A Regularization Method for Fourier Series of Discontinuous Functions.

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Summary. – A regularization procedure, based on the punctual Padé approximants, is used to treat Fourier series of discontinuous functions. The arising difficulties for such series are easily overcome by this method. Probably the same approach can be used to regularize general orthogonal expansions.

Introduction. – The problem of the approximation of discontinuous functions by harmonic analysis occurs in many fields of mathematics and physics, as for instance in electromagnetism.

Generally the arising difficulties are twofold: the poor convergence rate of the harmonic series which is not uniformly convergent in a neighbourhood of the functions discontinuity points and the presence of Gibbs phenomenon for the series partial sums in a neighbourhood of the same points.

Even though we are able to eliminate the oscillations of the partial sums by arithmetical summation methods, it is important to use summation methods which are widely applicable. We are referring to rational approximants. These prove useful to sum power series and Legendre series too.

The purpose of this work is to study the ability of rational approximants to solve the problems mentioned above. In this way with a single approach and, from a practical point of view, with a single routine, it should be possible to have a powerful method to accelerate and regularize power series as well as orthogonal expansions.

A regularization procedure for a Fourier series of discontinuous functions. – Let us suppose that a function $f(x)$ can be expanded in Fourier series and has a first-kind discontinuity at a point $x = x_0$. We can always write

$$f(x) = f_c(x) + \Delta(x - x_0)\,,$$

where $f_c(x)$ is a continuous function and $\Delta(x - x_0) = \alpha\theta(x - x_0)$, $\theta(x - x_0)$ being the step function. Therefore, the Fourier series for $f(x)$ is the sum of the Fourier expansions of $f_c(x)$ and $\Delta(x-x_0)$. Since low convergence and Gibbs phenomena [1] arise from the expansion of $\Delta(x-x_0)$, we need only to study the regularization of this expansion to treat the most general case. Then, without loss of generality, we can consider the step function

$$\Delta(x) = \begin{cases} -\pi/4\,, & -\pi < x < 0\,, \\ \pi/4\,, & 0 < x < \pi\,, \\ \Delta(-\pi) = \Delta(\pi) = \Delta(0) = 0\,, & \end{cases}$$

and its Fourier series

$$\sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1}\,, \tag{1}$$

which is not uniformly convergent in a neighbourhood of the point $x = 0$.

The convergence can be regularized by using the punctual Padé approximants applied to the sequence of the partial sums of the series (1).

The regularization procedure is based on the correspondence between the formal power series

$$C(z) = \sum_{r=0}^{\infty} b_r z^r \tag{2}$$

and the sequence $\{C_m\}$ of partial sums of the series

$$C(1) = \sum_{r=0}^{\infty} b_r\,. \tag{3}$$

If we set $C_m = \sum_{r=0}^{\infty} b_r$, for each expansion (2) we have an associated sequence $\{C_m\}$, by defining $b_0 = C_0$, $b_m = C_m - C_{m-1}$ $(m \geqslant 1)$.

This correspondence allows one to consider without distinction power series summation methods and procedures to approximate the limits of sequences. Therefore, we can use the Padé approximants for the series $C(z)$ to find the limit of a sequence. Starting from the sequence $\{C_m\}$, we define another sequence:

$$E_n(C_m) = [n, n+m]_{C(1)}\,,$$

where $[n, n+m]_{C(1)}$ is the punctual Padé approximant (PPA) to the series $C(z)$ for $z = 1$, which can have sometimes a more rapid convergence rate to the limit value of the given sequence $\{C_m\}$. Such approximants, which are determinantal quotients, are nonreadily computable for large n. Bauer's η algorithm [2] allows one to overcome this difficulty. Such an algorithm is a nonlinear series transformation in which one

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[1] H. S. CARLSLAW: *Introduction to the Theory of Fourier's Series and Integrals* (London, 1930), p. 297.

[2] F. L. BAUER: in *Approximation of Functions* (Amsterdam, 1965), p. 134.

uses only the coefficients b_r of the series (2). On constructs the table

$$\begin{array}{ccccc}
\eta_0^0 & & & & \\
 & \eta_1^0 & & & \\
\eta_0^1 & & \eta_2^0 & & \\
 & \eta_1^1 & & \eta_3^0 & \\
\eta_0^2 & & \eta_2^1 & & \eta_4^0 \cdots \\
 & \eta_1^2 & & \eta_3^1 & \vdots \\
\eta_0^3 & & \eta_2^2 & \vdots & \\
\vdots & \eta_1^3 & \vdots & & \\
 & \vdots & & &
\end{array}$$

by using the triangular recurrence formulae

$$\left\{\begin{aligned} &\eta_{2n+1}^{m} = \eta_{2n}^{m+1} - \eta_{2n}^{m} + \eta_{2n-1}^{m+1}\,, \\ &(\eta_{2n+2}^{m})^{-1} = (\eta_{2n+1}^{m+1})^{-1} - (\eta_{2n+1}^{m})^{-1} + (\eta_{2n}^{m+1})^{-1} \end{aligned}\right. \tag{4}$$

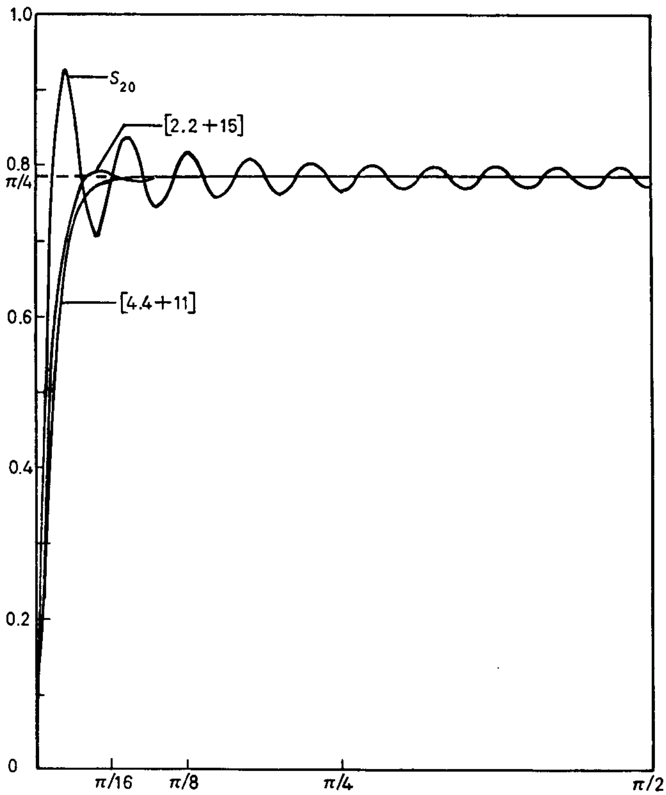

Fig. 1.

with the initial values $\eta_0^m = \infty$ and $\eta_1^m = b_m$. Then

$$(5)\qquad \begin{cases} E_n(C_m) = [n, n+m]_{C(1)} = \sum\limits_{k=0}^{m-1} \eta_1^k + \sum\limits_{k=1}^{2n+1} \eta_k^m\,, \\ E_{n+1}(C_{m-1}) = [n+1, n+m]_{C(1)} = \sum\limits_{k=0}^{m-1} \eta_1^k + \sum\limits_{k=1}^{2n+1} \eta_k^m\,, \end{cases}$$

where the first sum should be omitted for $m = 0$ and $C_{-1} = 0$.

From formulae (4) and (5) we can see that, when calculating $[n, n+m]_{C(1)}$, the information contained in the first m terms of the series $C(1)$ is resumed in

$$C_{m-1} = \sum_{k=0}^{m+1} \eta_1^k = \sum_{k=0}^{m+1} b_k\,.$$

The regularized procedure by PPA is applied in our case to the series

$$(6)\qquad \sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1}\, z^{k-1}\,.$$

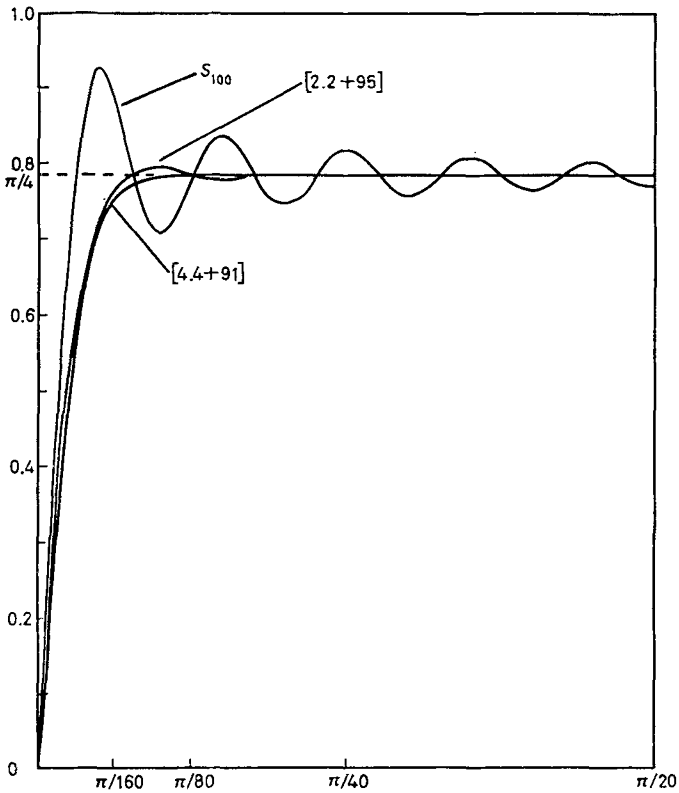

Fig. 2.

The PPA $[n, n+m]$ increasing with fixed n and m can be calculated by the η algorithm which allows us to consider very high m values without stability problems. In fig. 1 are shown the results obtained with the partial sum of 20 waves S_{20} and the approximants $[2, 2+15]$ and $[4, 4+11]$. Since the series (1) is a sinus series and the approximants have the same symmetry (as can be seen by observing that such approximants are determinantal quotients), it is sufficient to show the results only in the interval $(0, \pi/2)$. We do not consider the approximants $[1, 1+m]$ and $[3, 3+m]$ because these latter have a behaviour similar to that shown in the case of Legendre series ([3]): for a fixed number of waves and for some values of the variable x, the denominator of the approximant becomes very small and the error diverges. One can see that the convergence is remarkably increased with a few iterations and that the Gibbs phenomenon is present in the partial sums, but not in the approximants. This effect is more clear if we use a great number of waves, as is shown in fig. 2, where are considered the partial sum with 100 waves S_{100} and the approximants $[2, 2+95]$ and $[4, 4+91]$. For $\pi/80 \leqslant x \leqslant \pi/2$ the approximants are completely flat and converge to the exact value.

The increasing convergence rate of the approximants is shown in fig. 3, where we consider the approximants $[4, 4+11]$, $[4, 4+31]$ and $[4, 4+51]$ corresponding to 20, 40 and 60 terms of the series, respectively.

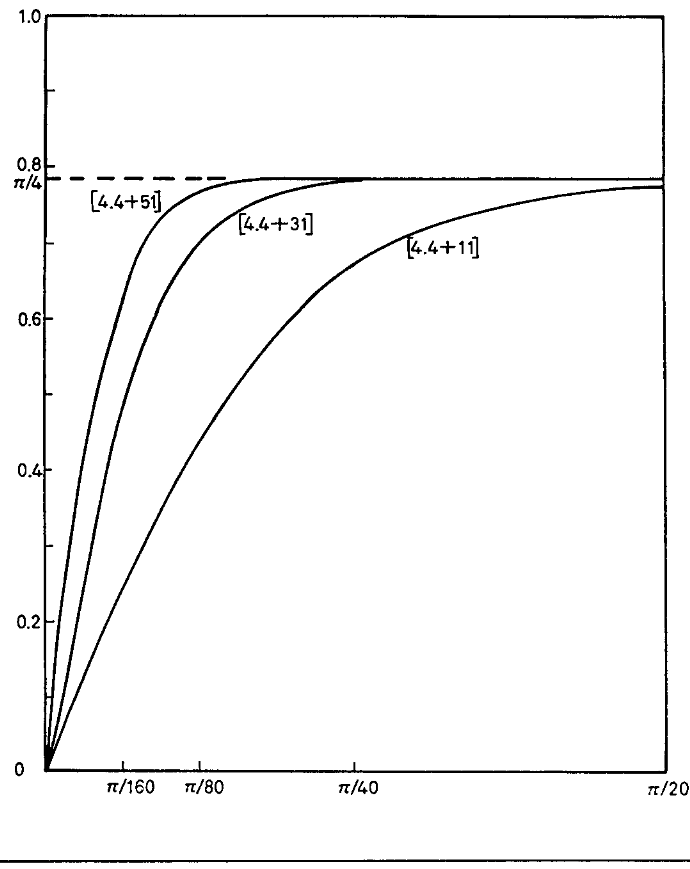

Fig. 3.

([3]) C. R. Garibotti, F. F. Grinstein and J. E. Miraglia: *Z. Phys. A*, **297**, 189 (1980).

Conclusion. – As is well known, the Gibbs phenomenon is present not only in the Fourier series, but also in general orthonormal series [4]. In this work only the regularization procedure for the Fourier series has been studied, but it is possible that the present approach is much more widely applicable. We have omitted the convergence proof for the PPA in the case of Fourier series, but we believe that the theorems proved in recent papers [5-8] can be slightly modified to fit our case: in fact, for the case of Legendre series the convergence proofs are stated by the behaviour of terms of high angular momentum which behave as harmonic terms with coefficients $a_n \sim n^{-\beta}$ similar to those of the Fourier series.

([4]) P. M. MORSE and H. FESHBACH: *Methods of Theoretical Physics*, Part I (New York, 1953), p. 745.

([5]) C. R. GARIBOTTI and F. F. GRINSTEIN: *J. Math. Phys.* (*N. Y.*), **19**, 821 (1978).

([6]) C. R. GARIBOTTI and F. F. GRINSTEIN: *J. Math. Phys.* (*N. Y.*), **19**, 2405 (1978).

([7]) C. R. GARIBOTTI and F. F. GRINSTEIN: *J. Math. Phys.* (*N. Y.*), **20**, 141 (1979).

([8]) F. F. GRINSTEIN: *J. Math. Phys.* (*N. Y.*), **21**, 112 (1980).