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CLUSTER OF NUCLEONS AS ELEMENTARY MODES OF EXCITATION IN NUCLEI

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Abstract: Conditions which must be fulfilled by clusters of nucleons to qualify as elementary modes of excitation are analysed in terms of simple criteria involving experimental binding energies. It is found that the most complex possible mode is the α-like cluster.

1. Introduction

Nuclear spectra have frequently been described in terms of elementary modes of excitation involving clusters of nucleons [1]). An example of such modes is the use of pairs of nucleons coupled to $J = 0$, $T = 1$ used in both nuclear superfluidity and pairing vibrations [2,3]). This cluster of two nucleons forms the building block which accounts for the collective aspects associated with two-particle correlations in nuclei. Pairs of nucleons coupled to higher angular momenta were also considered many years ago [4,5]) and revitalized more recently within the IBA [6]) in which the ground states of even nuclei are described as a condensate of pairs of nucleons carrying angular momenta 0(s-bosons) and 2(d-bosons).

A microscopic approach in the same direction is provided by the multi-step shell-model method [7]) (MSM), where clusters are successively added to an inert core. The size of the cluster can be left free within reasonable limits. The MSM provides an amenable procedure to deal with heavy systems where standard shell-model calculations are not feasible.

The existence of excitation modes with larger number of particles was considered within the quartet model [8]) some years ago. Within this picture the building blocks are α-clusters. The reason for this choice was clearly suggested by the large stability of the α-particle. Models of this type have been applied since the early times of nuclear physics [9]). The quartet model had a reasonable success in the description of low energy features of nuclear spectra of light nuclei ($A \lesssim 50$). In ref. [10]) the quartet picture is found to be consistent with the behaviour of nuclear masses throughout the periodic table. Its implications are also discussed in connection with

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excited states of many particles and many holes in the region of $A \approx 100$ and $A \approx 150$. A microscopic approach suggested in ref. [11]) is extended in a recent paper [12]) in which the symmetry and pairing terms of semiempirical mass formulae are replaced by one of the Casimir operators of SU(4). This quartetting scheme imposes a very special dependence of T (isospin) and the even-odd staggering effect. The good RMS obtained from a χ^2 fitting of experimental binding energies corrected by shell effects favours the existence of this kind of correlation in the nuclear ground-state wave functions.

In the problem of analyzing clustering as a way of providing elementary modes of excitation one should also mention the recent developments in transfer reactions between heavy ions that can efficiently probe this kind of phenomena.

The task of determining whether a given cluster can be considered as an "elementary mode" is based upon the delicate balance between shell effects and the residual nuclear interaction among the valence nucleons. For such modes to be considered "elementary" it must be possible to superimpose the clusters with small further corrections and they must contain the largest possible correlation energy.

The purpose of the present paper is to explore in a systematic way all possible modes of excitation based upon modes involving the transfer of several nucleons. To this purpose we extract from experimental binding energies the energy of the correlated states of k particles and k holes, and thereby the correlation energy involved in the cluster of k nucleons. We first analyze the pairing case and later we generalize the discussion to many-nucleon clusters.

2. Definitions

Elementary modes of excitation must be constructed so that the effective interaction among them is as small as possible while the energy gained (correlation energy) in their construction is as large as possible. Our interest here is to give simple criteria to determine when a dressed cluster of particles in nuclei may be considered as such a building block. To achieve this we define

$$S(A, \delta A) = B(A+\delta A) + B(A-\delta A) - 2B(A)\,, \tag{1}$$

where B is minus the binding energy (i.e., they are negative numbers). Eq. (1) gives the energy corresponding to the promotion of δA correlated nucleons across the Fermi surface. In fact, if δA is taken to be one nucleon, $S(A, 1)$ represents the two-quasiparticle energy. This, far from closed-shell nuclei, can be related to the pairing gap, while for closed-shell nuclei it gives the energy gap among major shells.

An important characteristic of the function S is that it approximately corresponds to the excitation energy of a state with δA particles-δA holes in the system with A nucleons [10,13]), except for corrections arising from the Coulomb and nuclear interaction between the δA particles and the δA holes. Furthermore the fact that

all binding energies are taken from experiments ensures that most of the rearrangement corrections that correspond to the Coulomb and nuclear interaction between the particles (holes) are properly included in S.

Eq. (1) is generally used to support the interpretation of pairing modes as elementary degrees of freedom in nuclei [14]), however it does not give a direct measure of the interaction between the correlated clusters δA. The importance of this interaction is better seen by defining the quantity

$$C(A, \delta A_1, \delta A_2) = S(A, \delta A_1) + S(A, \delta A_2) - S(A, \delta A_1 + \delta A_2)\,, \qquad (2)$$

where S is from eq. (1). In order to make it clear what we mean by "free modes" and "interaction among modes", we give in fig. 1 the Feyman diagrams corresponding to eqs. (1) and (2) in normal systems. It is clear from this figure that C is closely related to the interaction among the elementary modes. As an example, let us consider neutron excitations in ^{208}Pb. In this case, to fig. 1a corresponds $S(^{208}\text{Pb}, 1) = 3.431$ MeV while $S(^{208}\text{Pb}, 2) = 4.985$ MeV. Therefore, the correlation energy is $C(^{208}\text{Pb}, 1, 1) = 1.877$ MeV. Since C is positive (although not very big) one sees that it is advantageous to use the normal pairing modes (which contains the pairing interaction) as elementary degrees of freedom [3]) in this region, to provide a description of 2p-2h configurations in ^{208}Pb.

Eq. (2) gives, if positive (see the sign in fig. 1b), the energy gained by constructing a cluster of size $\delta A_1 + \delta A_2$ instead of considering the clusters δA_1 and δA_2 separately. In other words, for a given partition of the cluster A into the clusters δA_1

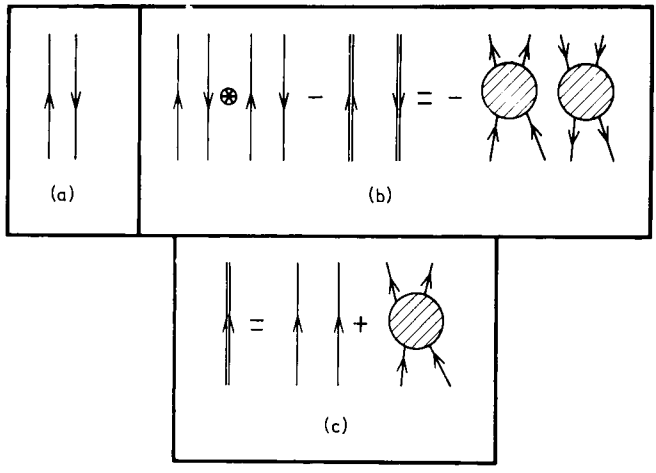

Fig. 1. Graphical representation corresponding to eqs. (1) and (2). In all cases clusters of one nucleon are assumed (i.e. $\delta A = \delta A_1 = \delta A_2 = 1$) while A is a doubly magic nucleus. (a) Zero-order particle–hole excitation with energy given by eq. (1). Each line (one-particle cluster) represents a dressed (correlated) single-particle propagator. (b) Graphical representation corresponding to the process given by $C(A, 1, 1)$ [eq. (2)]. The double line represents the correlated two-particle propagator, as seen in (c). (c) Graphical representation of the two-particle propagator.

and δA_2, eq. (2) indicates whether the uncorrelated configuration $(\delta A_1, \delta A_2)$ is dominant over the correlated configuration δA. This "energy criterion" to determine relevant configurations is common to all microscopic descriptions of nuclear spectra [15]).

A comment should be made stressing that the correlation energy C may be more directly related to experimentally measurable quantities than S.

As quoted above the S-functions have to be corrected by an amount $\Delta E(\delta A)$ to get the excitation energy of a physical state in the A-nucleon system [16]). This correction ΔE is given by the nuclear and Coulomb interactions between the cluster of particles and the cluster of holes. The quantities defined in eq. (2) are physically meaningful if $\Delta E(\delta A_1) + \Delta E(\delta A_2) \approx \Delta E(\delta A_1 + \delta A_2)$. This is in fact the case to a large extent. For the Coulomb part of ΔE this equality is fulfilled when δA_1 or δA_2 is a chargeless fragment, up to terms of the order $R(\delta A_1) - R(\delta A_2)$, where $R(\delta A)$ is the correlation distance between the δA particles and δA holes. In general this difference is a small quantity. As for the nuclear part this is also expected to approximately cancel for $\delta A \geq 2$ due to the short and saturating nature of the nucleon–nucleon interaction.

3. Two-particle clusters

In fig. 2 it is shown the correlation energy C for the three possible combinations of two nucleons. Fig. 2c shows that the neutron–proton correlation is generally small compared with the neutron–neutron or proton–proton correlations shown in fig. 2a and 2b, respectively†.

Ground states of nuclei away from closed shells are usually described as condensates of Cooper pairs. From the fact that the neutron–proton correlations are much smaller than the proton–proton or neutron–neutron correlations it follows that the Cooper pairs must be built upon identical particles.

Fig. 2a shows that C is closely related to the pairing gap Δ. In this figure one can clearly distinguish the superfluid from the normal region. In the superfluid region C is large, almost four times Δ (as expected from fig. 1b). In the normal region Δ goes to zero, while C (which in this region is small) represents the energy gained by constructing the pairing vibration bosons, as previously mentioned. The fact that C is small in the normal region indicates that this quantity [eq. (2)] is more convenient for analysing pairing modes than S [eq. (1)].

All these features are not as clearly present for protons, as seen in fig. 2b. In fact one may say that protons tend to be superfluid in almost all the periodic table. On the other hand, for the neutron–proton case of fig. 2c one sees that the correlation

† The data used to construct figs. 2 to 7 were taken from ref. [17]). For nuclei far from the stability valley the errors in the binding energies may be rather large. However these do not wash out the features that are discussed in the present paper.

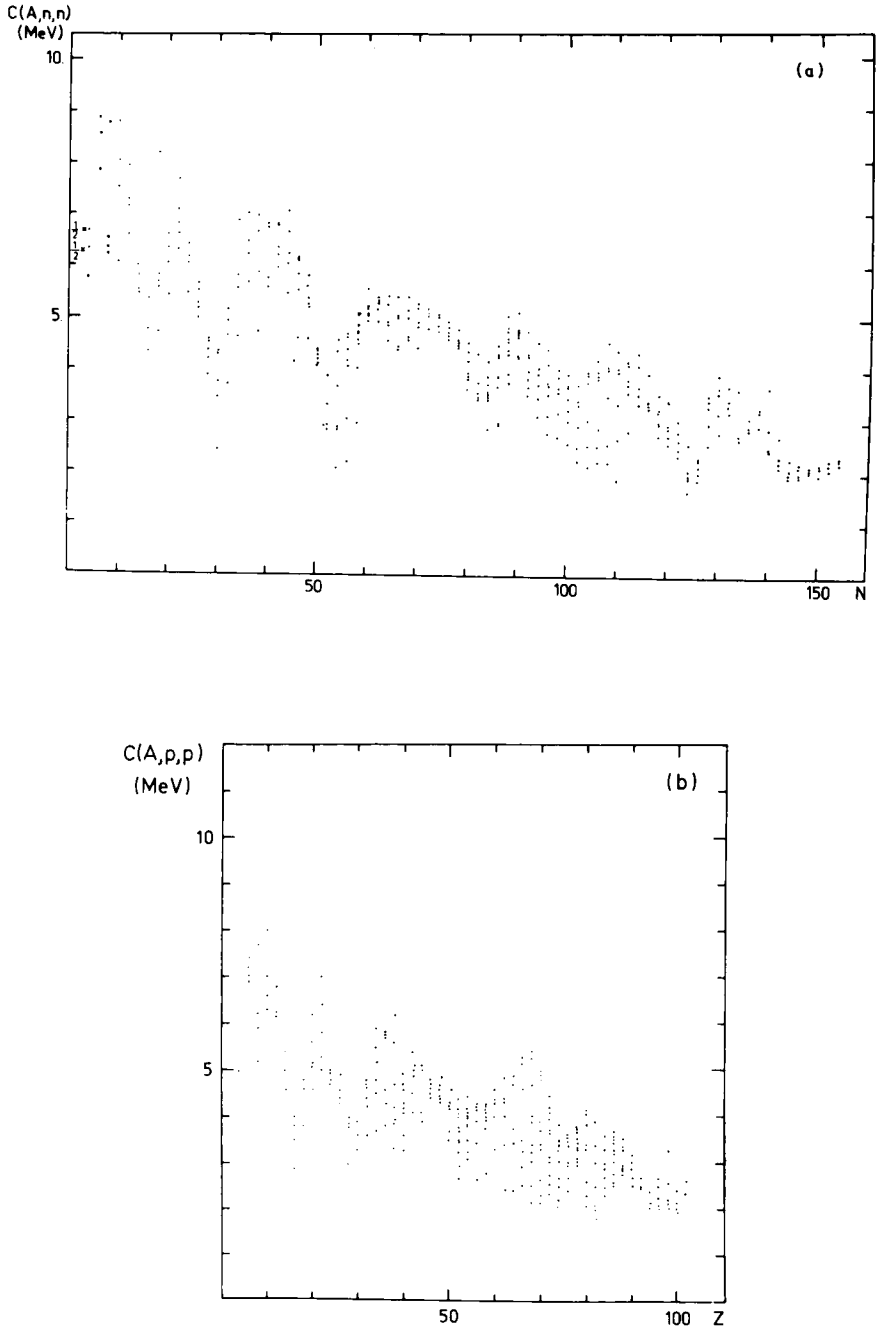

Fig. 2. Correlation energy $C(A, \delta A_1, \delta A_2)$ for two-nucleon clusters for $\delta A_1 = \delta A_2 = 1$. (a) δA_1 and δA_2 are neutrons. (b) δA_1 and δA_2 are protons. (c) δA_1 is a neutron and δA_2 is a proton. The circles correspond to nuclei with $N = Z$. In this and the other figures binding energies were taken from ref. [17]) and A corresponds to an even–even system.

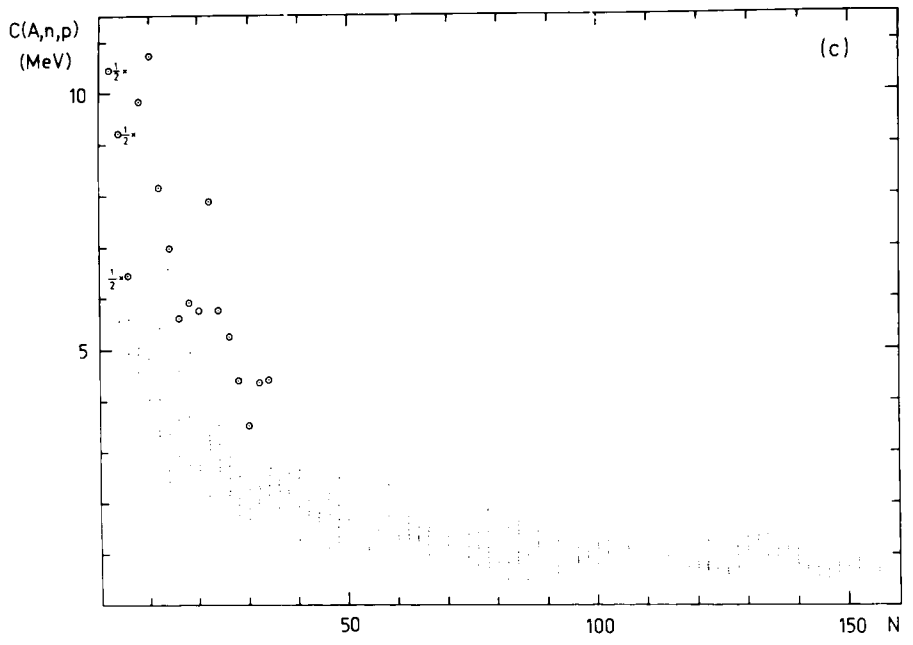

Fig. 2. (cont.)

energy corresponds to the energy gained by one pair (as for normal systems) and therefore shell effects are washed out.

An alternative way of analyzing the occurrence of a superfluid phase is to evaluate the ratios

$$R(A) = S(A, \delta A = 4)/S(A, \delta A = 2)\,, \tag{3}$$

where the cluster δA consists of identical nucleons. For superconducting systems B goes quadratically with the number of pair of particles [7]) and therefore $R = 4$ in this case. For normal systems, within the harmonic pairing–vibration model, B goes linearly with the number of pairs, giving $R = 2$ when A is a closed shell system and $R = \infty$ when A is a system with two particles away from a closed shell nucleus.

In figs. 3a and 3b the ratios R are shown for neutrons and protons respectively, while table 1 displays the few neutron systems that yield values of R outside the figure scale. The value $R = 2$ is approached in all well-known neutron pairing vibrators. In these nuclei R is very large when two neutrons are added to the closed shell. All these features are in agreement with the pairing–vibration model, as mentioned above. Nevertheless, the value $R = 2$ is never precisely reached because there are anharmonicities due to the blocking induced by the Pauli principle. The value $R = 4$, corresponding to the superfluid phase, is reached in the middle of the major shells for both neutrons and protons, as expected. However, it is noted that for protons this value has a much smaller dispersion than for neutrons. In addition,

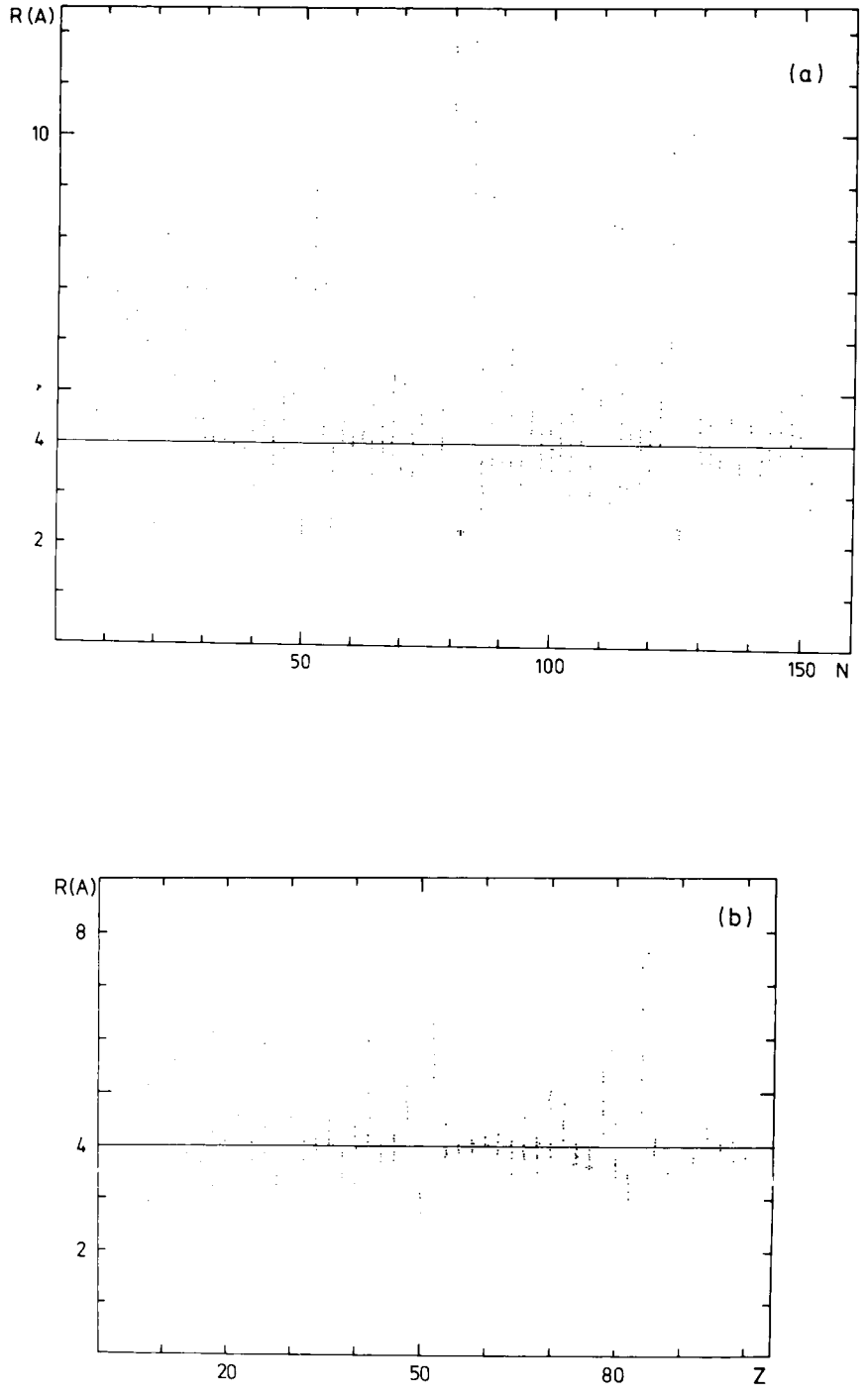

Fig. 3. The ratio $R(A)$ [eq. (3)] for the clusters (a) neutron–neutron, and (b) proton–proton.

TABLE 1

Values of $R(A)$ [eq. (3)] for nuclei where the value of $R(A)$ for neutrons is outside the range of fig. 3a

N	Z	$R(A)$	N	Z	$R(A)$
10	8	30.35	88	64	−9.55
22	20	15.34	90	68	14.05
26	20	13.74	110	80	19.54
48	38	15.00	128	82	35.46
80	52	16.30	128	84	15.80
84	54	12.81	128	86	18.13
88	62	330.0	128	88	21.87
90	62	43.1			

for protons, the values of R in closed shell nuclei are far from the corresponding harmonic values, marking a striking difference with the neutron case. This confirms the superconductive character of protons throughout the periodic table, as suggested above.

The large (and sometimes negative) values obtained at $N = 88$ and $N = 90$ are due to the vanishing of $S(A, 2)$ at the phase transition from spherical to deformed nuclei.

The larger tendency of protons to be in a superconductive phase is also associated, as pointed out in ref. [14]), to a larger abundance of odd-N nuclei in the stability valley as compared with the odd-Z ones. Correspondingly there is a slightly larger odd–even mass difference for odd-Z nuclei than for odd-N ones. This feature has been customarily reproduced in BCS calculations by choosing $G_p > G_n$ by 10 to 20%.

The reason for this difference is a result of a very delicate balance between different effects and should require a detailed investigation that falls outside the scope of the present paper. On the one hand the Coulomb repulsion among two protons outside the close core favours configurations in which both particles have the least possible spacial overlap, thus favouring unpaired states. On the other hand [18]) it is possible to argue that a greater net attraction between two protons close to the Fermi surface is expected because of the lesser kinetic energy due to the presence of the Coulomb potential. This fact makes less effective the repulsive hard-core of the nucleon–nucleon interaction.

4. Four-nucleon clusters

The discussion carried out in the previous section can be extended to more complicated clusters.

A cluster δA can be considered as a relevant mode of excitation when $C(A, \delta A_1, \delta A_2)$ is positive for all possible partitions of δA into the fragments δA_1

and δA_2. It is enough that one combination of δA_1 and δA_2 gives negative values of C to invalidate the cluster δA as a possible building block, because in this case it would be energetically favorable to use the uncorrelated configuration. The only cluster of four nucleons to fulfill this condition throughout the periodic table is the α-like cluster, which includes two neutrons and two protons. One may think that other cases, like the four-neutron cluster, should have some relevance since the pairing interaction is very important in this case. However, since for identical nucleons

$$C(A, 2, 2) = 2\,S(A, 2) - S(A, 4) = 2\,S(A, 2)(1 - \tfrac{1}{2}R(A)) \tag{4}$$

and R is always larger than 2, as seen in the previous section, $C(A, 2, 2)$ turns out to be always negative. As already mentioned, this is a manifestation of the blocking induced by the Pauli principle. In fact, of all possible four-nucleon clusters the α-like cluster is the one which feels that blocking less.

In figs. 4a and 4b we show $C(A, \mathrm{n}^2, \mathrm{p}^2)$ and $C(A, \mathrm{d}, \mathrm{d})$, respectively, where $\mathrm{n}^2(\mathrm{p}^2)$ indicates the two-neutron (proton) cluster and d indicates the deuteron cluster. In both cases C is always positive, showing that the α-cluster gains energy over the uncorrelated $T = 1$ and $T = 0$ pairs. The quantities $C(A, \mathrm{t}, \mathrm{p})$ and $C(A, {}^3\mathrm{He}, \mathrm{n})$ are also always positive and of the same order of magnitude as $C(A, \mathrm{d}, \mathrm{d})$. One can thus consider the α-cluster as an elementary mode of excitation that includes simultaneously the $T = 0$ and $T = 1$ channels of the residual interaction. In fact, when an α-cluster is built as two $T = 1$ pairs (fig. 4a) one may expect that the pairing correlations are already contained in these pairs giving a small value of C. The remaining part of the interaction (that which contributes to C) may be due to the neutron–proton interaction acting upon the two deuteron-like clusters included in the α-like cluster. This picture is confirmed, as shown in fig. 5, by the fact that

$$C(A, \mathrm{n}^2, \mathrm{p}^2) \approx 2C(A, \mathrm{n}, \mathrm{p})\,, \tag{5a}$$

$$C(A, \mathrm{d}, \mathrm{d}) \approx C(A, \mathrm{n}, \mathrm{n}) + C(A, \mathrm{p}, \mathrm{p})\,. \tag{5b}$$

These relations also explain why in fig. 4a no shell structure is present, since such structure is absent in $C(A, \mathrm{n}, \mathrm{p})$ (fig. 2c).

In order to confirm the relevance of the α-cluster as an elementary mode of excitation we consider

$$R_2(A) = S(A, 2\alpha)/S(A, \alpha)\,, \tag{6a}$$

$$R_3(A) = S(A, 3\alpha)/S(A, \alpha)\,, \tag{6b}$$

which, as in the two-particle cluster case, can give us information on the character of the α-cluster. That is, if $R_n(A) \approx n$, the system behaves as the head of an α-vibration band. On the other hand, if the system behaves as a condensate of α-clusters, the binding energies would be quadratic in the number of clusters [7]) and, therefore, $R_2(A) \approx 4$ and $R_3(A) \approx 9$.

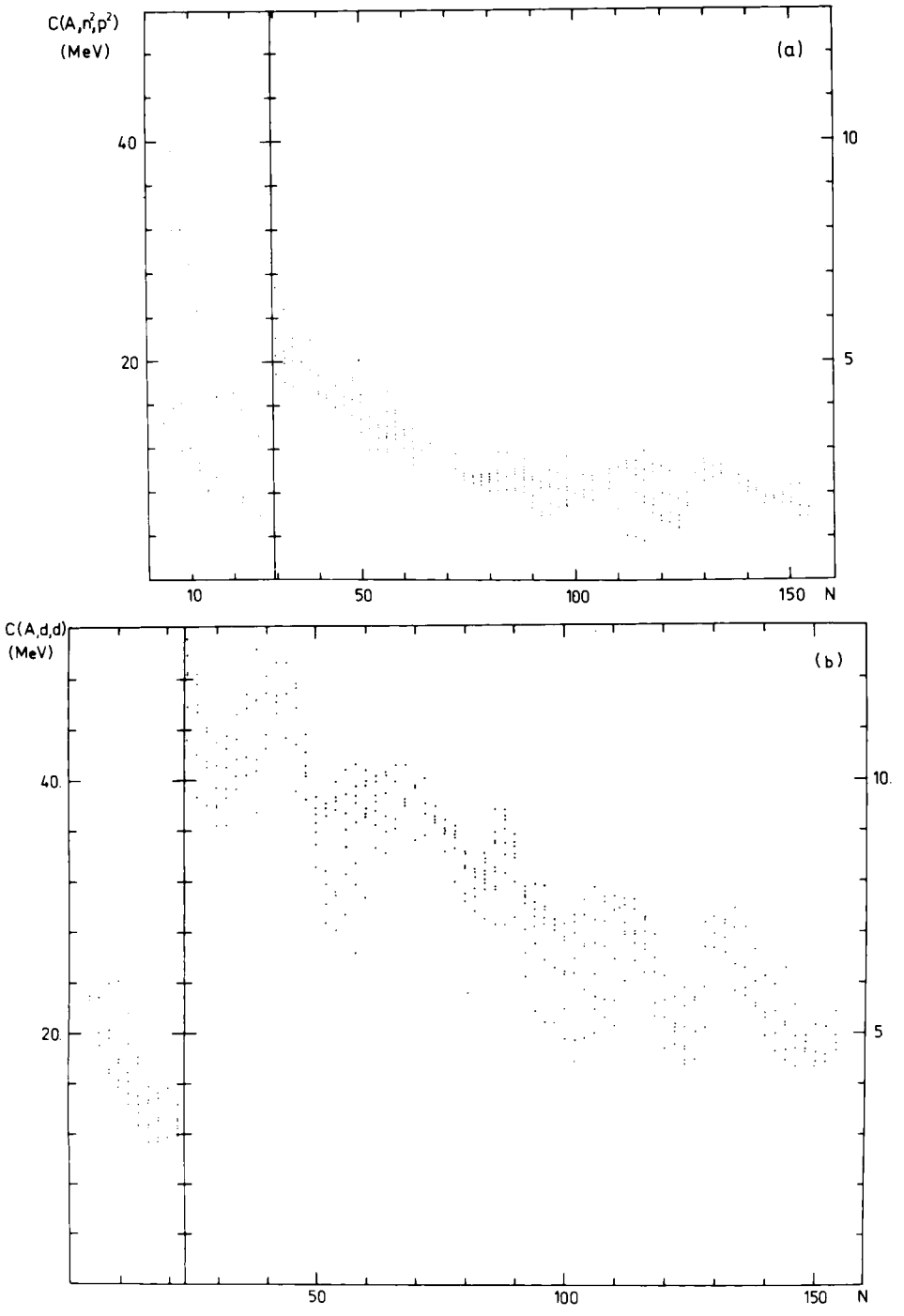

Fig. 4. Correlation energy $C(A, \delta A_1, \delta A_2)$ for $\delta A = \delta A_1 + \delta A_2$ being an α-like cluster: (a) $\delta A_1 (\delta A_2)$ is a neutron–neutron (proton–proton) cluster. (b) δA_1 and δA_2 are deuteron-like clusters. Note the different energy scale for light and heavy nuclei.

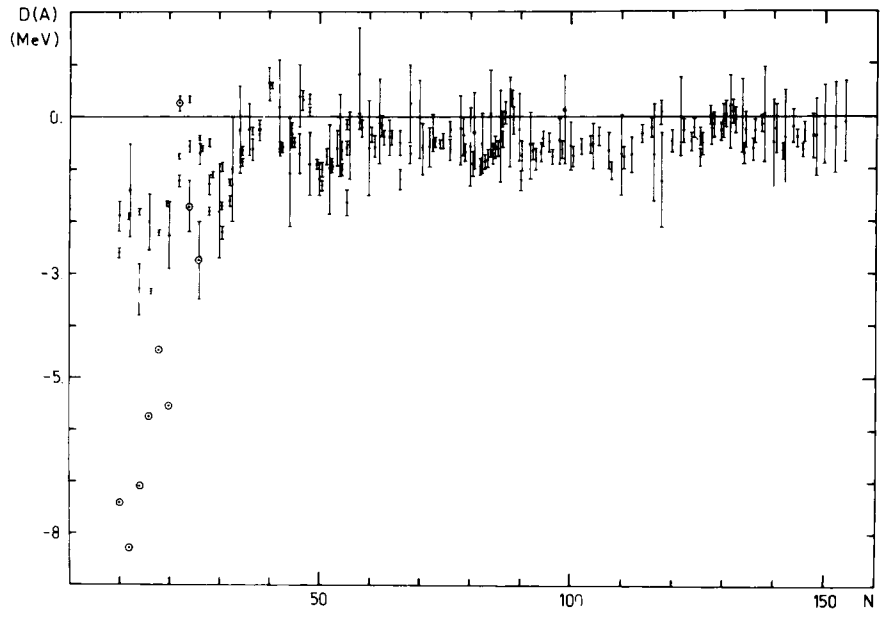

Fig. 5. Difference $D(A) = C(A, n^2, p^2) - 2C(A, n, p)$ [eq. (5a)]. The vertical bars indicate experimental errors. Only nuclei with errors in D smaller than 1 MeV are plotted. Systematic errors of ref. [17]) were taken to be equal to 300 keV. The circles correspond to nuclei with $N = Z$.

TABLE 2

Value of $R_2(A)$ and $R_3(A)$ [eq. (6)] for nuclei which may be considered as band-heads of α-vibration bands

Nucleus	$R_2(A)$	$R_3(A)$	Nucleus	$R_2(A)$	$R_3(A)$
^{50}Ti	2.250	3.312	^{118}Sn	1.947	3.241
^{56}Ni	1.997	3.047	^{120}Sn	2.027	3.346
^{64}Ni	2.307	3.441	^{128}Ba	2.023	3.379
^{66}Ni	1.850	2.727	^{138}Ba	2.129	3.467
^{88}Sr	1.878	2.713	^{136}Xe	2.062	
^{90}Zr	1.892	2.730	^{142}Nd	2.139	3.471
^{92}Mo	1.976	3.030	^{144}Sm	2.129	3.358
^{94}Ru	2.097	3.281	^{146}Gd	1.982	3.101
^{96}Zr	1.715	3.001	^{148}Dy	2.198	3.333
^{106}Sn	2.014		^{208}Pb	2.035	
^{112}Sn	2.173	3.468	^{212}Rn	2.189	
^{114}Sn	1.949	3.050	^{226}Th	1.799	3.183
^{116}Sn	1.979	3.085			

Experimental binding energies are taken from ref. [17]).

In table 2 we show all nuclei with $A \geq 50$ for which $R_2(A)$ and $R_3(A)$ are simultaneously in the range $1.65 \leq R_2(A) \leq 2.35$ and $2.55 \leq R_3 \leq 3.5$. These nuclei can be considered as the head of α-vibration bands. For comparison the values of $R_n(A)$ for the Pb isotopes are shown in table 3.

TABLE 3
As table 2, but for lead isotopes

A	$R_2(A)$	$R_3(A)$
192	2.436	4.456
194	2.459	4.403
196	2.981	5.023
198	3.018	5.158
200	2.908	4.983
202	2.752	4.862
204	2.747	5.437
206	3.124	5.108
208	2.035	

Note in table 2 that the head of the α-vibration band corresponds to a magic number of neutrons or protons, except for the number 56 and ^{226}Th. However the analysis of α-condensate nuclei (which should be away from closed shell nuclei) shows that R_2 and R_3 generally depart significatively from the values 4 and 9 respectively. This is mostly due to zeros in the function $S(A, \alpha)$. To avoid these singularities we analyze

$$T(A) = B(A) - B(A + \alpha) \tag{7}$$

which is proportional to the first derivative of the binding energies. For α-condensate nuclei the function defined in eq. (7) should behave as a straight line.

As seen in fig. 6, $T(A)$ has a rather constant slope for rare earth, transuranic and light lead isotopes. This supports a possible interpretation of deformed heavy systems as α-condensate. This contention however requires further investigation. A step in this direction has been taken [19]) attempting the description of rotational features in the rare earth region within an interacting quartet boson model. A possible experimental probe with which to check this picture is a systematic analysis of α-transfer cross sections although same ambiguities should be expected due to strong superfluid effects.

5. Three- and more particle modes

We have calculated eq. (2) for three-particle clusters of identical nucleons and obtained negative values of C throughout the periodic table. This is due, as for the four-particle case, to the blocking induced by the Pauli principle. The ^{3}He- and t-like clusters have positive values of C for all the possible partitions (i.e. for the t-like cluster these partitions would be (i) $\delta A_1 = \mathrm{n}$; $\delta A_2 = \mathrm{d}$ and (ii) $\delta A_1 = \mathrm{n}^2$; $\delta A_2 = \mathrm{p}$).

For the t-like cluster the partition that would compete most with the correlated configuration is $\delta A_1 = \mathrm{n}^2$; $\delta A_2 = \mathrm{p}$ while for the ^{3}He-like cluster that partition is

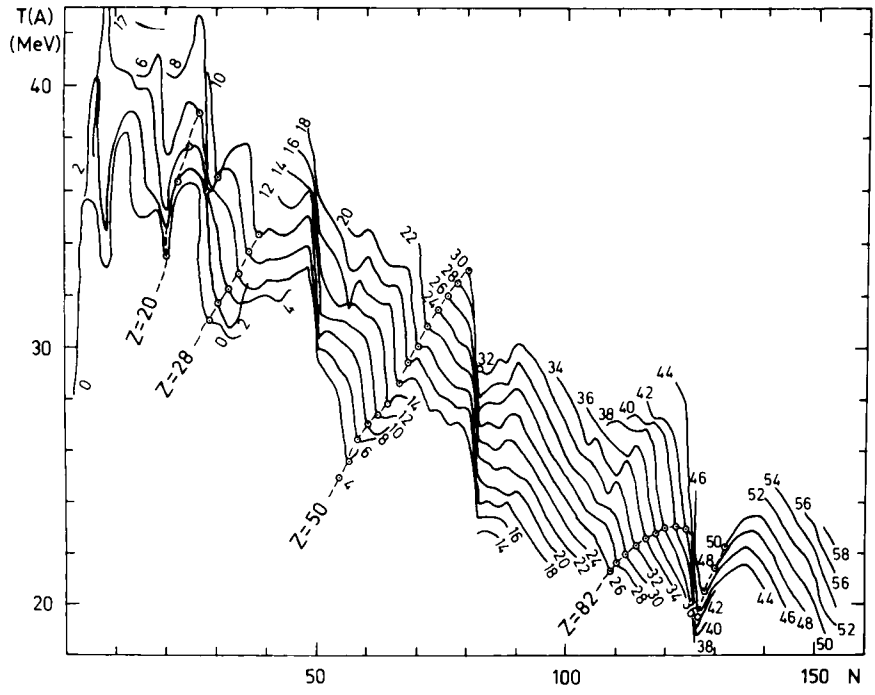

Fig. 6. Separation energy $T(A)$ [eq. (7)]. The lines join nuclei with the same $T_z = \frac{1}{2}(N-Z)$. These lines are labelled by the number $2T_z$. The circles correspond to nuclei having a magic number of protons.

$\delta A_1 = \mathrm{n}$; $\delta A_2 = \mathrm{p}^2$. The corresponding values of $C(A, \mathrm{n}^2, \mathrm{p})$ and $C(A, \mathrm{n}, \mathrm{p}^2)$ are shown in fig. 7. As seen in this figure C is rather small (except in light nuclei) and therefore three-particle clusters do not quite qualify as "building blocks" to describe nuclear spectra.

We have also evaluated eq. (2) for many-particle clusters and found C to be generally negative, which indicates that no cluster beyond the α-like cluster is eligible as "elementary modes" throughout the periodic table. This conclusion is rather surprising since one would think that, for instance, the ^{8}Be-like cluster may have some relevance. However, the correlation energy $C(A, \alpha, \alpha)$ is only positive in a few cases for light nuclei ($A \leqslant 40$) and for some nuclei between masses 74 and 90. In addition, that value of C is also positive for N or Z equal to 88 and 90. Yet, this correlation energy is always small.

6. Conclusions

The concept of elementary mode of excitation in nuclei assumes that a large amount of the nuclear correlation is already included in these modes. In addition one also requires that the interaction among the elementary modes must be described in a simple form. In this paper we have studied different possible modes

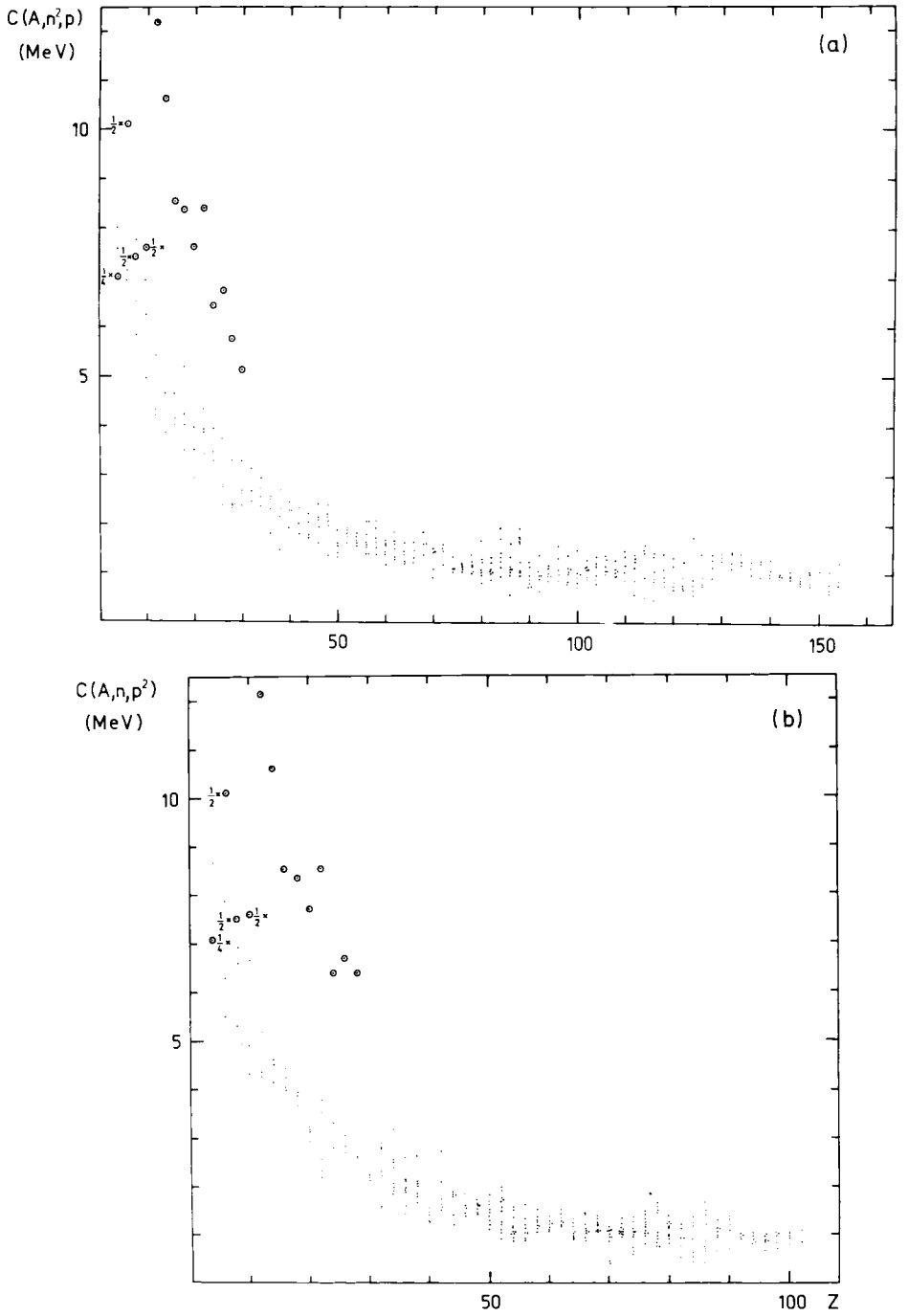

Fig. 7. Correlation energy for three-nucleon clusters (a) $C(A, n^2, p)$, and (b) $C(A, n, p^2)$. The circles correspond to nuclei with $N = Z$.

of excitation as clusters with more than one particle. The correlation aspects as well as the energy gained by these modes were analysed in terms of simple criteria associated with experimental binding energies. We have thus found for the two-particle cluster the known normal and superconductive behaviour of nuclei. However, we have also found that protons tend to behave as better superconductors than neutrons.

The application of the same criteria to more complex clusters showed that only α-like clusters can be considered as elementary modes. This suggests that the best way to study microscopically nuclei away from magic numbers, would be first to maximize the number of α-clusters. Afterwards one must bring in pairs of identical particles and, if necessary, consider, finally, the corresponding one-, two- or three-particle cluster.

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