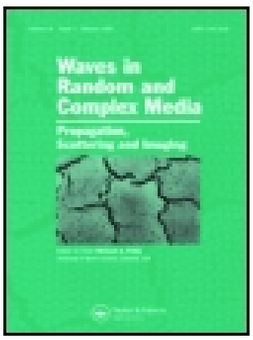

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Functional integral approach to the transfer function of a stochastic scattering channel

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ABSTRACT

We apply the formalism of functional integration to the calculation of the transfer function of a stochastic scattering channel formed by stationary, non-interacting point scatterers. The channel is described through a scattering amplitude density, defined over space, whose random component is characterized by a functional probability distribution. This random component induces in turn a probability distribution for the scattering transfer function, which we compute by means of functional integration in the case of Gaussian distributions. Some geometric configurations relevant to radar operation are worked out, as well as the statistical properties of the transfer function in the large-frequency limit. Extensions of the formulation in order to include scattering phase shifts, i.e. complex scattering amplitudes, and to consider non-Gaussian probability distributions are outlined.

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1. Introduction

In most of its applications, ranging from target detection to terrain characterization, radar operation is based on comparing emitted and received signals. Reflections at material objects in the environment modify the original emission, so that differences between emitted and received signals bear information on the position, the state of motion, and the nature of the scatterers where reflections occurred. Modeling and estimation of the scattering channel embodied by the environment is thus a crucial prerequisite for performance prediction, and interpretation of radar measurements by means of signal processing. Standard models for signal propagation and radar cross-section distributions [1,2] are based on phenomenological considerations derived from both heuristic arguments and empirical results. First-principle formulations, on the other hand, can be used to construct models better grounded on physical bases, although they are usually restricted to stylized representations of scatterer distributions [3]. Both approaches, in any case, must be able to account for random ingredients in the scattering channel, whose consideration is virtually unavoidable when processing radar signals [4,5].

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In this paper, in order to evaluate how stochasticity in the environmental distribution of radar cross section affects signal transmission, we apply functional integral techniques to calculate the probability distribution of the transfer function of a scattering channel. Functional integrals constitute a standard, but very powerful, mathematical tool in the computation of physical quantities of interest to such areas as quantum field theory, statistical mechanics, and solid-state physics [6–8]. They are based on extending the procedures of integral calculus to mathematical sets whose elements are functions – instead of the real or complex variables of ordinary integrals. In our specific case, as explained in detail in Section 2, the transfer function $\tau(\omega)$ that characterizes the scattering channel for each frequency component in the transmitted signal, can be formally expressed as an integral involving the distribution of scattering amplitudes over space, $\rho(\mathbf{r})$, in the form $\tau = J[\rho]$. The integral operator J acts on the set of functions $\rho(\mathbf{r})$ to yield a function $\tau(\omega)$ in a different set. Starting from a probability distribution defined on the first set, which represents the uncertainty that stochastic elements induce in the scattering amplitude distribution $\rho(\mathbf{r})$, functional integrals make it possible to calculate the corresponding probability distribution for $\tau(\omega)$ in the second set.

In the next section, we define the transfer function of a channel formed by non-interacting stationary point scatterers arbitrarily distributed over space, and provide an integral form based on the spatial distribution of scattering amplitudes, separating deterministic and stochastic contributions. Assuming that random ingredients in the environment determine a probability distribution over the space of scattering amplitude densities, in Section 3 we apply the functional integral approach to find the corresponding probability distribution for the stochastic part of the transfer function. Emphasis is put on a Gaussian probability distribution for the scattering amplitude density, which can be given a full analytical solution within the present formulation. In Section 4, we first work out some explicit examples for the transfer function probability distribution, within different radar configurations, and discuss the large-frequency limit in rather generic cases. Then, we outline extensions of functional integrals to take into account random phase shifts in scattering events, and to consider non-Gaussian distributions within the steepest-descent approximation. Finally, conclusions are drawn in Section 5. For the reader's convenience, we include an appendix summarizing the basic procedures of functional integral calculus.

2. Transfer function for stationary point scatterers

Consider a signal of amplitude $T(t)$ emitted at position $\mathbf{r}_T$ and received at position $\mathbf{r}_R$. Transmitter and receiver are both assumed to be stationary and isotropic. If propagation occurs in the presence of N non-interacting stationary point scatterers, the contribution of the reflections at the scatterers to the received signal amplitude is [2]

$$R_s(t) = \sum_{k=1}^{N} \frac{\sqrt{\sigma_k}}{d_T^{(k)} d_R^{(k)}} T(t - t_k). \tag{1}$$

Here $d_T^{(k)} = |\mathbf{r}_k - \mathbf{r}_T|$ and $d_R^{(k)} = |\mathbf{r}_k - \mathbf{r}_R|$ are the distances from the scatterer k, situated at position $\mathbf{r}_k$, respectively to the transmitter and the receiver. The time $t_k = (d_T^{(k)} + d_R^{(k)})/c$ is the total delay of the echo coming from scatterer k, assuming that the signal velocity is c.

The factor $\sqrt{\sigma_k}$, where σ_k denotes the radar cross section (RCS) of scatterer k, is the *scattering amplitude* associated to that scatterer.

Fourier transforming Equation (1), we obtain the relation between the transmitted signal and the scattering contribution to the received signal in the frequency domain:

$$\tilde{R}_s(\omega) = \tau(\omega)\tilde{T}(\omega). \tag{2}$$

The quantity

$$\tau(\omega) = \sum_{k=1}^{N} \frac{\sqrt{\sigma_k}}{d_T^{(k)} d_R^{(k)}} \exp\left[-\mathrm{i}\frac{\omega}{c}\left(d_T^{(k)} + d_R^{(k)}\right)\right] \tag{3}$$

is the *transfer function* of the scattering channel. The transfer function is a frequency-by-frequency characterization of the effect of scatterers on the transmitted signal. This makes it possible to readily extend the definition of $\tau(\omega)$ in order to include a dependence on frequency in the RCS: $\sigma_k \to \sigma_k(\omega)$. The frequency-by-frequency description provided by the transfer function, moreover, allows us to omit explicit indication of the dependence on ω, as we do in the following to simplify the notation.

We now introduce a scattering amplitude density over space, $\rho(\mathbf{r})$, formally defined by the identity

$$\rho(\mathbf{r})\,\mathrm{d}\mathbf{r} = \sum_{k \text{ in } \mathrm{d}\mathbf{r}} \sqrt{\sigma_k}, \tag{4}$$

where the sum in the right-hand side runs over all the scatterers inside the volume element $\mathrm{d}\mathbf{r}$ situated at position $\mathbf{r}$. Using this definition, the transfer function can be expressed as the integral

$$\tau_\rho = \int \rho(\mathbf{r}) H(\mathbf{r})\,\mathrm{d}\mathbf{r} \equiv u_\rho + \mathrm{i}v_\rho, \tag{5}$$

where

$$H(\mathbf{r}) = \frac{\exp[-\mathrm{i}\omega(|\mathbf{r}-\mathbf{r}_T| + |\mathbf{r}_R-\mathbf{r}|)/c]}{|\mathbf{r}-\mathbf{r}_T||\mathbf{r}_R-\mathbf{r}|} \equiv f(\mathbf{r}) + \mathrm{i}g(\mathbf{r}) \tag{6}$$

accounts for the phase change and attenuation of the transmitted signal in the path $\mathbf{r}_T \to \mathbf{r} \to \mathbf{r}_R$. The subindex ρ in Equation (5) emphasizes, for future convenience, that this is the expression for the transfer function of the channel associated with the scattering amplitude density $\rho(\mathbf{r})$. In the above equations, moreover, we have introduced the complex decomposition of τ_ρ and $H(\mathbf{r})$ into their real and imaginary parts.

In the integrand of Equation (5), the function $H(\mathbf{r})$ encompasses both geometric and signal attenuation effects associated with reflection at a scatterer with generic position $\mathbf{r}$, while the density $\rho(\mathbf{r})$ captures all the information on the number and physical nature of scatterers in the volume element $d\mathbf{r}$. It is $\rho(\mathbf{r})$, in fact, which accounts for any random ingredients in the scattering channel, associated with the distribution and properties of scatterers. Generally, we can split $\rho(\mathbf{r})$ into a deterministic and a random component, $\rho(\mathbf{r}) \equiv \rho^D(\mathbf{r}) + \rho^R(\mathbf{r})$. As advanced in the introduction, moreover, the first identity in Equation (5) can be written as $\tau_\rho = J[\rho]$, where J is a linear integral operator. This entails a homologous separation of the transfer function into a deterministic and a stochastic component, $\tau_\rho \equiv \tau_\rho^D + \tau_\rho^R$, with $\tau_\rho^{D,R} = J[\rho^{D,R}]$. The random density component $\rho^R(\mathbf{r})$ can be statistically characterized by

specifying a probability distribution over the functional space which contains all the possible forms that $\rho^R(\mathbf{r})$ can adopt. Through the relation $\tau_\rho^R = J[\rho^R]$, in turn, the probability distribution for ρ^R induces a probability distribution for τ_ρ^R defined on the corresponding functional space.

3. Functional integral calculation of the transfer function probability

Starting from the relation $\tau_\rho^R = J[\rho^R]$, in this section, we use functional integral procedures to establish the connection between the probability distributions associated with the random components of the scattering amplitude density, $\rho^R(\mathbf{r})$, and of the transfer function, τ_ρ^R. For clarity in the notation, from now on we drop the superindex R, on the understanding that the following analysis involves the random components only. For brevity, moreover, we respectively call ρ-space and τ-space the functional spaces containing the random components of each function.

We assume that the probability distribution over ρ-space is given by the functional

$$Q[\rho] = \frac{F[\rho]}{\int F[\rho]\,\mathrm{D}\rho}. \tag{7}$$

The functional $F[\rho]$ defines the profile of the probability distribution, while the integral in the denominator stands for its normalization. The notation $\int \cdots \mathrm{D}\rho$ indicates functional integration over ρ-space. In Appendix 1, we summarize how this operation is implemented in practice, as a limit of summations of $\rho(\mathbf{r})$ over a discrete set of positions $\mathbf{r}$.

The distribution $P[\tau]$ for the probability that the transfer function adopts a certain value $\tau = u + \mathrm{i}v$ in τ-space can be expressed as the mean value of the Dirac delta $\delta(\tau_\rho - \tau) = \delta(u_\rho - u)\delta(v_\rho - v)$ over ρ-space, where u_ρ and v_ρ are the real and imaginary parts of τ_ρ [9]; cf. Equation (5). Namely,

$$P[\tau] = \langle \delta(u_\rho - u)\delta(v_\rho - v)\rangle_\rho = \frac{\int \delta(u_\rho - u)\delta(v_\rho - v)F[\rho]\,\mathrm{D}\rho}{\int F[\rho]\,\mathrm{D}\rho}. \tag{8}$$

Using the Fourier representation for the Dirac deltas in the integrand, $\delta(x-\mu) = (2\pi)^{-1}\int_{-\infty}^{\infty} \exp[\mathrm{i}\nu(x-\mu)]\,\mathrm{d}\nu$, we write

$$P[\tau] = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Z(\alpha,\beta)\exp[\mathrm{i}(\alpha u + \beta v)]\,\mathrm{d}\alpha\,\mathrm{d}\beta, \tag{9}$$

with

$$Z(\alpha,\beta) = \frac{\int F[\rho]\exp\left(W[\rho]\right)\,\mathrm{D}\rho}{\int F[\rho]\,\mathrm{D}\rho} \tag{10}$$

the characteristic function of the random variable τ [9]. Here,

$$W[\rho] = -\mathrm{i}(\alpha u_\rho + \beta v_\rho) = -\mathrm{i}\int \rho(\mathbf{r})h(\mathbf{r})\,\mathrm{d}\mathbf{r}, \tag{11}$$

with $h(\mathbf{r}) = \alpha f(\mathbf{r}) + \beta g(\mathbf{r})$; cf. Equation (6).

3.1. Gaussian scattering channel

Functional integrals such as those appearing in Equations (7), (8), and (10) can be explicitly calculated for a handful of integrands only [10]. A solvable case relevant to our problem is that of a Gaussian profile for the probability distribution in ρ-space,

$$F[\rho] = \exp\left[-\frac{1}{2}\iint \rho(\mathbf{r})K(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{r}'\right], \tag{12}$$

where the integrals on $\mathbf{r}$ and $\mathbf{r}'$ extend over the spatial domain relevant to the scattering amplitude density of the physical problem under study. This form for $F[\rho]$ represents a distribution centered at $\rho(\mathbf{r}) = 0$. The kernel $K(\mathbf{r},\mathbf{r}')$ – whose form must insure that the double integral in Equation (12) is non-negative for all $\rho(\mathbf{r})$ – plays the role of an inverse covariance for the distribution [9], in the sense that the two-point correlation function averaged over ρ-space is

$$\langle \rho(\mathbf{r})\rho(\mathbf{r}')\rangle_\rho = \bar{K}(\mathbf{r},\mathbf{r}'), \tag{13}$$

with

$$\int \bar{K}(\mathbf{r},\mathbf{s})K(\mathbf{s},\mathbf{r}')\,\mathrm{d}\mathbf{s} = \delta(\mathbf{r}-\mathbf{r}'). \tag{14}$$

Moreover, $K(\mathbf{r},\mathbf{r}')$ – and, consequently, $\bar{K}(\mathbf{r},\mathbf{r}')$ – can always be chosen symmetric with respect to the two variables, $K(\mathbf{r},\mathbf{r}') = K(\mathbf{r}',\mathbf{r})$, as the antisymmetric part does not contribute to the integral in Equation (12).

Replacing Equation (12) into Equation (10), the characteristic function turns out to be

$$Z(\alpha,\beta) = \frac{\int \exp\left[W[\rho] - \frac{1}{2}\iint \rho(\mathbf{r})K(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{r}'\right]\mathrm{D}\rho}{\int \exp\left[-\frac{1}{2}\iint \rho(\mathbf{r})K(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{r}'\right]\mathrm{D}\rho}. \tag{15}$$

The argument of the exponential in the upper integral of this equation is a quadratic functional of $\rho(\mathbf{r})$, with linear terms coming from $W[\rho]$ – see Equation (11) – and quadratic terms coming from the exponent in the Gaussian profile $F[\rho]$. The form of this quadratic functional can be simplified by 'completing the square', just as in elementary algebra, with respect to the variable ρ. In fact, writing $\rho = \hat{\rho} + \rho_0$, with

$$\rho_0(\mathbf{r}) = -\mathrm{i}\int h(\mathbf{r}')\bar{K}(\mathbf{r},\mathbf{r}')\,\mathrm{d}\mathbf{r}', \tag{16}$$

we get

$$\begin{aligned} W[\rho] - \frac{1}{2}\iint \rho(\mathbf{r})K(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{r}' \\ = -\frac{1}{2}\iint \hat{\rho}(\mathbf{r})K(\mathbf{r},\mathbf{r}')\hat{\rho}(\mathbf{r}')\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{r}' - \frac{1}{2}\iint h(\mathbf{r})\bar{K}(\mathbf{r},\mathbf{r}')h(\mathbf{r}')\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{r}'. \end{aligned} \tag{17}$$

The change of variables $\hat{\rho} = \rho - \rho_0$ is equivalent to a Popov–Faddeev canonical transformation [8], of standard use both in quantum and classical applications of functional integration [6,7]. It amounts to shifting the origin in ρ-space to ρ_0, so that $\mathrm{D}\rho = \mathrm{D}\hat{\rho}$. Consequently, when the right-hand side of Equation (17) is replaced into Equation (15), the

functional integral of its first term over ρ-space exactly cancels the denominator in the same equation. As a result of this cancelation we obtain [6,7]

$$Z(\alpha,\beta) = \exp\left[-\frac{1}{2}\iint h(\mathbf{r})\bar{K}(\mathbf{r},\mathbf{r}')h(\mathbf{r}')\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{r}'\right]. \tag{18}$$

At this point, all functional integrals have been computed. The calculation of the characteristic function $Z(\alpha,\beta)$ from Equation (18) and, subsequently, of $P[\tau]$ from Equation (8) now involves ordinary integration only. In terms of the variables α and β, we have

$$Z(\alpha,\beta) = \exp\left[-\frac{1}{2}\left(\Sigma_{11}\alpha^2 + 2\Sigma_{12}\alpha\beta + \Sigma_{22}\beta^2\right)\right], \tag{19}$$

where the coefficients in the exponent are

$$\begin{aligned}
\Sigma_{11} &= \iint f(\mathbf{r})\bar{K}(\mathbf{r},\mathbf{r}')f(\mathbf{r}')\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{r}',\\
\Sigma_{12} &= \iint f(\mathbf{r})\bar{K}(\mathbf{r},\mathbf{r}')g(\mathbf{r}')\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{r}',\\
\Sigma_{22} &= \iint g(\mathbf{r})\bar{K}(\mathbf{r},\mathbf{r}')g(\mathbf{r}')\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{r}',
\end{aligned} \tag{20}$$

which can be arranged as a symmetric matrix $\Sigma = \left(\begin{smallmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{12} & \Sigma_{22}\end{smallmatrix}\right)$. Replacing Equation (19) into Equation (8), the probability distribution for the random component of the transfer function can be explicitly evaluated, as

$$P[\tau] = \frac{1}{\sqrt{4\pi^2|\Sigma|}}\exp\left[-\frac{1}{2}\left(\bar{\Sigma}_{11}u^2 + 2\bar{\Sigma}_{12}uv + \bar{\Sigma}_{22}v^2\right)\right], \tag{21}$$

in terms of the real and imaginary parts of $\tau = u + \mathrm{i}v$. The coefficients $\bar{\Sigma}_{ij}$ are the elements of the matrix $\bar{\Sigma} = \Sigma^{-1}$, and $|\Sigma|$ denotes the determinant of Σ. For the Gaussian scattering channel, $P[\tau]$ is therefore a bivariate normal distribution on the variables u and v, centered at $u = v = 0$ and with covariance matrix Σ. The respective expectation values, variances, and covariance [9] are

$$\begin{aligned}
\mathrm{E}[u] = \mathrm{E}[v] &= 0,\\
\mathrm{Var}[u] = \Sigma_{11}, \quad \mathrm{Var}[v] &= \Sigma_{22},\\
\mathrm{Cov}[u,v] &= \Sigma_{12}.
\end{aligned} \tag{22}$$

Recall that, in general, these variances and covariance depend on the frequency ω through the functions $f(\mathbf{r})$ and $g(\mathbf{r})$ in Equations (20); cf. Equation (6). The quantities in Equations (22) are to be interpreted as averages over realizations of the random component of the scattering amplitude density following the Gaussian distribution of Equation (12), which induces the probability distribution of Equation (21) on τ-space.

4. Examples and extensions

4.1. Delta-correlated scattering amplitude densities

The computation of the integrals in Equations (20) generally requires to resort to numerical techniques. A case where they can be analytically evaluated is that of delta-correlated

scattering amplitude densities,

$$\bar{K}(\mathbf{r},\mathbf{r}') = \kappa_0 \delta(\mathbf{r}-\mathbf{r}'), \tag{23}$$

where κ_0 weights the autocorrelation of $\rho(\mathbf{r})$; cf. Equation (13). Although this choice may a priori seem trivial, it already induces nontrivial statistical correlations in the real and imaginary parts of the transfer function. In fact, u and v are both given by the same density $\rho(\mathbf{r})$ and are therefore statistically interdependent.

With the kernel given by Equation (23), the elements of the transfer function covariance matrix read

$$\Sigma_{11} = \kappa_0 \int f(\mathbf{r})^2 \,\mathrm{d}\mathbf{r}, \quad \Sigma_{12} = \kappa_0 \int f(\mathbf{r}) g(\mathbf{r}) \,\mathrm{d}\mathbf{r}, \quad \Sigma_{22} = \kappa_0 \int g(\mathbf{r})^2 \,\mathrm{d}\mathbf{r}. \tag{24}$$

These integrals can be calculated in a variety of geometrical configurations, as shown in the following. Generally, however, they yield finite values if the integration domain does not include the positions of the transmitter and the receiver, $\mathbf{r}_T$ and $\mathbf{r}_R$, where $f(\mathbf{r})$ and $g(\mathbf{r})$ are singular; cf. Equation (6). These singularities are related to the divergence of the contributions of scatterers infinitely close to those positions [11]. In practice, scatterers in the immediate vicinity of transmitter and receiver have to be treated separately from our continuous description of scattering amplitudes. The integrals in Equations (24) therefore must be calculated by excluding a specified, typically small, domain around $\mathbf{r}_T$ and $\mathbf{r}_R$.

4.1.1. Monostatic configuration in three-dimensional space

In a monostatic radar, transmitter and receiver are collocated [12]. For this configuration, we take $\mathbf{r}_T = \mathbf{r}_R = \mathbf{0}$, so that $f(\mathbf{r}) = r^{-2}\cos(2\omega r/c)$ and $g(\mathbf{r}) = -r^{-2}\sin(2\omega r/c)$. If the relevant spatial domain is three-dimensional space, with a spherical excluded volume of radius r_0 centered at the origin, we find

$$\begin{aligned}
\Sigma_{11} &= \frac{2\pi\kappa_0}{r_0}\left(1 + \cos x + x\,\mathrm{si}\,x\right), \\
\Sigma_{12} &= \frac{2\pi\kappa_0}{r_0}\left(-\sin x + x\,\mathrm{Ci}\,x\right), \\
\Sigma_{22} &= \frac{2\pi\kappa_0}{r_0}\left(1 - \cos x - x\,\mathrm{si}\,x\right),
\end{aligned} \tag{25}$$

with $x = 4r_0\omega/c$ and $\mathrm{si}\,x = -\int_x^\infty y^{-1}\sin y\,\mathrm{d}y$, $\mathrm{Ci}\,x = -\int_x^\infty y^{-1}\cos y\,\mathrm{d}y$ the complementary sine integral and the cosine integral, respectively. Note that the sum $\Sigma_{11} + \Sigma_{22}$ is independent of the frequency. This is a general property when the scattering amplitude densities are delta-correlated. In fact, Equations (24) show that $\Sigma_{11} + \Sigma_{22} = \kappa_0 \int |H(\mathbf{r})|^2 \,\mathrm{d}\mathbf{r}$, with $H(\mathbf{r})$ given by Equation (6), whose modulus does not depend on ω.

Figure 1 shows the rescaled elements of the covariance matrix, $r_0\Sigma_{ij}/2\pi\kappa_0$ ($i,j = 1,2$), as functions of the rescaled frequency $4r_0\omega/c$. Due to the fact that the transfer function is a real quantity for $\omega = 0$, $\Sigma_{11}(0)$ is positive, while $\Sigma_{12}(0)$ and $\Sigma_{22}(0)$ vanish. For large ω, in turn, the variances Σ_{11} and Σ_{22} asymptotically approach finite values. Meanwhile Σ_{12} goes to zero: in the large-frequency limit, the real and imaginary parts of τ become mutually uncorrelated. In all cases, the behavior is oscillatory, with period 2π in the rescaled variable. The amplitude of oscillations decays as ω^{-1}. Note that, since the transfer function generally decreases with ω for large frequencies, the fact that the variances Σ_{11} and Σ_{22} remain

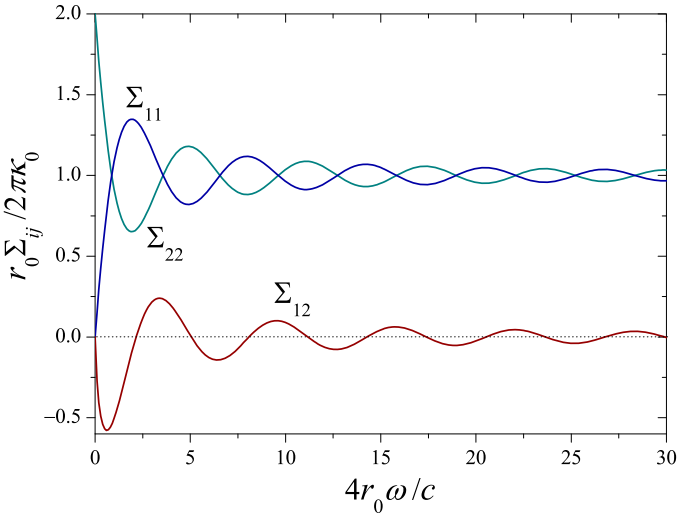

Figure 1. Rescaled elements of the covariance matrix, $r_0\Sigma_{ij}/2\pi\kappa_0$ ($i,j = 1,2$), as functions of the rescaled frequency $4r_0\omega/c$ for a monostatic configuration in three-dimensions, with an excluded spherical volume of radius r_0 centered at the position of transmitter and receiver.

finite implies that relative fluctuations in the transfer function become larger as ω grows; cf. Equation (22).

4.1.2. Monostatic configuration in two-dimensional space

When scatterers are coplanar, and transmitter and receiver are located at the same position and on the same plane as the scatterers, the calculation of Σ_{ij} is similar to the three-dimensional case considered above. Excluding a disk of radius r_0 centered at the position of transmitter and received, we get

$$\begin{aligned}
\Sigma_{11} &= \frac{\pi\kappa_0}{2r_0^2}\left(1+\cos x - x\sin x + x^2\,\mathrm{Ci}\,x\right),\\
\Sigma_{12} &= \frac{\pi\kappa_0}{2r_0^2}\left(-\sin x - x\cos x - x^2\,\mathrm{si}\,x\right),\\
\Sigma_{22} &= \frac{\pi\kappa_0}{2r_0^2}\left(1-\cos x + x\sin x - x^2\,\mathrm{Ci}\,x\right),
\end{aligned} \tag{26}$$

with $x = 4r_0\omega/c$. In spite of the differences between the functional dependences of the covariance elements given in Equations (25) and (26), their behavior is qualitatively the same. Even more, the rescaled elements $2r_0^2\Sigma_{ij}/\pi\kappa_0$ calculated from Equation (26) have exactly the same asymptotic values for $\omega \to 0$ and $\omega \to \infty$ as those shown in Figure 1, and they also oscillate with frequency 2π in the rescaled frequency $4r_0\omega/c$. Figure 1, therefore, provides a close depiction of the rescaled covariance elements also for the monostatic two-dimensional configuration.

4.1.3. Bistatic configuration in two-dimensional space

In bistatic configurations, transmitter and received are located at different positions, $\mathbf{r}_T \neq \mathbf{r}_R$ [12]. The distance between the two points, which we take to be $|\mathbf{r}_T - \mathbf{r}_R| = 2d_0$, is called

baseline of the radar setup. Over two-dimensional space, in this case, the integrals in Equations (24) can be computed using elliptical coordinates with foci at $\mathbf{r}_T$ and $\mathbf{r}_R$. To preserve the elliptic symmetry of the problem, the excluded integration domain is the interior of an ellipse of major semi-axis a_0 ($\gtrsim d_0$), with the same foci as the coordinate system. Along the line which passes through the transmitter and received, thus, the distance between the foci and the inner boundary of the integration domain is $a_0 - d_0$.

The elements of the covariance matrix turn out to be

$$\begin{aligned}
\Sigma_{11} &= \frac{\pi\kappa_0}{2d_0^2}\left[s_0 + 2\mathrm{Ci}\,X - (\mathrm{Ci}\,x_+ + \mathrm{Ci}\,x_-)\cos x - (\mathrm{si}\,x_+ - \mathrm{si}\,x_-)\sin x\right],\\
\Sigma_{12} &= \frac{\pi\kappa_0}{2d_0^2}\left[-2\mathrm{si}\,X - (\mathrm{Ci}\,x_+ - \mathrm{Ci}\,x_-)\sin x + (\mathrm{si}\,x_+ + \mathrm{si}\,x_-)\cos x\right],\\
\Sigma_{22} &= \frac{\pi\kappa_0}{2d_0^2}\left[s_0 - 2\mathrm{Ci}\,X + (\mathrm{Ci}\,x_+ + \mathrm{Ci}\,x_-)\cos x + (\mathrm{si}\,x_+ - \mathrm{si}\,x_-)\sin x\right],
\end{aligned} \tag{27}$$

with $s_0 = \ln[a_0^2/(a_0^2 - d_0^2)]$, $x_\pm = 4(a_0 \pm d_0)\omega/c$, $x = (x_+ - x_-)/2 = 4d_0\omega/c$, and $X = (x_+ + x_-)/2 = 4a_0\omega/c$. In contrast with the monostatic cases considered previously, results for the bistatic configuration do not admit a simple rescaling of the frequency variable, due to the presence of two mutually independent characteristic lengths, d_0 and a_0.

Although the expressions in Equations (27) involve oscillations with different periods in the variable ω, the dominant contribution comes from the factors $\cos x$ and $\sin x$, which oscillate with period 2π in the rescaled frequency $4d_0\omega/c$. Moreover, the overall behavior of the rescaled covariances $2d_0^2\Sigma_{ij}/\pi\kappa_0$ is similar to the monostatic cases considered above, with decaying oscillations which converge to a constant value for $\omega \to \infty$: $\Sigma_{11}, \Sigma_{22} \to s_0$ and $\Sigma_{12} \to 0$. Again, the plots in Figure 1 provide a qualitatively correct representation of the rescaled covariances. The main difference with the monostatic cases is that, now, the ratio a_0/d_0 controls the rate at which oscillations decay as the frequency grows. This is exemplified in Figure 2 with Σ_{11}, for three values of a_0/d_0. In the plot, Σ_{11} is both rescaled and normalized by its large-frequency limit s_0.

4.2. The large-frequency limit

Beyond the case of delta-correlated scattering amplitude densities, the results obtained in Section 4.1 are expected to correctly describe the covariance properties of the transfer function for sufficiently small frequencies when the two-point density correlation $\bar{K}(\mathbf{r}, \mathbf{r}')$, Equation (13), decays within some typical distance $|\mathbf{r} - \mathbf{r}'| \sim \lambda_0$. In fact, for wavelengths much larger than this distance, $\lambda = 2\pi c/\omega \gg \lambda_0$, local details in the profile of the function $\bar{K}(\mathbf{r}, \mathbf{r}')$ should be irrelevant to the calculation of the covariances Σ_{ij} from Equations (20).

The opposite limit of large frequencies, on the other hand, can be obtained from standard asymptotic expansions of rapidly oscillating integrals. In particular, we recall that, if $q(t)$ is a C^n function ($n > 1$) in the interval $[a, b]$, the leading large-ω term of the integral $\int_a^b \exp(-\mathrm{i}\omega t)q(t)\,\mathrm{d}t$ is [13]

$$\int_a^b \exp(-\mathrm{i}\omega t)q(t)\,\mathrm{d}t \approx \frac{\mathrm{i}}{\omega}\left[\exp(-\mathrm{i}\omega b)q(b) - \exp(-\mathrm{i}\omega a)q(a)\right]. \tag{28}$$

The integration interval $[a, b]$ can be extended to infinity provided that the right-hand side of this equation returns a well defined value. This approximation can be used to obtain the

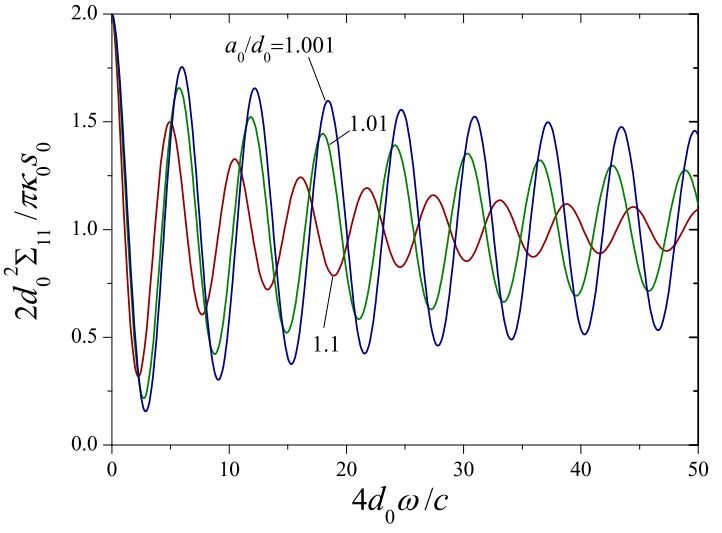

Figure 2. Rescaled and normalized covariance matrix, $r_0\Sigma_{11}/2\pi\kappa_0 s_0$, as a function of the rescaled frequency $4d_0\omega/c$ for a bistatic configuration in three-dimensions and three values of the ratio a_0/d_0, as indicated by the labels. The distances d_0 and a_0 are, respectively, half the radar baseline and the major semi-axis of the excluded ellipse around transmitter and receiver.

dominant large-frequency behavior of Σ_{ij} for a broad class of correlation functions $\bar{K}(\mathbf{r},\mathbf{r}')$ (note, however, that the expansion does not apply to the delta-like correlation considered in Section 4.1, because of its non-differentiability).

For the sake of concreteness, we exemplify the use of the large-ω approximation for two-point density correlations of the form $\bar{K}(\mathbf{r},\mathbf{r}') = \bar{K}(|\mathbf{r}-\mathbf{r}'|)$, which only depend on the distance between the two points: $|\mathbf{r}-\mathbf{r}'| = \sqrt{r^2+r'^2-2rr'\cos\theta}$, with θ the angle formed by $\mathbf{r}$ and $\mathbf{r}'$. In three-dimensional monostatic configurations with an excluded spherical volume of radius r_0 around the position of transmitter and receiver, Equation (28) makes it possible to compute

$$\Sigma_{11} \approx \frac{\pi^2 c^2 I_K}{4\omega^2}(1-\cos x), \quad \Sigma_{12} \approx -\frac{\pi^2 c^2 I_K}{4\omega^2}\sin x, \quad \Sigma_{22} \approx \frac{\pi^2 c^2 I_K}{4\omega^2}(1+\cos x), \tag{29}$$

with $x = 4r_0\omega/c$, and

$$I_K = \int_{-1}^{1} \bar{K}\left[r_0\sqrt{2(1-y)}\right] \mathrm{d}y. \tag{30}$$

We see that the leading large-ω contribution to Σ_{ij} decreases as ω^{-2}. This decay is modulated by oscillations of period 2π in the rescaled frequency $4r_0\omega/c$, exactly the same as in the oscillations found for delta-correlated densities in the monostatic two- and three-dimensional configurations considered in Section 4.1.

The integral I_K in Equation (30) encompasses all the information referred to the scattering amplitude density correlation, in particular, the dependence on a correlation length. To illustrate this, we report the explicit form of I_K for three choices of $\bar{K}(|\mathbf{r}-\mathbf{r}'|) \equiv \bar{K}(\eta)$ with different decays in η – respectively, Gaussian, exponential, and Lorentzian – all of them within

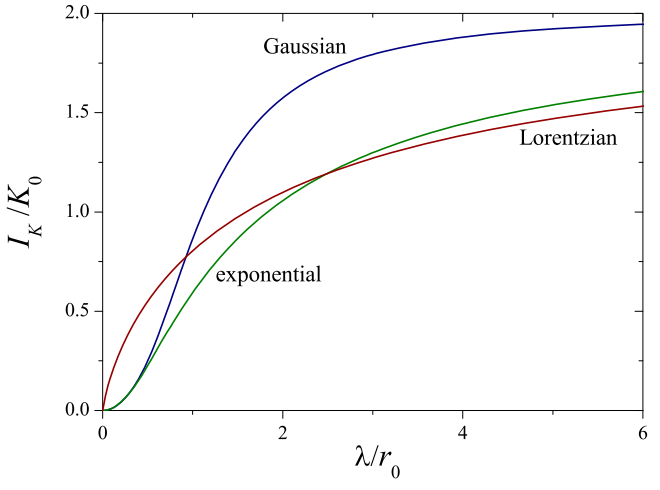

Figure 3. Rescaled integral I_K/K_0, Equation (30), as a function of the rescaled correlation length λ/r_0, for the three correlation kernels of Equations (31).

a typical distance λ:

$$\begin{aligned}
&\bar{K}(\eta) = K_0 \exp(-\eta^2/2\lambda^2): \quad I_K = K_0 \frac{\lambda^2}{r_0^2}\left[1 - \exp\left(-\frac{2r_0^2}{\lambda^2}\right)\right],\\
&\bar{K}(\eta) = K_0 \exp(-\eta/\lambda): \quad I_K = K_0 \frac{\lambda}{r_0}\left[\frac{\lambda}{r_0} - \left(2 + \frac{\lambda}{r_0}\right)\exp\left(-\frac{2r_0}{\lambda}\right)\right],\\
&\bar{K}(\eta) = K_0 \left(1 + \eta^2/\lambda^2\right)^{-1}: \quad I_K = K_0 \frac{\lambda}{2r_0} \ln\left(1 + \frac{4r_0}{\lambda}\right).
\end{aligned} \tag{31}$$

Curves in Figure 3 show the rescaled integral I_K/K_0 as a function of the rescaled correlation length λ/r_0 for the three kernels of Equations (31). We see that, in spite of their diverse functional forms, their overall behavior is the same. In particular, all of them start at zero for $\lambda = 0$ and grow monotonically, at different rates, toward $I_K = 2K_0$ for large λ. This large-λ limit, in fact, readily derives from Equation (30) if $K(\eta)$ is assumed to become increasingly flat as λ increases.

4.3. Scattering phase shifts

As it stands in Equation (3), the transfer function $\tau(\omega)$ does not take into account the possibility that the transmitted signal is shifted in phase at each scattering event. This effect can be incorporated to our description by adding a complex phase to the scattering amplitude, $\sqrt{\sigma_k(\omega)} \to \sqrt{\sigma_k(\omega)}\exp[\mathrm{i}\phi_k(\omega)]$, where $\phi_k(\omega)$ is the frequency-dependent phase shift introduced by scatterer k. In turn, when defining the scattering amplitude density through Equation (4), the scattering phase shift induces the appearance of a phase in the density, $\rho(\mathbf{r}) \to \rho(\mathbf{r})\exp[\mathrm{i}\phi(\mathbf{r})]$. Thus, the scattering amplitude density becomes a complex quantity.

A convenient way to consider a complex random component of scattering amplitude density within the present formulation consists in splitting $\rho(\mathbf{r})$ into its real and imaginary

part or, alternatively, in using $\rho(\mathbf{r})$ and its complex conjugate $\rho^*(\mathbf{r})$ as two independent quantities [6]. In both cases, functional integration has now to be performed over two functional variables. In terms of $\rho(\mathbf{r})$ and $\rho^*(\mathbf{r})$, the most general form for a Gaussian-like probability distribution profile over functional space, corresponding to Equation (12), reads

$$F[\rho,\rho^*] = \exp\left\{-\frac{1}{2}\iint \begin{bmatrix}\rho(\mathbf{r}) & \rho^*(\mathbf{r})\end{bmatrix}\begin{bmatrix}K(\mathbf{r},\mathbf{r}') & k(\mathbf{r},\mathbf{r}')\\ k(\mathbf{r},\mathbf{r}') & K^*(\mathbf{r},\mathbf{r}')\end{bmatrix}\begin{bmatrix}\rho(\mathbf{r}')\\ \rho^*(\mathbf{r}')\end{bmatrix}\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{r}'\right\}, \tag{32}$$

where the integrand is computed as an ordinary matrix product. The result of this product is a real function if $K(\mathbf{r},\mathbf{r}')$ and $k(\mathbf{r},\mathbf{r}')$ are, respectively, complex and real.

The corresponding characteristic function,

$$Z(\alpha,\beta) = \frac{\int F[\rho,\rho^*]\exp\left(W[\rho]\right)\,\mathrm{D}\rho\,\mathrm{D}\rho^*}{\int F[\rho,\rho^*]\,\mathrm{D}\rho\,\mathrm{D}\rho^*}, \tag{33}$$

involves now double functional integrals, although $W[\rho]$ is still a function of $\rho(\mathbf{r})$ only. Each double integral can be factorized into two independent integrals by means of a linear change of functional variables $(\rho,\rho^*)\to(\rho_1,\rho_2)$ such that, in the new variables, the matrix in the integrand of Equation (32) is diagonal. With this transformation, in fact, the probability distribution profile becomes

$$\begin{aligned}F[\rho_1,\rho_2] &= \exp\left\{-\frac{1}{2}\iint\left[\rho_1(\mathbf{r})K_1(\mathbf{r},\mathbf{r}')\rho_1(\mathbf{r}') + \rho_2(\mathbf{r})K_2(\mathbf{r},\mathbf{r}')\rho_2(\mathbf{r}')\right]\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{r}'\right\}\\ &\equiv F_1[\rho_1]F_2[\rho_2],\end{aligned} \tag{34}$$

where $K_1(\mathbf{r},\mathbf{r}')$ and $K_2(\mathbf{r},\mathbf{r}')$ are the diagonal elements of the transformed matrix. We have, moreover, $\rho(\mathbf{r}) = \int[C_1(\mathbf{r},\mathbf{r}')\rho_1(\mathbf{r}') + C_2(\mathbf{r},\mathbf{r}')\rho_2(\mathbf{r}')]\,\mathrm{d}\mathbf{r}'$, where the functions $C_1(\mathbf{r},\mathbf{r}')$ and $C_2(\mathbf{r},\mathbf{r}')$ are elements of the functional matrix that defines the change of variables. Taking into account Equation (11), we can write

$$W[\rho] = -\mathrm{i}\iint h(\mathbf{r})\left[C_1(\mathbf{r},\mathbf{r}')\rho_1(\mathbf{r}') + C_2(\mathbf{r},\mathbf{r}')\rho_2(\mathbf{r}')\right]\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{r}' \equiv W_1[\rho_1] + W_2[\rho_2]. \tag{35}$$

Finally, replacing Equations (34) and (35) into (33) yields

$$\begin{aligned}Z(\alpha,\beta) &= \frac{\int F_1[\rho_1]\exp(W_1[\rho_1])\,\mathrm{D}\rho_1\int F_2[\rho_2]\exp(W_2[\rho_2])\,\mathrm{D}\rho_2}{\int F_1[\rho_1]\,\mathrm{D}\rho_1\int F_2[\rho_2]\,\mathrm{D}\rho_2}\\ &\equiv Z_1(\alpha,\beta)Z_2(\alpha,\beta),\end{aligned} \tag{36}$$

where $Z_1(\alpha,\beta)$ and $Z_2(\alpha,\beta)$ have the same form as $Z(\alpha,\beta)$ for the case of real $\rho(\mathbf{r})$, Equation (10). From this point on, the calculation can proceed exactly as described in Section 3.1.

4.4. Steepest-descent method for non-Gaussian channels

In ordinary integration, the steepest-descent (or stationary phase [14]) approximation applies to integrals of the form

$$I_\Delta = \int q(x) \exp\left[\frac{1}{\Delta} p(x)\right] \mathrm{d}x \approx \sum_{x_{\max}} \sqrt{\frac{2\pi\,\Delta}{|p''(x_{\max})|}}\, q(x_{\max}) \exp\left[\frac{1}{\Delta} p(x_{\max})\right], \tag{37}$$

where Δ is a small parameter, and the sum runs over the points $x_{\max}$ at which $p(x)$ attains local maxima. For $\Delta \to 0$, in fact, I_Δ is dominated by the integrand evaluated at those points. The approximated identity in the rightmost part of Equation (37) is obtained by expanding $p(x)$ up to order $(x - x_{\max})^2$ around each maximum, and integrating the exponential $\exp(p/\Delta)$ as if it were a narrow Gaussian, an approximation that improves as Δ decreases.

The steepest-descent approximation can be straightforwardly applied to the calculation of functional integrals when they are calculated as a limit of multiple ordinary integrals (see Appendix 1). The method makes it possible to consider integrands for which, unlike the Gaussian distributions dealt with above, exact integration is not possible [6]. Here, we illustrate the results of the approximation for a scattering amplitude probability distribution with a generic exponential profile

$$F[\rho] = \exp\left[\frac{1}{\Delta} \int R[\rho]\, \mathrm{d}\mathbf{r}\right], \tag{38}$$

where $R(\rho)$ is a real-valued function. Note that, taking $R(\rho) = -(\Delta/2)\rho^2$, we recover the Gaussian distribution for delta-correlated densities considered in Section 4.1. For $\Delta \to 0$, $F[\rho]$ represents a profile increasingly concentrated around the maxima of $\int R[\rho(\mathbf{r})]\, \mathrm{d}\mathbf{r}$.

Following the steps outlined in Appendix 1 to compute the functional integrals in the numerator and the denominator of the characteristic function, Equation (10), and applying the steepest-descend approximation, we find

$$Z(\alpha, \beta) = \frac{\sum_{\rho_{\max}} z(\rho_{\max}) \exp\left[-\mathrm{i}\rho_{\max} \int h(\mathbf{r})\, \mathrm{d}\mathbf{r}\right]}{\sum_{\rho_{\max}} z(\rho_{\max})}, \tag{39}$$

with $z(\rho_{\max}) = 1/\sqrt{|R''(\rho_{\max})|}$, and where $\rho_{\max}$ are the points at which $R(\rho)$ attains its maxima. Note that the dependence on Δ in the approximated result of Equation (37) disappears in $Z(\alpha, \beta)$, as it is canceled between numerator and denominator.

The integral in Equation (39) can be explicitly computed in a variety of cases. Here, as an example, we consider the two-dimensional monostatic configuration with an excluded domain of radius r_0 around the collocated transmitter and receiver, already analyzed in Section 4.1.2. In this situation, we get

$$Z(\alpha, \beta) = \frac{\sum_{\rho_{\max}} z(\rho_{\max}) \exp\left[2\pi\,\mathrm{i}\rho_{\max}(\alpha \mathrm{Ci}\, x - \beta \mathrm{si}\, x)\right]}{\sum_{\rho_{\max}} z(\rho_{\max})}, \tag{40}$$

with $x = 2r_0\omega/c$. From Equation (9), the resulting probability density for the transfer function turns out to be

$$P[\tau] = \frac{\sum_{\rho_{\max}} z(\rho_{\max}) \delta(u + 2\pi\rho_{\max} \mathrm{Ci}\, x) \delta(v - 2\pi\rho_{\max} \mathrm{si}\, x)}{\sum_{\rho_{\max}} z(\rho_{\max})}. \tag{41}$$

5. Conclusion

Functional integrals have been applied in the past to a variety of problems related to wave transport through random media [15–17]. In contrast with applications where integration is performed over propagation paths – i.e. in the path-integral version of the same formulation – we have here employed this method to evaluate how randomness in the scattering amplitude density of an environmental transmission channel translates into stochastic properties for the corresponding transfer function, which characterizes the ratio between received and transmitted signals in a given radar configuration. In this application, thus, functional integration occurs over the space of scattering amplitude densities and, ultimately, determines a probability distribution over the space of transfer functions.

Because of the possibility of obtaining exact results, we have focused the attention on Gaussian scattering channels, where probability over the scattering amplitude density space is normally distributed around some deterministic function. We have applied this exact result to some radar configurations of practical interest, including monostatic and bistatic geometries. The large-frequency limit for the statistical properties of the transfer function has also been studied, and extensions to consider complex scattering amplitudes – taking into account scattering phase shifts – and non-Gaussian channels have been outlined. Beyond situations of the kind presented here as illustrative examples, the use of this formulation in more realistic cases would require to resort to numerical techniques.

By no means do these applications exhaust the use of functional integrals for characterization of stochastic factors relevant to radar operation. Calculations of transfer functions in the presence of multiple scattering [18], as well as of other quantities routinely used in radar signal processing – such as, for instance, ambiguity functions [1] – fit within the class of problems that can be dealt with using the same techniques.

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Appendix 1. Basic procedures of functional integration

Operationally, it is convenient to conceive functional integration as a limit of multiple integration over the values that the functions $\rho(\mathbf{r})$ of a certain functional space (called ρ-space in the main text) can adopt when evaluated at different points $\mathbf{r}$ [7,19]. For the sake of concreteness, we assume that the variable $\mathbf{r}$ identifies the points of the real d-dimensional space $\mathbb{R}^d$ (examples in Sections 4.1 and 4.2 correspond to $d = 2$ and 3). Let us now consider a hypercubic lattice in $\mathbb{R}^d$, with lattice constant ϵ, such that each unit cell has volume ϵ^d. Let $\mathbf{r}_n$ be the point at the center of a generic cell n, and $\rho_n = \rho(\mathbf{r}_n)$ the value that one of the functions belonging to ρ-space adopts on that point. Functional integration over ρ-space is defined in terms of ordinary integration over the set of variables ρ_n, as

$$\int \mathrm{D}\rho = \lim_{\epsilon\to 0} \int \cdots \int \prod_n \frac{\mathrm{d}\rho_n}{\mu(\epsilon)}. \tag{A1}$$

The integration measure $\mu(\epsilon)$ is chosen ad hoc, depending on the kind of problem being dealt with. Within our particular applications, the choice of $\mu(\epsilon)$ is irrelevant, because in the calculation of the probability distribution for the transfer function, Equation (8), the integration measure cancels as probabilities are normalized; see also Equation (10).

As an illustration of how Equation (A1) works in a concrete calculation, we explicitly compute the functional integral $\int F[\rho]\mathrm{D}\rho$ for the Gaussian profile of Equation (12). Over the d-dimensional lattice, the kernel $K(\mathbf{r}, \mathbf{r}')$ transforms into a definite non-negative symmetric matrix $K = \{K_{nm}\}$, and the integrals in the right-hand side of Equation (12) reduce to sums over the indexes n and m. Thus, according to Equation (A1), we have

$$\int F[\rho]\mathrm{D}\rho = \lim_{\epsilon\to 0} \int \cdots \int \exp\left[-\frac{\epsilon^{2d}}{2}\sum_{n,m}\rho_n K_{nm}\rho_m\right]\prod_n \frac{\mathrm{d}\rho_n}{\mu(\epsilon)}. \tag{A2}$$

It is useful to express the double summation in the integrand of this equation in the base of coordinates $\hat{\rho}_n$ that diagonalize the matrix K, $\sum_{n,m}\rho_n K_{nm}\rho_m = \sum_n \hat{K}_n\hat{\rho}_n^2$, where $\hat{K}_n$ are the eigenvalues of

K. In this way, in fact, the exponential splits into factors, each of them depending on one of the variables $\hat{\rho}_n$. Moreover, the change of variables $\rho_n \to \hat{\rho}_n$ is a unitary rotation, so that $\prod_n \mathrm{d}\rho_n = \prod_n \mathrm{d}\hat{\rho}_n$. Assuming that the functions $\rho(\mathbf{r})$ adopt values over the whole real axis, the integral over each variable $\hat{\rho}_n$ can be immediately computed, yielding

$$\int F[\rho]\mathrm{D}\rho = \lim_{\epsilon\to 0}\prod_n \frac{1}{\mu(\epsilon)}\sqrt{\frac{2\pi}{\epsilon^d \hat{K}_n}} = \lim_{\epsilon\to 0}\frac{1}{\sqrt{|K|}}\prod_n \frac{\sqrt{2\pi/\epsilon^d}}{\mu(\epsilon)}, \tag{A3}$$

where $|K|$ is the determinant of K. In applications where this functional integral is relevant, it is customary to choose a measure $\mu(\epsilon)$ which insures that the limit in Equation (A3) is well defined, specifically, $\mu(\epsilon) = \sqrt{2\pi/\epsilon^d}$. In that limit, in fact, $|K|$ becomes a property of the kernel $K(\mathbf{r}, \mathbf{r}')$ and is therefore independent of ϵ. With the above choice of $\mu(\epsilon)$, we finally have

$$\int \exp\left[-\frac{1}{2}\iint \rho(\mathbf{r})K(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{r}'\right]\mathrm{D}\rho = \frac{1}{\sqrt{|K|}}. \tag{A4}$$