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## On the Quantization of Fermion Fields with Zero Mass.

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**Summary.** — The formulation of quantum field theory for massless tensor fields given in a previous paper, is here extended to fermion fields with zero mass. The fermion field, which in the Rarita-Schwinger formalism is represented by a tensor spinor, is here expressed in terms of simple Dirac spinors. These Dirac spinors are free components of the field and the usual anticommutation relations are imposed on them. The corresponding anticommutation relations of the field components are given. They are compatible with the supplementary conditions.

### 1. - Introduction.

In the Rarita-Schwinger formalism <sup>(1)</sup> the general fermion field is represented by means of a symmetric tensor-spinor  $\Psi_{v_1 \dots v_s}$ . When this field corresponds to massless particles with zero mass, it satisfies the equations

$$(1.1) \quad \gamma_\mu \partial_\mu \Psi_{v_1 \dots v_s} = 0,$$

$$(1.2) \quad \gamma_{v_1} \Psi_{v_1 \dots v_s} = 0.$$

The matrices  $\gamma_\mu$  satisfying the relations

$$(1.3) \quad \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}.$$

<sup>(1)</sup> W. RARITA and J. SCHWINGER: *Phys. Rev.*, **60**, 61 (1941).

It is easily seen that these equations imply.

$$(1.4) \quad \hat{\gamma}_{\nu_1} \Psi_{\nu_1 \dots \nu_s} = 0,$$

$$(1.5) \quad \delta_{\nu_1 \nu_2} \Psi_{\nu_1 \dots \nu_s} = 0.$$

(1.1) is the field equation and (1.2) is the supplementary condition which reduces to  $2s+2$  the number of independent components of the tensor-spinor  $\Psi_{\nu_1 \dots \nu_s}$ . This is the right number of possible orientations of a particle with spin  $s+\frac{1}{2}$ . The gauge invariance condition reduces further this number to two <sup>(2)</sup>.

We intend to take (1.2) as constraint equations and so our first aim is the extraction of the independent components from the field entity (in a covariant manner). To get this result we are going to use a method for the tensorial field, already developed in a previous paper <sup>(3)</sup> (hereafter referred to as I).

## 2. - Decomposition of the field entity.

We first arbitrarily choose a time-like unit vector:  $n_\mu n_\mu = -1$ . Then, let us define by recurrence the following tensor spinors: (see also I (2.1) (2.2) (2.3) (2.4))

$$\begin{aligned} \Psi^{(0)} &= n_{\nu_1} \dots n_{\nu_s} \Psi_{\nu_1 \dots \nu_s} \\ \Psi_{\nu_1 \dots \nu_s}^{(0)} &= \partial^{-s} \partial_{\nu_1} \dots \partial_{\nu_s} \Psi^{(0)} \\ (2.1) \quad \Psi_{\nu_1 \dots \nu_r}^{(r)} &= n_{\nu_{r+1}} \dots n_{\nu_s} \left( \Psi_{\nu_1 \dots \nu_s} - \sum_{t=0}^{r-1} \Psi_{\nu_1 \dots \nu_s}^{(t)} \right) \\ (2.2) \quad \Psi_{\nu_1 \dots \nu_s}^{(r)} &= \partial^{-(s-r)} \partial_{\nu_{r+1}} \dots \partial_{\nu_s} \Psi_{\nu_1 \dots \nu_r}^{(r)} + (\text{symm.}) \end{aligned} \quad \left. \vphantom{\begin{aligned} \Psi^{(0)} \\ \Psi_{\nu_1 \dots \nu_s}^{(0)} \\ \Psi_{\nu_1 \dots \nu_r}^{(r)} \\ \Psi_{\nu_1 \dots \nu_s}^{(r)} \end{aligned}} \right\} r = 0, 1, \dots, s.$$

Here  $\hat{\partial} = n_\nu \partial_\nu$  and  $\hat{\partial}^{-1}$  is the inverse operator. «Symm.» means terms necessary to make symmetric the right hand side of (2.2).

The quantities defined by (2.1) and (2.2), all satisfy (1.1), (1.2).

It can be proved by recurrence that

$$(2.3) \quad n_{\nu_{r+1}} \dots n_{\nu_s} \Psi_{\nu_1 \dots \nu_s}^{(r)} = \Psi_{\nu_1 \dots \nu_r}^{(r)}$$

$$(2.4) \quad n_{\nu_1} \Psi_{\nu_1 \dots \nu_r}^{(r)} = 0.$$

<sup>(2)</sup> J. S. DE WET: *Phys. Rev.*, **58**, 236 (1940).

<sup>(3)</sup> C. G. BOLLINI: *Nuovo Cimento*, **6**, 1034 (1957).

The last equation of the recurrence chain is

$$\Psi_{\nu_1 \dots \nu_s}^{(s)} = \Psi_{\nu_1 \dots \nu_s} - \sum_0^{s-1} \Psi_{\nu_1 \dots \nu_s}^{(t)}$$

which brings the decomposition

$$(2.5) \quad \Psi_{\nu_1 \dots \nu_s} = \sum_{t=0}^s \Psi_{\nu_1 \dots \nu_s}^{(t)}$$

of the field quantities in  $s+1$  new tensor-spinors satisfying (1.1), (1.2), (2.3) and (2.4). We are going to prove that each one of this new tensor-spinors has only one independent component which is a simple spinor (without tensorial index).

### 3. - Polarization matrices.

In I (Eq. (3.1)) we introduced two derivation operators  $\alpha_\nu$ ,  $\beta_\nu$ , such that

$$(3.1) \quad \begin{cases} n_\nu \alpha_\nu = 0, & n_\nu \beta_\nu = 0, \\ \alpha_\nu \partial_\nu = 0, & \beta_\nu \partial_\nu = 0, \\ \alpha_\nu \alpha_\nu = 1, & \beta_\nu \beta_\nu = 1, \\ & \alpha_\nu \beta_\nu = 0. \end{cases}$$

By means of these operators and the  $\gamma_\mu$  matrices, we define now

$$(3.2) \quad \Gamma_\nu = \alpha_\nu + \beta_\nu \gamma \cdot \alpha \gamma \cdot \beta,$$

$$(3.3) \quad \Gamma_{\nu_1 \dots \nu_r} = \Gamma_{\nu_1} \dots \Gamma_{\nu_r}.$$

(From now on, we put  $\gamma_\mu \alpha_\mu = \gamma \cdot \alpha$ , etc.).

The  $\Gamma_\mu$  are commuting matrices:

$$(3.4) \quad \Gamma_\mu \Gamma_\nu - \Gamma_\nu \Gamma_\mu = 0.$$

It follows from (3.1) and (3.2) that

$$(3.5) \quad n_\nu \Gamma_\nu = 0, \quad \partial_\nu \Gamma_\nu = 0.$$

We also have

$$(3.6) \quad \gamma_\nu \Gamma_\nu = 0, \quad \Gamma_\nu \Gamma_\nu = 0,$$

because

$$(3.7) \quad \begin{cases} \gamma \cdot \alpha \gamma \cdot \beta = -\gamma \cdot \beta \gamma \cdot \alpha, \\ \gamma \cdot \alpha \gamma \cdot \alpha = \gamma \cdot \beta \gamma \cdot \beta = 1. \end{cases}$$

According to these equations (3.3) defines a symmetric set of matrices satisfying

$$(3.8) \quad \begin{cases} \delta_{v_1 v_2} \Gamma_{v_1 \dots v_r} = 0, \\ \gamma_{v_1} \Gamma_{v_1 \dots v_r} = 0, \\ \partial_{v_1} \Gamma_{v_1 \dots v_r} = 0, \\ n_{v_1} \Gamma_{v_1 \dots v_r} = 0, \end{cases}$$

also

$$(3.9) \quad \gamma \cdot \partial \Gamma_{v_1 \dots v_r} = \Gamma_{v_1 \dots v_r} \gamma \cdot \partial.$$

Analogously, putting

$$(3.10) \quad \tilde{\Gamma}_v = \alpha_v + \beta_v \gamma \beta \gamma \cdot \alpha$$

and

$$(3.11) \quad \tilde{\Gamma}_{v_1 \dots v_r} = \tilde{\Gamma}_{v_1} \dots \tilde{\Gamma}_{v_r},$$

we have

$$(3.12) \quad \begin{cases} \delta_{v_1 v_2} \tilde{\Gamma}_{v_1 \dots v_r} = 0, \\ \tilde{\Gamma}_{v_1 \dots v_r} \gamma_{v_1} = 0, \\ \partial_{v_1} \tilde{\Gamma}_{v_1 \dots v_r} = 0, \\ n_{v_1} \tilde{\Gamma}_{v_1 \dots v_r} = 0, \\ \gamma \cdot \partial \tilde{\Gamma}_{v_1 \dots v_r} = \tilde{\Gamma}_{v_1 \dots v_r} \gamma \cdot \partial. \end{cases}$$

#### 4. - Field co-ordinates.

The tensor-spinors defined by (2.1) are orthogonal to  $n_v$ ,  $\partial_v$  and  $\gamma_v$ . Because of this we can put

$$(4.1) \quad \begin{aligned} 0 &= \gamma_{v_1} \Psi_{v_1 \dots v_r}^{(\sigma)} = \gamma_\mu (\alpha_\mu \alpha_{v_1} + \beta_\mu \beta_{v_1}) \Psi_{v_1 \dots v_r}^{(\sigma)} \\ &= \gamma \cdot \alpha \alpha_{v_1} \Psi_{v_1 \dots v_r}^{(\sigma)} + \gamma \cdot \beta \beta_{v_1} \Psi_{v_1 \dots v_r}^{(\sigma)} \\ &\gamma \cdot \alpha \alpha_{v_1} \Psi_{v_1 \dots v_r}^{(\sigma)} = -\gamma \cdot \beta \beta_{v_1} \Psi_{v_1 \dots v_r}^{(\sigma)}. \end{aligned}$$

Then

$$\begin{aligned} \Gamma_\mu \alpha_{v_1} \Psi_{v_1 \dots v_r}^{(r)} &= \alpha_\mu \alpha_{v_1} \Psi_{v_1 \dots v_r}^{(r)} + \beta_\mu \gamma \cdot \alpha \gamma \cdot \beta \alpha_{v_1} \Psi_{v_1 \dots v_r}^{(r)} \\ &= \alpha_\mu \alpha_{v_1} \Psi_{v_1 \dots v_r}^{(r)} + \beta_\mu \beta_{v_1} \Psi_{v_1 \dots v_r}^{(r)} \\ &= \Psi_{\mu v_2 \dots v_r}^{(r)}. \end{aligned}$$

In the same way we can deduce that

$$(4.2) \quad \Gamma_{\mu_1 \dots \mu_r} \alpha_{v_1} \dots \alpha_{v_r} \Psi_{v_1 \dots v_r}^{(r)} = \Psi_{\mu_1 \dots \mu_r}^{(r)}.$$

Equivalently, but more symmetrically

$$(4.3) \quad 2^{-r} \Gamma_{\mu_1 \dots \mu_r} \tilde{\Gamma}_{v_1 \dots v_r} \Psi_{v_1 \dots v_r}^{(r)} = \Psi_{\mu_1 \dots \mu_r}^{(r)}$$

on account of

$$\begin{aligned} \tilde{\Gamma}_{v_1} \Psi_{v_1 \dots v_r}^{(r)} &= \alpha_{v_1} \Psi_{v_1 \dots v_r}^{(r)} + \gamma \cdot \beta \gamma \cdot \alpha \beta_{v_1} \Psi_{v_1 \dots v_r}^{(r)} \\ &= \alpha_{v_1} \Psi_{v_1 \dots v_r}^{(r)} + \alpha_{v_1} \Psi_{v_1 \dots v_r}^{(r)} \\ &= 2\alpha_{v_1} \Psi_{v_1 \dots v_r}^{(r)}. \end{aligned}$$

The equation (4.2) allows then to express the tensor spinor  $\Psi_{\mu_1 \dots \mu_r}^{(r)}$  as a function of one spinor, namely:

$$(4.4) \quad \Psi^{(r)} = \alpha_{v_1} \dots \alpha_{v_r} \Psi_{v_1 \dots v_r}^{(r)},$$

$$(4.5) \quad \Psi_{\mu_1 \dots \mu_r}^{(r)} = \Gamma_{\mu_1 \dots \mu_r} \Psi^{(r)}.$$

(4.5) shows an explicit separation of the tensorial part from  $\Psi_{\mu_1 \dots \mu_r}^{(r)}$ .

The tensor-matrix operator  $\Gamma_{\mu_1 \dots \mu_r}^{(r)}$  carries all the subsidiary properties of  $\Psi_{\mu_1 \dots \mu_r}^{(r)}$ , whereas the purely spinorial part  $\Psi^{(r)}$  satisfies only the field equation:

$$(4.6) \quad \gamma \cdot \partial \Psi^{(r)} = 0$$

without any other condition.

The  $s+1$  Dirac spinors  $\Psi^{(r)}$  are then free components of the field. (4.5), (2.2) and (2.5), allow one to express the field entity as a function of these co-ordinates.

When a gauge transformation is made, all but one of these spinors are changed. The only really free component is  $\Psi^{(s)}$ , i.e. the completely « transverse » component of  $\Psi_{\nu_1 \dots \nu_s}$  (see I).

### 5. - Adjoint components.

The adjoint spinor of  $\Psi^{(r)}$  is defined by (4)

$$\overline{\Psi^{(r)}} = \Psi^{(r)*} A,$$

$\Psi^{(r)*}$  being the transposed conjugate, and  $A$  the matrix such that

$$(5.1) \quad \gamma_\mu^* = -A \gamma_\mu A^{-1}. \quad A^* = A$$

From (4.6) we deduce

$$(5.2) \quad \Psi^{(r)*} \gamma^* \cdot \partial = 0.$$

(In the expressions like (5.2) it is understood that the derivation operator operates on the left).

With (5.1) equation (5.2) is transformed into

$$-\Psi^{(r)*} A \gamma \cdot \partial A^{-1} = -\overline{\Psi^{(r)}} \gamma \cdot \partial A^{-1} = 0,$$

i.e.

$$(5.3) \quad \overline{\Psi^{(r)}} \gamma \cdot \partial = 0.$$

(5.3) is the equation adjoint to (4.6).

Applying the star operation to the matrix  $\Gamma_\nu$ , we have

$$\begin{aligned} \Gamma_\nu^* &= \alpha_\nu + \beta_\nu \gamma^* \cdot \beta \gamma^* \cdot \alpha, \\ &= \alpha_\nu + \beta_\nu A \gamma \cdot \beta A^{-1} A \gamma \cdot \alpha A^{-1}, \\ &= \alpha_\nu + \beta_\nu A \gamma \cdot \beta \gamma \cdot \alpha A^{-1}, \\ &= A(\alpha_\nu + \beta_\nu \gamma \cdot \beta \gamma \cdot \alpha) A^{-1}, \\ \Gamma_\nu^* &= A \tilde{\Gamma}_\nu A^{-1}. \end{aligned}$$

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(4) J. M. JAUCH and F. ROHRlich: *The Theory of Photons and Electrons* (Cambridge, Mass., 1955), p. 53.

In general

$$(5.4) \quad \Gamma_{\nu_1 \dots \nu_r}^* = A \tilde{\Gamma}_{\nu_1 \dots \nu_r} A^{-1}.$$

Transposing (4.5) and taking the complex conjugate

$$\begin{aligned} \Psi_{\mu_1 \dots \mu_r}^{(r)*} &= \Psi^{(r)*} \Gamma_{\mu_1 \dots \mu_r}^* = \Psi^{(r)*} A \tilde{\Gamma}_{\mu_1 \dots \mu_r} A^{-1}, \\ &= \overline{\Psi^{(r)}} \tilde{\Gamma}_{\mu_1 \dots \mu_r} A^{-1}. \end{aligned}$$

Putting

$$\overline{\Psi^{(r)}}_{\mu_1 \dots \mu_r} = \Psi_{\mu_1 \dots \mu_r}^{(r)*} A$$

one has

$$(5.5) \quad \overline{\Psi^{(r)}}_{\mu_1 \dots \mu_r} = \overline{\Psi^{(r)}} \tilde{\Gamma}_{\mu_1 \dots \mu_r}.$$

The decomposition (2.5) is now transformed into the adjoint decomposition

$$(5.6) \quad \overline{\Psi}_{\nu_1 \dots \nu_s} = \sum_{t=0}^s \overline{\Psi^{(t)}}_{\nu_1 \dots \nu_s},$$

where

$$(5.7) \quad \overline{\Psi^{(r)}}_{\nu_1 \dots \nu_s} = \Psi_{\nu_1 \dots \nu_s}^{(r)*} A = \partial^{-(s-r)} \partial_{\nu_{r+1}} \dots \partial_{\nu_s} \overline{\Psi^{(r)}}_{\nu_1 \dots \nu_r} + (\text{symm.}).$$

## 6. - Quantization.

The  $s+1$  spinors  $\Psi^{(r)}$  are free co-ordinates satisfying the Dirac equation (with zero mass). It seems logical then to impose the anticommutation relations:

$$(6.1) \quad \{\Psi^{(r)}(x), \overline{\Psi^{(r')}}(x')\} = i \delta^{rr'} \gamma \cdot \partial D(x - x'),$$

where we take

$$\delta^{rr'} = 0 \quad \text{when } r \neq r'$$

and

$$\delta^{rr} = \varepsilon^{(r)} \quad (\text{not necessarily } 1).$$

$\varepsilon^{(r)}$  is the « weight » of the  $r$ -th free co-ordinate and depends on the way

in which it appears in the field Lagrangian. (4.5), (5.5) and (6.1) give:

$$\begin{aligned} \{\Psi_{\mu_1 \dots \mu_r}^{(r)}(x), \overline{\Psi}_{\nu_1 \dots \nu_r}^{(r)}(x')\} &= \{\Gamma_{\mu_1 \dots \mu_r} \Psi^{(r)}(x), \overline{\Psi}^{(r)}(x') \tilde{\Gamma}_{\nu_1 \dots \nu_r}\} = \\ &= \Gamma_{\mu_1 \dots \mu_r} \{\Psi^{(r)}(x), \overline{\Psi}^{(r)}(x')\} \tilde{\Gamma}_{\nu_1 \dots \nu_r} = i \delta^{rr'} \Gamma_{\mu_1 \dots \mu_r} \gamma \cdot \partial \tilde{\Gamma}_{\nu_1 \dots \nu_r} D(x - x'), \\ (6.2) \quad \{\Psi_{\mu_1 \dots \mu_r}^{(r)}(x), \overline{\Psi}_{\nu_1 \dots \nu_r}^{(r)}(x')\} &= i \delta^{rr'} \Gamma_{\mu_1 \dots \mu_r} \tilde{\Gamma}_{\nu_1 \dots \nu_r} \gamma \cdot \partial D(x - x'). \end{aligned}$$

Now, with (2.2), we have

$$(6.3) \quad \{\Psi_{\mu_1 \dots \mu_s}^{(r)}(x), \overline{\Psi}_{\nu_1 \dots \nu_s}^{(r)}(x')\} = i \delta^{rr'} \Gamma_{\mu_1 \dots \mu_s}^{(r)} \tilde{\Gamma}_{\nu_1 \dots \nu_s}^{(r)} \gamma \cdot \partial D(x - x'),$$

where

$$(6.4) \quad \Gamma_{\mu_1 \dots \mu_s}^{(r)} = \partial^{-(s-r)} \partial_{\mu_{r+1}} \dots \partial_{\mu_s} \Gamma_{\mu_1 \dots \mu_s} + (\text{symm.}).$$

From (6.3) and the decompositions (2.5) and (5.6) we obtain

$$(6.5) \quad \{\Psi_{\mu_1 \dots \mu_s}(x), \overline{\Psi}_{\nu_1 \dots \nu_s}(x')\} = i \Gamma_{\mu_1 \dots \mu_s; \nu_1 \dots \nu_s} \gamma \cdot \partial D(x - x')$$

with

$$(6.6) \quad \Gamma_{\mu_1 \dots \mu_s; \nu_1 \dots \nu_s} = \sum_{t=0}^s \varepsilon^{(t)} \Gamma_{\mu_1 \dots \mu_s}^{(t)} \tilde{\Gamma}_{\nu_1 \dots \nu_s}^{(t)}.$$

The anticommutation relations (6.5) are compatible with the supplementary condition.

## 7. - Example spin $\frac{3}{2}$ .

The decomposition (2.5) is now

$$\begin{aligned} \Psi_\nu &= \partial^{-1} \partial_\nu \Psi^{(0)} + \Gamma_\nu \Psi^{(1)}, \\ \Psi^{(3)} &= n_\nu \Psi_\nu; \quad \Psi^{(1)} = \alpha_\nu \Psi_\nu. \end{aligned}$$

And the anticommutation relations are

$$\begin{aligned} \{\Psi^{(1)}(x), \overline{\Psi}^{(0)}(x')\} &= i \varepsilon^{(0)} \gamma \cdot \partial D(x - x'), \\ \{\Psi^{(1)}(x), \overline{\Psi}^{(1)}(x')\} &= i \varepsilon^{(1)} \gamma \cdot \partial D(x - x'), \\ \{\Psi_\mu(x), \overline{\Psi}_\nu(x')\} &= i \Gamma_{\mu; \nu} \gamma \cdot \partial D(x - x'), \\ \Gamma_{\mu; \nu} &= \varepsilon^{(0)} \partial^{-2} \partial_\mu \partial_\nu + \varepsilon^{(1)} \Gamma_\mu \tilde{\Gamma}_\nu \\ \Gamma_\mu \tilde{\Gamma}_\nu &= \alpha_\mu \alpha_\nu + \beta_\mu \beta_\nu + (\alpha_\mu \beta_\nu - \beta_\mu \alpha_\nu) \gamma \cdot \beta \gamma \cdot \alpha. \end{aligned}$$

It is easily seen that the last expression is actually independent of a particular choice for  $\alpha_\nu$  and  $\beta_\nu$ . In fact

$$\Gamma_\mu \tilde{F}_\nu = \gamma_\nu \gamma_\mu - \partial^{-2} \partial_\mu \partial_\nu - \partial^{-1} (\partial_\mu \gamma_\nu \gamma \cdot n + \gamma \cdot n \partial_\nu \gamma_\mu).$$

When both sides act on  $\gamma \cdot \partial D(x - x')$ .

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#### RIASSUNTO (\*)

Nel presente lavoro si estende ai campi di fermioni di massa nulla la formulazione della teoria quantistica dei campi di tensori privi di massa. Il campo fermionico che nel formalismo di Rarita-Schwinger è rappresentato da uno spinore tensoriale è espresso qui in termini di semplici spinori di Dirac. Tali spinori di Dirac sono componenti libere del campo e si impongono loro le solite relazioni di anticommutazione. Si danno le corrispondenti relazioni di anticommutazione delle componenti del campo, che sono compatibili con le condizioni supplementari.

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(\*) *Traduzione a cura della Redazione.*

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