading to the states (qm). One first calculates the amplitude for the transition to a basis component $(n_i = 1; m, 1)$ including only those graphs in which all intermediate states are excluded from appearing as initial or final states by the rule IV of ref. [1]. This exclusion reflects the fact that the diagonalization procedure has included all interaction effects that link these allowed states. The final amplitude for the transition to the state (qm) is obtained by summing the amplitudes to $(n_i = 1; m, 1)$ each weighted by the amplitude ξ_{iqm} given by eq. (13).

Inelastic scattering. The lower order contributions to the "inelastic scattering" process are displayed in fig. 2(a). For $m \neq m'$ only the phonon part of the operator $a^+_{m',1}a_{m',-1}$ (i.e. graph (a) in fig. 2(a) gives a contribution. It is equal to

$$\langle qm | a^+_{m',1} a_{m',-1} | m, 1\rangle$$

$$= \sum_i \frac{\Lambda_i \xi_{iqm}}{\epsilon_{m'} - \omega_i} = N_{qm} \frac{E_{qm} - \epsilon_m}{E_{qm} - \epsilon_m}, \qquad (16)$$

and agrees with the exact result. For $m = m'$ the different contributions of fig. 2(a) give, when summed to all orders,

$$\langle qm | a^+_{m,1} a_{m,-1} | m, 1\rangle$$

$$= \sum_i \xi_{iqm} \frac{\Lambda_i}{\epsilon_m - \omega_i} + \sum_i \xi_{iqm} M_{iqm} = 0 \qquad (17)$$

as expected. In fact, the matrix element (17) measures the amount of the "Pauli principle violating" component $|m, 1; m, -1; m, 1\rangle$ that appears in the physical state (qm).

The inelastic sum rule

$$\sum_q |\langle qm | a^+_{m',1} a_{m',-1} | m, 1\rangle|^2 = 1 - \delta(m, m'), \qquad (18)$$

is obtained from (16) and (17) and again reproduces the exact result. It expresses the completeness of the states (qm).

One particle transfer. The lower order contribu-

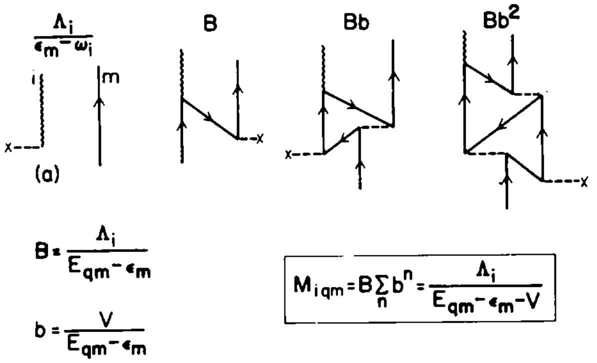

Fig. 2(a) Lower order contributions to the inelastic process induced by $a^+_{m,1} a_{m,-1}$. The iteration of the different contributions to all orders in $1/\Omega$ gives as result the transition matrix element M_{iqm}.

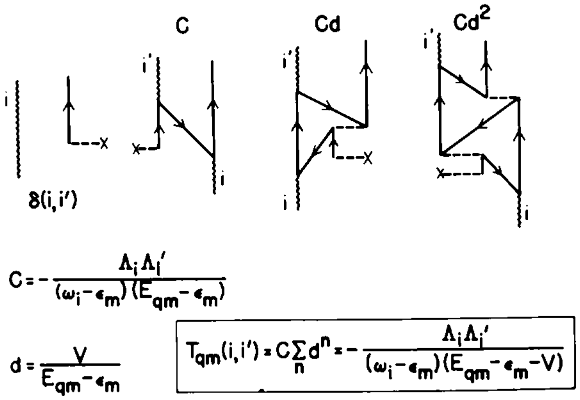

Fig. 2(b) Lower order contributions to the one-particle transfer reaction induced by $a^+_{m,1}$. The result of iterating the different contributions to all orders in $1/\Omega$ is equal to $T_{qm}(ii')$.

tions to the one-particle transfer amplitude between the state $|n_i = 1\rangle$ and the state $|qm\rangle$ are displayed in fig. 2(b). They can be summed up to all orders of $1/\Omega$, the result being equal to

$$\langle qm | a^+_{m,1} | n_i = 1 \rangle = \sum_{i'} \xi_{i'mq} [\delta(i, i') - T_{qm}(i, i')]$$

$$= N_{qm} \Lambda_i \frac{E_{qm} - \epsilon_m}{(E_{qm} - \omega_i)(\omega_i - \epsilon_m)} \tag{19}$$

This quantity is zero for the spurious roots (i.e. $E_{qm} = \epsilon_m$) and agrees with the exact result for the $\Omega - 1$ remaining physical roots. The one-particle transfer sum rule is given by

$$\sum_q |\langle qm | a^+_{m,1} | n_i = 1 \rangle|^2 = 1 - \frac{\Lambda_i^2}{(\epsilon_m - \omega_i)^2}, \tag{20}$$

which is also the exact answer.

Based on the above empirical evidence it is our belief that the rules formulated in [1] represent a general prescription for evaluating all coupling and anharmonicity effects involved when superimposing particles and phonons, also when the solution of the physical system requires a diagonalization. However, in most situations where the nuclear field theory will be applied, the problem admits a perturbation treatment.

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References

[1] D.R. Bes et al., Phys. Lett. 52B (1974) 253.
[2] D.R. Bes et al., Nucl. Phys. A260 (1976) 1. See especially Appendix B.