Series Development of the Solution of an Eigenvalue Problem

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With 2 Figures

(*Received November 3, 1964*)

Summary

In view of the divergencies of the field theory, the convergence of series developments of bound eigenfunctions in powers of the coupling constant is studied. Born approximation represents a suitable development in cases where an unperturbated state can be defined and in which the energy shift due to the interaction is finite. Another series development, at fixed energy, can be deviced. This latter development leads necessarily to coefficients which represent non-regular functions in configuration space, nevertheless, those series converge to a regular function if the fixed energy coincides with an eigenvalue of the considered problem or, in cases where the energy of the systems is given, furnish an eigenvalue problem for the coupling constant.

1. Introduction

The physical problem which we have in mind refers to the scattering of light by light, which has been treated [1] as a fourth order problem in quantum electrodynamics. Two incident photons are successively absorved by an electron pair and become subsequently reemited by annhilation of the intermediate pair. In this treatment, the components of the electron pair which occur in the perturbation treatment, are considered as free particles. We have, however, reasons to believe, that electromagnetic interaction between the charged particles will lead, at convenient energies, to the formation of bound positronium states of known energy breath and will produce resonance scattering of photons by photons in narrow energy bands.

In the center of gravity system, those resonances are expected to occur at photon energies of

$$h\nu = m c^2 - \frac{1}{2} E_n$$

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where E_n denote the binding energies of positronium states. Though the scattering cross sections to be expected turn out to be of the order of the square of Compton's wave length $\lambda = 1/\mu$ this resonance phenomenon is still far outside the present experimental possibilities, because of the extremely narrow resonance energy bands which correspond to a life time of 10^{-10} sec.

As we know from available experimental evidence concerning the inverse process, pair annhilation, the predictions of ordinary perturbation theory agree, in the resonance region, fairly well with experiments. We may therefore suppose that, in the resonance region, perturbation theory represents, as an intermediate state of the scattering process, approximately the unperturbated ground state of positronium.

Outside of the resonance region no experimental evidence is, so far, available. What we have to expect, in this region, is that the intermediate configurations remain finite in the scattering process, but that they are different from a single positronium state and that they have small amplitudes. We ignore whether or not a power development in the coupling constant, accounting for this behaviour, can converge. We shall therefore, restrict our attention to the resonance region.

Quantum field theory accounts for the interaction between particles by successive emissions and absorptions of, in our case, mainly longitudinal photons. If quantum field theory in its present form, is able to account for the considered resonance phenomenon, the interaction and the resulting bound positronium states will be contained in the higher order terms which include photon exchanges between the virtual pair particles.

Examining the configurations which correspond to the virtual pairs, say in second order, in configuration space, we find singular functions which are related to the divergencies of perturbation energy. On the other hand, we know that the eigenfunctions describing bound positronium states which should intervene in a resonance phenomenon, are well known regular functions. We have therefore, reasons to believe, that the inclusion of interaction may, without further assumptions, regularize the divergencies with which we meet.

In this paper we shall put forward arguments refering to a simplified case, tending to show that there exist series deveolpments in power of the coupling constant, which converge everywhere to regular eigenfunctions, though all individual terms or partial sums, which do not take into account the complete interaction, show inevitable singularities.

2. The Pair Configuration According to Perturbation Theory

Perturbation theory leads, say up to second order, to the state vector

$$\phi = \phi^{(0)} + \phi^{(1)} + \phi^{(2)} \tag{1}$$

where

$$\phi^0 = |\vec{\kappa}, -\vec{\kappa}, 0, 0\rangle \tag{2}$$

$$\phi^{(1)} = \sum' \frac{H_{\vec{\kappa},\vec{p},\vec{p}'}^{\vec{\kappa},\vec{\kappa}}}{(E^0 - E_{\kappa,p,p'})} |\vec{\kappa}, 0, \vec{p}, \vec{p}'\rangle + \sum' \frac{H_{-\vec{\kappa},\vec{p},\vec{p}'}^{\vec{\kappa},-\vec{\kappa}}}{(E^0 - E_{-\kappa,p,-p})} |0, -\vec{\kappa}, \vec{p}, \vec{p}'\rangle$$

$$\phi^{(2)} = \sum' \frac{H_{\vec{\kappa},\vec{p},\vec{p}'}^{\vec{\kappa},-\vec{\kappa}} H_{\vec{p},-\vec{p}}^{\vec{\kappa},\vec{p},\vec{p}'}}{(E^0 - E_{\kappa,p,p'})(E^0 - E_{\vec{p},-p})} |0, 0, \vec{p}, -\vec{p}\rangle + \ldots$$

refer respectively to partial states in which two photons of opposite momenta, $\vec{\kappa}$ and $-\vec{\kappa}$, one photon and a pair $\vec{p}$, $\vec{p}'$ and, finally, a pair of zero total momentum are present.

Instead of ϕ we may consider the quantities

$$\psi^{\sigma\rho} = \sum' \frac{H_{\vec{\kappa},\vec{p},\vec{p}'}^{\vec{\kappa},-\vec{\kappa}} H_{\vec{p},-\vec{p}}^{\vec{\kappa},\vec{p},\vec{p}'}}{(E^0 - E_{\kappa,p,p'})(E^0 - E_{\kappa,p,-p})} u_{\vec{p}}^{\sigma}(\vec{r})\, v_{\vec{p}}^{\rho}(\vec{r}') \tag{3}$$

which represent the virtual electrons $u_{\vec{p}}$ and positrons $v_{\vec{p}}$ contributing to the solution (1) by means of DIRAC spinors. Though (3) is not a legitimate quantity of field theory, it allows to form expressions which appear in field theory as [2]

$$\langle \varrho(\vec{r})\, \varrho(\vec{r}') \rangle. \tag{4}$$

The perturbation energies in second and in fourth order belonging to (2) diverge. In particular, the fourth order terms contain contributions from the two-particles configurations given by (3). We emphazise, that the divergencies of the perturbation energies are closely connected with the fact, that the corresponding configurations (3), (4) are given by non regular functions in space.

The evaluation of (3) leads to rather involved complex integrals, which can be brought to the form

$$\psi = A(k,\mu) \exp\{-2(\mu^2 - \kappa^2)^{1/2} \varrho\}/\varrho + F(\mu, \varrho). \tag{5}$$

(5), in the center-of-gravity system of the pair, depends on the relative coordinates $\varrho = |\vec{r} - \vec{r}'|$ of the two particles, only. The second term, $F(\mu, \varrho)$, represents a non-regular function of spread $1/\mu$ which may be taken to account for the first order structure of the colliding photons, while the main term in which we are interested is the first one.

For $\kappa > \mu$ this term represents a free outgoing wave, which describes real pair production (to this order of the perturbation) and, therefore, the inverse process of the well established phenomenon of pair annihilation. For $\kappa < \mu$ the same term describes a bound, non-interacting pair. Its configuration is spread over a distance $[4(\mu^2 - \kappa^2)]^{-1/2}$ which, for k-values comparable to μ exceeds considerably the COMPTON wave length, and in the region in which we are interested becomes of the order of the positronium BOHR radius $b = 2\hbar^2/m e^2$.

It cannot be expected that the perturbation treatment (1) will account for the resonances at positronium levels, but we can expect for physical reasons that a more complete treatment, taking into account interaction between the virtually created particles will replace, at least at resonance energies, the first term of (5) by the corresponding positronium eigenfunction.

In the following we shall study the question how, in a simplified model, interaction between created particles can be described by a power expansion in terms of non-interacting configurations.

3. The Perturbation Treatment of the Kepler Problem

It is well known [3], that perturbation theory and, in particular, BORN approximation applies in scattering problems at high energies, but it is generally assumed that the series of this theory do not converge in the case of bound states. We shall show now, that the difficulties regarding bound states do not come from the method of perturbation theory but rather from the fact, that discrete bound states have to be isolated from the initially continuous spectrum of free particles. In order to overcome this difficulty, we shall consider a COULOMB field within an arbitrarily large sphere of radius R and postulate eigenfunctions wich vanish at $r = R$. This procedure ensures that the energy spectrum becomes discrete already for positive energies. It modifies sensibly the eigenfunctions in the immediate neighbourhood of the zero energy accumulation point, but it hardly affects the physically significant parts of the positive and negative spectrum.

It is inmediately clear that, after introducing a COULOMB field of variable strength perturbation theory applies and converges rapidly at sufficiently high energies. It is, therefore, sufficient to consider the extreme case of the 1s ground state.

The rigorous solution of the non-relativistic KEPLER problem is given, for an s-state, by

$$u = \exp(-i\,k\,r)\,F(1 + i/k\,b', 2, 2\,i\,k\,r) \tag{6}$$

where $F(1 + i/k\,b', 2, 2\,i\,k\,r)$ denotes the confluent hypergeometric series and $b' = \hbar^2/m\,e'^2$. The charge e' shall be considered, in the following, as a variable parameter. F is, then, an analytic function of the variables b', k and r.

The eigenvalue condition in our sphere is given by

$$F(1 + i/k\,b', 2, 2\,i\,k\,R) = 0 \tag{7}$$

and possesses a discrete set of solutions

$$k_1(b', R), \qquad k_2(b', R), \ldots .$$

The lowest mode of vibration is characterized by the eigenfunction

$$u_1(b', r) = \exp(-i\,k_1 r)\,F(1 + i/k_1\,b', 2, 2\,i\,k_1 r) \tag{8}$$

and the energy eigenvalue becomes

$$E_1(b') = \hbar^2\,k_1^2/2\,m. \tag{9}$$

According to (7), (8) and (9), both eigenfunctions and eigenvalues are analytic functions of parameter b' and vary smoothly as e' increases from zero. We note, in particular, the values

$$b' = \infty \qquad u_1(\infty, r) = \frac{\sin k_1 r}{r}, \qquad E_1 = \pi^2 \hbar^2 / 2\, m\, R^2$$

$$b' = \frac{8\,R}{{x_{1,1}}^2} \qquad u_1(b, r) = \left(\frac{b}{2\,r}\right)^{1/2} J_1\left[2\left(\frac{2\,r}{b}\right)^{1/2}\right], \qquad E_1 = 0, \qquad J_1(x_{1,1}) = 0$$

$$b' = b \qquad u_1(b, r) \cong \exp(-r/b), \qquad E_1 \cong -\frac{1}{2}\, e^2/b, \qquad R \gg b.$$

The comparison with the usual perturbation theory follows from the fact, that the analytic functions (8) and (9) can be uniquely developed into power series in the parameter $1/b'$ and coincide with the perturbation development. In general, these series will converge in a certain domain and may be extended, if necessary, by analytic continuation.

We note, in particular, that the lowest eigenvalue is shifted, in this case, by a considerable, but finite amount, which is approximately equal to the corresponding binding energy.

4. Power Development at Fixed Energy

The preceeding section shows, that the perturbation method does not represent an obstacle for the definition of an eigenvalue problem, provided that an appropriate initial state can be chosen.

It is this latter condition which is lacking in field theory. In our example the configuration of two photons initially present cannot be modified continuosly into a configuration of two interacting electrons of different sign. As a matter of fact, experience shows that, according to perturbation theory, in such problems non-regular configurations and corresponding energy divergencies appear which require additional renormalisation hypothesis in order to be removed.

Nevertheless, we can easily device a procedure in order to satisfy, at least formally, the equations of an eigenvalue problem in terms of interaction-free configurations. For this purpose we choose a fixed energy value, which, in general, will not be an eigenvalue of our problem. The example of the ordinary KEPLER problem in free space, sugests the following procedure:

We introduce into the SCHRÖDINGER equation

$$(\Delta + k^2 + 2/b\,r)\, u = 0, \tag{10}$$

$$u(r) = \sum_{n=0}^{\infty} (1/k\,b)^n\, u^{(n)}(r) \tag{11}$$

and obtain the set of equations

$$\begin{aligned} (\Delta + k^2)\, u_{(0)} &= -4\pi\, \delta(\vec{r}) \\ (\Delta + k^2)\, u_{(1)} &= -2\, k/r\, u^{(0)} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{12}$$

$$(\Delta + k^2)\, u^{(n)} = -\, 2\, k/r\, u^{(n-1)}$$

which can be successively solved, for $l = 0$ by

$$u^{(0)} = \mp \exp(\pm i\, k\, r)/r \qquad k^2 > 0 \tag{13}$$

$$u^{(0)} = \exp(-\kappa r)/r \qquad k^2 = -\kappa^2 < 0 \tag{14}$$

and

$$u^{(n)} = \frac{(-i)^n}{2\pi i n!}\,\frac{1}{r}\int_C \frac{\exp(-2\, i\, k\, r\, t)}{(t+\frac{1}{2})(t-\frac{1}{2})}\ln^n\left(\frac{t+\frac{1}{2}}{t-\frac{1}{2}}\right)dt \tag{15}$$

where, for $k^2 > 0$ we refer to Fig. 1 and for $k^2 < 0$ to Fig. 2. The integration path C_+ corresponds to the series which begins with $\exp(i\, k\, r)/r$.

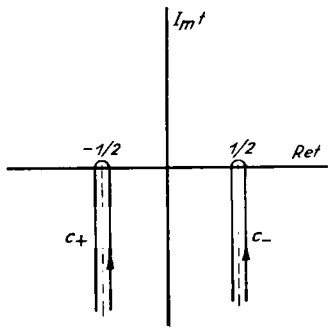

Fig. 1. The integration paths for $k^2 < 0$

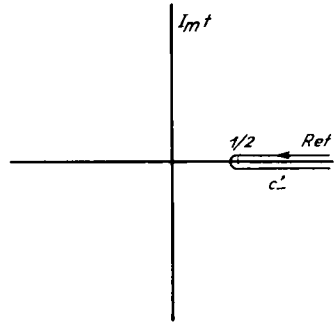

Fig. 2. The integration path for $k^2 \neq 0$

Similarly, the paths C_- and C_-' have to be used in order to obtain the series beginning with $\exp(-i\, k\, r)/r$ and $\exp(-\kappa\, r)/r$.

In the case $k^2 > 0$ we can choose linear combinations which are regular at $r = 0$ beginning with

$$u^{(0)}(r) = \sin k\, r/r \tag{16}$$

and which represent the eigenfunctions of the continuous spectrum of the KEPLER problem.

By inserting the terms (15) into (11) the summation can be performed and yields the well-known hydrogen solutions

$$u^{(\pm)}(r) = \frac{1}{2\pi i r}\int_C \frac{\exp(-2\, i\, k\, r\, t)}{(t+\frac{1}{2})(t-\frac{1}{2})}\left(\frac{t+\frac{1}{2}}{t-\frac{1}{2}}\right)^{-i/k\, b} dt =$$

$$= e^{-ikr} F^{(\pm)}(1 + i/k\, b, 2, 2\, i\, k\, r) \tag{17}$$

For $k^2 < 0$ we have to choose the path C_-' and obtain the exponentially decreasing solution

$$u^{(-)}(r) = \exp(-\kappa\, r)\, F^{(-)}(1 - 1/\kappa\, b, 2, 2\, \kappa\, r).$$

which is, in general, singular at $r = 0$. For discrete values of $k\,b$, however, the singularity at the origin disappears [4] and leads to the usual hydrogen eigenfunction.

We want to point out, that, in the case $k^2 < 0$, already the zero-order function $\exp(-\kappa r)/r$ is singular at the origin. The same is true for all higher order terms. At the eigenvalues each individual term in the series remains singular, while the entire series becomes regular. In this case we have, therefore, to deal with non-regular terms in each order of approximation. No regularisation of those terms, however, is required, since the regularisation is automatically performed by the summation of the entire series.

5. Conclusions

We have seen in paragraph 4 that there exist eigenvalue problems, the rigorous solution of wich can be expanded in power series. What is unusual of those series is the fact that the individual terms are singular at $r = 0$ and that no finite energy nor probability distribution can be attributed to them. Nevertheless, the series, once summed up, represent regular functions and belong to finite energies and probability distributions.

The situation encountered in field theory shows marked analogies with such behaviour. We cannot exclude therefore, the possibility that field theory may be formulated in such a way that the intrinsic divergencies at each order remain, but no regularisation is required for the complete solutions.

If this point of view should be found correct it implies immediately a condition for the coupling constant.

As is usually the case, an eigenvalue problem requires the energy, for fixed coupling constant, to be an eigenvalue. In many problems, e.g. the stability of the electron, the energy value $m\,c^2$ is determined beforehand. In such cases the coupling constant takes over the role of the eigenvalue parameter and permits, for a suitable value, to obtain finite solutions at given energy. Field theory would represent then, if it can be formulated as a mathematically well defined eigenvalue problem, an eigenvalue problem for the coupling constant.

As far as the above mentioned problem of the scattering of light by light is concerned, Prof. F. Beck (Darmstadt) has pointed out to the author that a more convenient, though phenomenological way exists, to account for the resonance phenomena to be expected at positronium bound states, by replacing in perturbation development the pairs described by free waves, by pair wave functions taking into account Coulomb attraction between the pair particles, what is very much in the aim of the Furry picture of quantum electrodynamics.

I wish to thank Professor Guido Beck for suggesting this work as well as his invaluable help in its development and in the preparation of the manuscript.

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