T E S I S

1971

CENTRO ATOMICO BARILOCHE
COMISION NACIONAL DE ENERGIA ATOMICA

INSTITUTO DE FISICA "Dr. J. A. BALSEIRO"
UNIVERSIDAD NACIONAL DE CUYO

SAN CARLOS DE BARILOCHE, RIO NEGRO - ARGENTINA

01.71.

1a. Parte

SOLUCIONES DEL SISTEMA ACOPLADO DE OSCILADORES Y CAMPOS ELECTROMAGNETICOS EXCITADOS POR UNA PARTICULA EXTERNA.

2a. Parte

ANALOGIAS ENTRE LA RESPUESTA DE UN OSCILADOR Y UN MEDIO DISPERSIVO A LOS CAMPOS DE UNA PARTICULA CARGADA EN MOVIMIENTO.

J.G. de VOOGHT

Tesis presentada ante la Universidad Nacional de Cuyo, Republica Argentina, para optar al título de Doctor en Física.

Guido Beck

Prof. Guido Beck

Asesor Científico

G. de Vooght

Lic. J.G. de Vooght

Doctorando

San Carlos de Bariloche (R.N.) - ARGENTINA

-1971-

RESUMEN

Este trabajo se divide en dos partes intimamente vinculadas.

La primera de ellas se refiere a las soluciones generales de un sistema acoplado de osciladores y campos electromagnéticos y al consiguiente analisis de las excitaciones creadas por una partícula cargada. Esta parte del trabajo tiene su origen en un importante trabajo de E.Fermi[1][2], en el cual este autor estudió la perdida de energía de una particula cargada rápida debida a la ionización del material a través del cual esta pasando. Tomando en cuenta la modificación que la polarizacion del medio introduce en los campos de la particula, Fermi encontro que esta perdida tiene dos contribuciones. En primer lugar una contribución que, cuando la absorción tiende a cero, permanece finita a medida que la distancia desde la trajectoria de la partícula tiende a infinito, y representa energía perdida en excitar el medio por un mecanismo que no era immediatamente aparente en la teoría de Fermi.

El tratamiento de Fermi era clasico, el medio fue tratado como un conjunto de osciladores distribuidos sobre todo el espacio, de una unica frecuencia de resonancia en el rango visible del espectro.

En un posterior trabajo Aage Bohr[3] dijo que, desde el punto de vista macroscópico, la perdida de energía de la partícula "parece tomar lugar" en dos modos esencialmente diferentes.

Para distinguir entre estos dos mecanismos, pensó que sería conveniente separar los campos electromagneticos producidos en la substancia por la partícula en una parte transversal (es decir de divergencia nula), y una parte longitudinal (es decir irrotational). La parte transversal debería dar cuenta de la parte radiativa de las perdidas, mientras que, la parte longitudinal correspondería a oscilaciones dejadas en "la estela" de la partícula.

El tratamiento de A.Bohr era tambien clasico pero sus conclusiones fueron obtenidas razonando desde un punto de vista microscópico, y sin dar expresiones explicitas que las substanciasen.

Despues de estos dos importantes trabajos un considerable esfuerzo fue dedicado a la investigación del efecto Cherenkov asi como al problema general de las excitaciones creadas por particulas cargadas, ya sea desde el punto de vista clasico o desde el punto de vista cuántico [4][5][6][7][8][9].

Sin embargo, y dentro de lo que es conocido por el presente autor, ningun analisis sistematico partiendo de primeros principios de las soluciones clasicas de un sistema acoplado de osciladores y campos electromagnéticos, y su vínculo con los resultados de Fermi y las conclusiones de A.Bohr, asi como con algunos problemas mas particulares, ha sido llevado a cabo.

A pesar de que no muchos resultados nuevos pueden esperarse de tal investigación, el presente trabajo ha sido motivado por el deseo de dar un tratamiento unificado y una comprehensión //

mas profunda de los mecanismos de excitación de un medio infinito, a traves de la demostracion explícita de la naturaleza y propiedades de los campos longitudinales y transversales creados por la partícula, asi como sus respectivas contribuciones a las perdidas de energía de la partícula.

Esta primera parte del trabajo sigue el siguiente esquema. El modelo utilizado, asi como sus ventajas y limitaciones, es discutido en el #1 . Un breve resumen de propiedades bien conocidas del sistema acoplado de osciladores y campos electromagnéticos es dado, que sirve asimismo para establecer la notación y conceptos a utilizarse en el resto del presente trabajo.

En # 2 el sistema acoplado inhomogeneo para una fuente general externa, con dependencia espacio-temporal, se analiza y separa en partes transversales y longitudinales. Una corta disgresion sobre la equivalencia de posibles metodos de separación aparece en el # 3 .

En # 4 se trata del caso particular en el cual la excitación de los osciladores es producida por una carga puntual en movimiento. Las soluciones longitudinales se estudian en # 5 y # 6 . En el primero de ellos encontramos las expresiones explicitas para los campos longitudinales. En el segundo se da su interpretacion física, valores en casos límites y comportamiento asintótico. Se muestra que, en general, las soluciones longitudinales tienen dos contribuciones intimamente //

relacionadas : a) un campo que puede pensarse como dando un apantallamiento de la carga y b) una onda que identificamos como una onda de plasma de osciladores. En principio la última contribucion corresponde a la parte oscilante predicha por A.Bohr, a pesar de que sus propriedades no son exactamente las mismas.

Las soluciones transversales se dan en el # 7. Estas soluciones aparecen como diferencias entre las soluciones totales (soluciones de Fermi) y las soluciones longitudinales encontradas en el # 5. Las propriedades de los campos totales y transversales se analizan en el # 8. Se revéen algunos resultados bien conocidos concerniente a los campos de radiación de Cherenkov de manera a proveer una base para los cálculos posteriores.

En el # 9 las soluciones totales son evaluadas por integración en el plano complejo de la frecuencia de manera a mostrar la naturaleza de las diferentes contribuciones, su dependencia espacio-temporal y su relación con las soluciones longitudinales.

Se muestra que, en el limite $v \ll c$ (es decir cuando no se consideran los efectos de retardo), solo dos contribuciones intimamente ligadas existen: el apantallamiento y la onda de plasma de osciladores. En este caso ambas contribuciones son longitudinales. Cuando se tienen en cuenta efectos de retardo estas dos contribuciones quedan, a pesar de que la primera//

ya no es de tipo longitudinal, mientras que aparecen además los campos de radiación de Cherenkov. El apantallamiento de la partícula se hace menos importante a medida que la velocidad aumenta y, en efecto, cambia de caracter en el umbral del efecto de saturacion predicho por la teoria de Fermi.

Finalmente, el tema de las perdidas de energía y su separación en diferentes modos se trata en el # 10, mientras que el # 11 da un resumen de resultados y conclusiones.

En la segunda parte del trabajo se estudian las analogías entre las respuestas de un unico oscilador y de un medio dispersivo representado por el modelo de osciladores de una única frecuencia, a los campos de una partícula cargada en movimiento.

Dentro de lo que es conocido por el presente autor, la respuesta de un oscilador a la fuerza ejercida por una partícula cargada en movimiento, fue analizada por primera vez por N.Bohr (1) , su interés en esa oportunidad siendo principalmente la energía transferida al oscilador y el posterior calculo de la pérdida de energía de una carga moviéndose a través del medio. El presente trabajo retoma el mismo problema pero analizado desde un diferente punto de vista, y aunque esta conectado con el problema de las pérdidas de energía, habiendo tenido por otra parte su orígen en el intento de comprender el mecanismo responsable de una parte de la perdida de energía como dada por la teoría de Fermi[(2)], y una posterior interpretacion del mecanismo de esta perdida por A.Bohr[(3)], su fin es //

estudiar el comportamiento cinemático del oscilador y su relacion con las perturbaciones creadas por una carga en movimiento en un medio dispersivo. Por comparacion de este comportamiento con las soluciones del sistema acoplado de un conjunto de osciladores y campos electromagnéticos, en presencia de una carga externa en movimiento, se muestra:

a) Como las propriedades asociadas a la respuesta del sistema acoplado surgen claramente de la respuesta de un unico oscilador.

b) Como la interacción entre los osciladores afecta esta respuesta.

El esquema general de esta parte del trabajo es el siguiente: en el # 1 las soluciones correspondientes al oscilador se presentan y analizan; # 2 trata de las soluciones del sistema acoplado y sus propriedades, mientras que finalmente el # 3 se dedica a una discusion y comparacion de resultados.

----0----

To A. and T.

CONTENTS

Part I

Part II

----o----

ON THE SOLUTIONS OF THE COUPLED SYSTEM OF OSCILLATORS AND ELECTROMAGNETIC FIELDS EXCITED BY AN EXTERNAL CHARGE.

J.G. de VOOGHT ,
Institut de Physique, Physique Théorique,
Université de Liège - Sart Tilman, Liège,
Belgium.

ABSTRACT :

A systematic study of the longitudinal, transverse and total solutions of a coupled system of oscillators and electromagnetic£ in the presence of an external point charge is carried out. The space-time dependence of the solutions as well as their values in specific cases and asymptotic behaviour are analyzed. It is shown that in general the longitudinal fields show two well defined contributions : a) a symmetric field surrounding the particle and carried convectively which is interpreted as a screening field. b) an excitation defined in principle in a whole semiespace and identified with an oscillator plasma wave which corresponds to the excitation predicted in A. Bohr's microscopic theory of energy loss although showing somewhat different properties. The transverse solutions appear as differences between the fields given in Fermi's macroscopic theory of energy losses and the longitudinal

Permanent address : Centro Atomico Bariloche - San Carlos de Bariloche - Rio Negro - Argentina.

solutions. Using methods of complex variable theory it is shown how we can separate the total perturbations created by the particle in a medium represented by oscillators into three intimately related contributions : screening, oscillator plasma excitation, Cherenkov radiation. The space-time configuration of these fields as well as their relation to the longitudinal solutions and their evolu-tion for different ranges of the velocity of the particle is given. The problem of the energy loss associated to the creation of the plasma wave is treated.

INTRODUCTION :

In a pioneering work Fermi (1)(2) studied the loss of energy of a fast charged particle due to the ionisation of the material through which it is passing. Taking into account the change in the electric field of the particle due to the polarization of the medium, he found that this energy loss had two contributions. Firstly, a contribution which for vanishing absorption remained finite as distance from the trajectory of the particle tended to infinite, and that could he identified with a loss due to Cherenkov radiation. Secondly, a contribution which due to an exponential dependence on distance from the trajectory tended to zero as that distance tended to infinite, and represented energy lost in exciting the medium by a mechanism which was not immediately apparent in Fermi's theory.

Fermi's treatment was classical, the medium being treated as a collection of oscillators distributed over all space, of one single resonance frequency in the visible range of the spectrum.

In a subsequent paper Aage Bohr (3) said that, from the macroscopic point of view, the energy loss of the particle "appears to take place" in two essentially different modes. In order to distinguish between these two mechanisms, he thought it convenient to separate the electromagnetic fields produced by the particle in the substance into a transverse (i.e. divergence free) part, and a longitudinal (i.e. irrotational) part. The transverse part would account for the radiation part of the loss, while the longitudinal part would correspond to oscillations left in "the wake" of the particle.

A. Bohr's treatment was also classical although his conclusions were derived by reasoning from a microscopic point of view, and without giving explicit expressions to substantiate them.

After those two important works a considerable amount of effort has been devoted to the investigation of Cherenkov effect as well as to the general subject of excitations created by charged particles, either from a classical or a quantum mechanical point of view.(4)(5)(6)(7)(8)(9)

However, as far as the present author knows, no systematic analysis starting from first principles of the classical solutions of a coupled system of oscillators and electromagnetic fields and their relation to Fermi's results and Aage Bohr's conclusions, as well as to some more particular problems, has been carried out.

Although not many new results may be expected from such an investigation, the present work has been stimulated by the desire to provide a unified treatment and a deeper understanding of the mechanisms of excitation of an infinite medium by showing explicitly the nature and properties of the longitudinal and transverse fields created by the particle, as well as their respective contributions to the energy loss of the particle.

The outline of the work is as follows. The model used as well as its advantages and limitations are discussed in § 1. A brief review of well known properties of the coupled system of oscillators and electromagnetic fields is given, which also serves to establish the notation to be used throughout the present paper.

In § 2 the inhomogeneous coupled system for a general space and time dependent external source is analyzed and separated into transverse and longitudinal parts. A short digression on the equivalence of possible methods of separation appears in paragraph § 3.

Paragraph § 4 deals with the particular case in which the excitation of the oscillators is produced by a moving point charge. The longitudinal solutions are studied in paragraphs § 5 and § 6. In the first of them we find the explicit expressions for the longitudinal fields. In the second one we give their physical interpretation, values in limiting cases and asymptotic behaviour. It is shown that, in general, the longitudinal solutions have two intimately related contributions : a) a field which may be thought of as providing a screening of the charge, and b) a wave which we identify as an oscillator plasma wave. In principle the latter contribution corresponds to the oscillating part predicted by A. Bohr, although its properties are somewhat different.

The transverse solutions are given in § 7. These solutions appear as differences between the <u>total</u> (Fermi's solutions) and the longitudinal solutions found in § 5. The properties of the transverse and total fields are analyzed in paragraph § 8. Some well known results concerning Cherenkov radiation fields are reviewed in order to provide a background for later developments.

In paragraph § 9 the total solutions are evaluated by integration in the complex frequency plane so as to show the nature of their different contributions, their space-time dependence and their relation to the longitudinal solutions.

It is shown that, in the limit when $V \ll C$ (i.e. when retardation effects are not considered), only two intimately related contributions exist : the screening and the oscillator plasma wave. In this case both contributions are longitudinal. When retardation effects are taken into account these two contributions remain, although the former one no longer of a longitudinal type, while in addition we have the Cherenkov radiation fields. The screening of the particle becomes less important as its velocity increases and in fact changes its character at the onset of the saturation effect predicted by Fermi's theory.

Finally, the subject of energy losses and its separation into different modes is treated in paragraph § 10, while § 11 gives a summary of results and conclusions.

§ 1. - THE HOMOGENEOUS COUPLED SYSTEM.

The homogeneous coupled system (no external sources) of oscillators and electromagnetic fields is given for a non magnetic medium by :

$$\ddot{\underset{\sim}{P}} + \omega_0'^2 \underset{\sim}{P} = \frac{\eta^2}{4\pi} \underset{\sim}{E}^* \qquad 1.1$$

$$\nabla . \underset{\sim}{E} = - 4\pi \nabla . \underset{\sim}{P} \qquad 1.2$$

$$\nabla . \underset{\sim}{B} = \nabla . \underset{\sim}{H} = 0 \quad (\mu = 1) \qquad 1.3$$

$$\nabla \times \underset{\sim}{E} = - \frac{1}{c} \frac{\partial \underset{\sim}{B}}{\partial t} \qquad 1.4$$

$$\nabla \times \underset{\sim}{B} = \frac{4\pi}{c} \frac{\partial \underset{\sim}{P}}{\partial t} + \frac{1}{c} \frac{\partial \underset{\sim}{E}}{\partial t} \qquad 1.5$$

where : $\eta^2 = \dfrac{4\pi N e^2}{m}$, N = N° of oscillators per unit volume. 1.6

As is well known this set of equations is the basis of Lorentz's theory of dispersion. The first of these equations describes the polarization $\vec{P}$ per unit volume due to the presence of sufficiently densely packed oscillators subject to an external force $\vec{E}^*$.

It must be emphasized that this represents a phenomenological system of equations which cannot be derived without the help of additional hypothesis from first principles. In order to account for elementary electrostatic phenomena we must equate the external force acting on the oscillators with the Lorentz expression :

$$\underset{\sim}{E}^{*} = \underset{\sim}{E} + \frac{4\pi}{3}\underset{\sim}{P} \qquad 1.7$$

where $\vec{E}$ = Average macroscopic value of the electric field.
Introducing scalar and vector potentials

$$\underset{\sim}{E} = -\,\mathrm{grad}\,\phi - \frac{1}{c}\frac{\partial \underset{\sim}{A}}{\partial t} \quad , \quad \underset{\sim}{B} = \underset{\sim}{H} = \nabla \times \underset{\sim}{A} \qquad 1.8$$

the system may be written :

$$\ddot{\underset{\sim}{P}} + \omega_0^2 \underset{\sim}{P} = \frac{\eta^2}{4\pi}\left(-\,\mathrm{grad}\,\phi - \frac{1}{c}\frac{\partial \underset{\sim}{A}}{\partial t}\right) \qquad 1.9$$

$$\Box \underset{\sim}{A} = -\frac{4\pi}{c}\frac{\partial \underset{\sim}{P}}{\partial t} \qquad 1.10$$

$$\Box \phi = 4\pi\,\mathrm{div}\,\underset{\sim}{P} \qquad 1.11$$

Together with the Lorentz condition :

$$\nabla . \underset{\sim}{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0 \qquad 1.12$$

where now :

$$\omega_0^2 = \omega_0'^2 - \frac{1}{3}\eta^2 \qquad 1.13$$

is the proper frequency of the oscillators corrected for the coupling with the electromagnetic field.

Some comments are pertinent concerning this system. Firstly, the description of the medium by a polarization per unit volume assumes implicitly that the system will be applied to the description of phenomena connected with excitations of long wavelength (i.e. greater than atomic dimensions). Secondly, the present work is concerned with the response of an infinite medium, i.e. no boundary effects such as transition radiation, or surface excitation, which would require a generalization of the present results will be considered.

Thirdly, no anharmonic effects are taken into account and this is obvious from the form of the equations which give the basic system.

In view of all this, it may be argued that the model assumed of oscillators of only one resonance frequency without even taking into account absorption effects is very idealized. However, it has the distinct advantage of showing clearly the nature of the excitations produced by the charged particle as well as the changes produced in the fields by the inclusion of dispersion effects. This would be extremely difficult if we were to consider a more realistic model.

As such, the results that will be derived, may apply just as well to gases, liquids or solids; to the case of an infrared resonance (ionic crystals), or to a resonance in the visible range of the spectrum (electronic motions), being in fact nothing more than the characteristic response of any system which as a first approximation may be represented by classical oscillators.

The results will also be valid in the limit when the proper frequency of the oscillators tends to zero (classical electron gas) and, as will be seen further on, are equivalent to some derived by Bohm and Pines [10,11,12] in their collective description of electron interactions in the limit when random motions may be neglected.

The solutions of the system may be found by a Fourier analysis in space and time where the transformation is defined by :

$$F(\underset{\sim}{x}, t) = \frac{1}{(2\pi)^2} \iiiint F(\underset{\sim}{k}, \omega)\, e^{i(\underset{\sim}{k}.\underset{\sim}{x} - \omega t)} d^3k\, d\omega \qquad 1.14$$

Given a certain vector field $\vec{F}(\vec{X}, t)$, separated into transverse and longitudinal parts :

$$\underset{\sim}{F}(\underset{\sim}{x}, t) = \underset{\sim}{F}_\ell(\underset{\sim}{x}, t) + \underset{\sim}{F}_t(\underset{\sim}{x}, t) \qquad 1.15$$

such that :

$$\nabla . \underset{\sim}{F}_\ell(\underset{\sim}{x}, t) \neq 0 \quad , \quad \nabla \times \underset{\sim}{F}_\ell(\underset{\sim}{x}, t) = 0 \qquad 1.16$$

$$\nabla . \underset{\sim}{F}_t(\underset{\sim}{x}, t) = 0 \quad , \quad \nabla \times \underset{\sim}{F}_t(\underset{\sim}{x}, t) \neq 0 \qquad 1.17$$

It is easily proved that the Fourier spectral amplitudes of the different parts will satisfy the conditions :

$\underset{\sim}{k} . \underset{\sim}{F}_t(\underset{\sim}{k}, \omega) = 0$ where : $\underset{\sim}{F}_t(\underset{\sim}{k}\omega) = \left(\frac{\underset{\sim}{k}}{k} \times \underset{\sim}{F}(\underset{\sim}{k}, \omega)\right) \times \frac{\underset{\sim}{k}}{k}$

$$= \underset{\sim}{F}(\underset{\sim}{k}, \omega) - \underset{\sim}{F}_\ell(\underset{\sim}{k}, \omega) \qquad 1.18$$

and $\underset{\sim}{k} \times \underset{\sim}{F}_\ell(\underset{\sim}{k}, \omega) = 0$ where : $\underset{\sim}{F}_\ell(\underset{\sim}{k}\omega) = F_\ell(\underset{\sim}{k}\omega)\frac{\underset{\sim}{k}}{k}$

$$= \left(\frac{\underset{\sim}{F}(\underset{\sim}{k}\omega) . \underset{\sim}{k}}{k}\right)\frac{\underset{\sim}{k}}{k} \qquad 1.19$$

These properties suggest the method to be employed for separating the total system into a transverse and a longitudinal subsystem.

Equations (1.10), (1.11) are not independent since they are connected by the Lorentz condition, therefore, in order to effect the separation, it is sufficient to deal with only two of equations 1.9, 1.10, 1.11 at a time. Bearing this in mind and using 1.18, 1.19, we obtain for the spectral amplitudes already separated :

<u>Transverse subsystem</u> :

$$(\omega_0^2-\omega^2)\,\underset{\sim}{P}_t(\underset{\sim}{k}\,\omega) - \frac{i\eta^2}{4\pi}\,\frac{\omega}{c}\,\underset{\sim}{A}_t(\underset{\sim}{k}\,\omega)=0 \qquad 1.20$$

$$\frac{4\pi}{c}\,i\,\omega\,\underset{\sim}{P}_t(\underset{\sim}{k}\,\omega)+\left(k^2-\frac{\omega^2}{c^2}\right)\underset{\sim}{A}_t(\underset{\sim}{k}\,\omega)=0 \qquad 1.21$$

where the condition for the existence of a solution is

$$k^2=\frac{\omega^2}{c^2}\left(1+\frac{\eta^2}{\omega_0^2-\omega^2}\right)=\frac{\omega^2}{c^2}\,\varepsilon(\omega) \qquad 1.22$$

and $\varepsilon(\omega)$ represents the dielectric constant of the medium as given by our model.

<u>Longitudinal subsystem</u> :

$$(\omega_0^2-\omega^2)\,\underset{\sim}{k}\cdot\underset{\sim}{P}(\underset{\sim}{k}\,\omega)=\frac{\eta^2}{4\pi}\left(-i\,k^2\phi(k\,\omega)+i\,\frac{\omega}{c}\,\underset{\sim}{k}\cdot\underset{\sim}{A}(\underset{\sim}{k}\,\omega)\right) \qquad 1.23$$

$$\left(-k^2+\frac{\omega^2}{c^2}\right)\phi(\underset{\sim}{k}\,\omega)=4\pi\,i\,\underset{\sim}{k}\cdot\underset{\sim}{P}(\underset{\sim}{k}\,\omega) \qquad 1.24$$

Together with the condition :

$$i\underset{\sim}{k}\cdot\underset{\sim}{A}(\underset{\sim}{k}\,\omega) - i\frac{\omega}{c}\phi(\underset{\sim}{k}\,\omega) = 0 \qquad 1.25$$

The condition for the existence of solutions with spectral amplitudes different from zero is either :

$$\omega^2 = \omega_0^2 + \gamma^2 = \omega_p^2 \qquad 1.26$$

or $$\mathcal{E}(\omega) = 0 \qquad 1.27$$

both conditions are equivalent.

Therefore, in the absence of boundaries, the most general solution of the homogeneous system will be given by expressions of the form :

$$\begin{Bmatrix} \underset{\sim}{A}(\underset{\sim}{x}\,t) \\ \underset{\sim}{P}(\underset{\sim}{x}\,t) \end{Bmatrix} = \begin{Bmatrix} \underset{\sim}{A}_l(\underset{\sim}{x}\,t) + \underset{\sim}{A}_t(\underset{\sim}{x}\,t) \\ \underset{\sim}{P}_l(\underset{\sim}{x}\,t) + \underset{\sim}{P}_t(\underset{\sim}{x}\,t) \end{Bmatrix} =$$

$$= \frac{1}{(2\pi)^2}\iiiint \begin{Bmatrix} \underset{\sim}{A}_l(\underset{\sim}{k}\,\omega) \\ \underset{\sim}{P}_l(\underset{\sim}{k}\,\omega) \end{Bmatrix} \delta(\omega^2 - \omega_p^2)\, e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x} - \omega t)}\, d^3k\, d\omega \; +$$

$$+ \frac{1}{(2\pi)^2}\iiiint \begin{Bmatrix} \underset{\sim}{A}_t(\underset{\sim}{k}\,\omega) \\ \underset{\sim}{P}_t(\underset{\sim}{k}\,\omega) \end{Bmatrix} \delta\left(k^2 - \frac{\omega^2}{c^2}\mathcal{E}(\omega)\right) e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x} - \omega t)}\, d^3k\, d\omega$$

1.28

and :

$$\phi(\underset{\sim}{x}\,t)=\frac{1}{(2\pi)^2}\iiiint \phi(\underset{\sim}{k}\,\omega)\,\delta(\omega^2-\omega_p^2)\,e^{i(\underset{\sim}{k}.\underset{\sim}{x}-\omega t)}\,d^3k\,d\omega \qquad 1.29$$

The transverse part of the contributions to the vector fields will be given by a superposition of plane waves :

$$\left.\begin{matrix}\underset{\sim}{P}_t\\ \underset{\sim}{A}_t\end{matrix}\right\}\sim \underset{\sim}{\varepsilon}\,e^{i(\underset{\sim}{k}.\underset{\sim}{x}-\omega t)}\;,\quad \underset{\sim}{\varepsilon}.\underset{\sim}{k}=0 \qquad 1.30$$

where the relation between $\underset{\sim}{k}$ and ω is defined by the dispersion relation (1.22).

The longitudinal part of the solutions is given, after integration over the frequencies, by a superposition of waves :

$$\left.\begin{matrix}\underset{\sim}{P}_l(\underset{\sim}{k})\\ \underset{\sim}{A}_l(\underset{\sim}{k})\end{matrix}\right\}\sim \frac{\underset{\sim}{k}}{k}\,e^{i(\underset{\sim}{k}.\underset{\sim}{x}\pm\omega_p t)} \qquad 1.31$$

$$\phi(\underset{\sim}{k})\sim e^{i(\underset{\sim}{k}.\underset{\sim}{x}\pm\omega_p t)} \qquad 1.32$$

with a group velocity :

$$\underset{\sim}{g}=\frac{\partial\omega}{\partial\underset{\sim}{k}}=\frac{\partial\omega_p}{\partial\underset{\sim}{k}}=0 \qquad 1.33$$

i.e. they cannot carry energy. The longitudinal part of the coupled system describes an <u>oscillator plasma</u>. In all the future work we shall call ω_p, as defined by equations $\left\{\begin{matrix}(1.26)\\(1.13)\end{matrix}\right.$, the <u>oscillator plasma frequency</u>.

If from the longitudinal fields we define a displacement vector :

$$\underset{\sim}{D}_l = \underset{\sim}{E}_l + 4\pi \underset{\sim}{P}_l = -\operatorname{grad} \phi - \frac{1}{c}\frac{\partial \underset{\sim}{A}_l}{\partial t} + 4\pi \underset{\sim}{P}_l \qquad 1.34$$

It can be proved by using equations 1.23, 1.24, 1.25 and integrating over the frequencies, that the spectral amplitudes in $\underset{\sim}{k}$ space of the longitudinal displacement vector satisfy the condition :

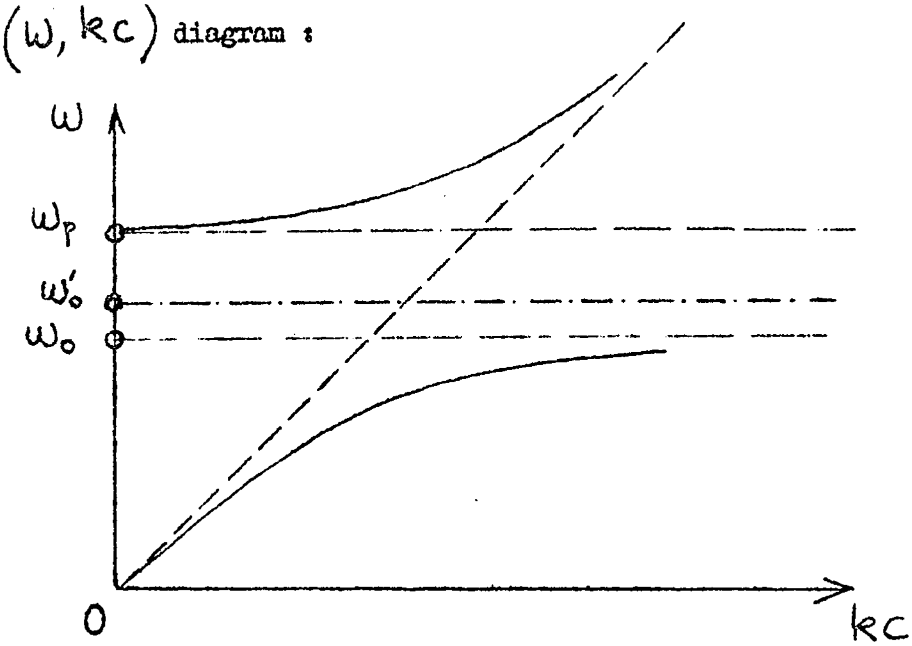
$$i\underset{\sim}{k} \cdot \underset{\sim}{D}_l(\underset{\sim}{k}) = 0 \quad \longrightarrow \quad \nabla \cdot \underset{\sim}{D}_l(\underset{\sim}{x}, t) = 0 \qquad 1.35$$

We infer from this result that, since by definition $\underset{\sim}{D}_l$ is irrotational, the displacement vector associated to the plasma solutions is null. It should be noted that the magnetic field for our case of an infinite distribution of oscillators and non magnetic medium will always be transverse.

In summary we may say : The homogeneous coupled system admits two types of solutions :

1) Longitudinal solutions which correspond to an oscillator plasma excitation of a fixed frequency ω_p.
2) Transverse solutions which correspond to mechanical and electromagnetic waves whose possible range is determined by the dispersion relation (1.22).

The properties of these solutions are clearly shown by the well known (ω, kc) diagram :



where the upper and lower branches represent mechanic and electromagnetic transverse solutions, whereas the horizontal line at ω_p corresponds to longitudinal solutions. Horizontal line at ω'_o corresponds to transverse and longitudinal solutions of the uncoupled oscillator plasma. This line, together with the straight line $\omega = kc$, represents the solutions of the uncoupled system. Region from ω_o to ω_p corresponds to the no-transmission zone.

We should bear in mind that in all rigorousness the range of allowed wave vectors for the validity of the model assumed should run from 0 to a certain k_{max} in order to satisfy the condition on the wave-lengths of possible excitations.

Therefore a solution by Fourier series with a more or less arbitrarily imposed cut-off would be more appropriate than the use of Fourier integrals. However, the use of the latter method in the present problem does not represent a great source of error, whereas the former would represent an undue refinement for our present purposes.

§ 2. - THE INHOMOGENEOUS COUPLED SYSTEM.

In order to study possible mechanisms of excitation of the coupled system we shall consider now its solution in the presence of space and time-dependent external sources. Writing in terms of scalar and vector potentials the system is now defined by :

$$\ddot{\underset{\sim}{P}} + \omega_0^2 \underset{\sim}{P} = \frac{\eta^2}{4\pi}\left(-\operatorname{grad}\phi - \frac{1}{c}\frac{\partial \underset{\sim}{A}}{\partial t}\right) \qquad 2.1$$

$$\Box \underset{\sim}{A} = -\frac{4\pi}{c}\underset{\sim}{j} - \frac{4\pi}{c}\frac{\partial \underset{\sim}{P}}{\partial t} \qquad 2.2$$

$$\Box \phi = -4\pi\rho + 4\pi \operatorname{div} \underset{\sim}{P} \qquad 2.3$$

Together with the Lorentz condition :

$$\nabla . \underset{\sim}{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0 \qquad 2.4$$

and the equation of charge conservation :

$$\nabla . \underset{\sim}{j} + \frac{\partial}{\partial t}\rho = 0 \qquad 2.5$$

where ρ and $\vec{j}$ are functions of space and time not specified for the moment.

We shall assume in what follows that the current may be separated into a transverse and a longitudinal part, i.e. :

$$\underset{\sim}{j}(\underset{\sim}{x}\, t) = \underset{\sim}{j}_l(\underset{\sim}{x}\, t) + \underset{\sim}{j}_t(\underset{\sim}{x}\, t) \qquad 2.6$$

where each part has the properties already indicated in § 1.

In this case the equation of charge conservation will be separated into the two subsidiary conditions :

$$\nabla \cdot \underset{\sim}{j}_l + \frac{\partial \rho}{\partial t} = 0 \tag{2.7}$$

$$\nabla \cdot \underset{\sim}{j}_t = 0 \tag{2.8}$$

carrying out a Fourier analysis of the system and separating into transverse and longitudinal components, we find the following expressions for the spectral amplitudes of the fields given as a function of the sources :

$$\phi(\underset{\sim}{k}\,\omega) = \frac{4\pi\,\rho(\underset{\sim}{k}\,\omega)}{\left(k^2 - \frac{\omega^2}{c^2}\right)\varepsilon(\omega)} \tag{2.9}$$

$$\underset{\sim}{A}_l(\underset{\sim}{k}\,\omega) = 4\pi\frac{\omega}{c}\,\frac{\rho(\underset{\sim}{k}\,\omega)}{k^2\left(k^2 - \frac{\omega^2}{c^2}\right)\varepsilon(\omega)}\,\underset{\sim}{k} = \frac{4\pi}{c}\,\frac{\underset{\sim}{j}_l(\underset{\sim}{k}\,\omega)}{\left(k^2 - \frac{\omega^2}{c^2}\right)\varepsilon(\omega)} \tag{2.10}$$

$$\underset{\sim}{A}_t(\underset{\sim}{k}\,\omega) = \frac{4\pi}{c}\,\frac{\underset{\sim}{j}_t(\underset{\sim}{k}\,\omega)}{\left(k^2 - \frac{\omega^2}{c^2}\varepsilon(\omega)\right)} \tag{2.11}$$

$$\underset{\sim}{P}_l(\underset{\sim}{k}\,\omega) = -i\,\frac{\varepsilon(\omega)-1}{\varepsilon(\omega)}\,\frac{\rho(\underset{\sim}{k}\,\omega)}{k^2}\,\underset{\sim}{k} \tag{2.12}$$

$$\underset{\sim}{P}_t(\underset{\sim}{k}\,\omega) = i\frac{\omega}{c^2}\left(\varepsilon(\omega)-1\right)\frac{\dot{\underset{\sim}{j}}_t(\underset{\sim}{k}\,\omega)}{\left(k^2-\frac{\omega^2}{c^2}\varepsilon(\omega)\right)}$$

2.13

$$\underset{\sim}{E}_l(\underset{\sim}{k}\,\omega) = -\frac{4\pi\,\rho(\underset{\sim}{k}\,\omega)}{\varepsilon(\omega)\,k^2}\,i\,\underset{\sim}{k}$$

2.14

$$\underset{\sim}{E}_t(\underset{\sim}{k}\,\omega) = \frac{4\pi}{c^2}\,i\,\omega\,\frac{\underset{\sim}{j}_t(\underset{\sim}{k}\,\omega)}{\left(k^2-\frac{\omega^2}{c^2}\varepsilon(\omega)\right)}$$

2.15

$$\underset{\sim}{B}(\underset{\sim}{k}\,\omega) = \frac{4\pi}{c}\,i\,\frac{\underset{\sim}{k}\times\underset{\sim}{j}_t(\underset{\sim}{k}\,\omega)}{\left(k^2-\frac{\omega^2}{c^2}\varepsilon(\omega)\right)}$$

2.16

where $\varepsilon(\omega) = 1 + \frac{\eta^2}{\omega_0^2-\omega^2}$ and the spectral amplitudes of the sources and scalar and vector potentials satisfy the conditions :

$$i\,\underset{\sim}{k}\cdot\underset{\sim}{j}_l(\underset{\sim}{k}\,\omega) - i\,\omega\,\rho(\underset{\sim}{k}\,\omega) = 0$$

2.17

$$i\,\underset{\sim}{k}\cdot\underset{\sim}{j}_t(\underset{\sim}{k}\,\omega) = 0$$

2.18

$$i\,\underset{\sim}{k}\cdot\underset{\sim}{A}_l(\underset{\sim}{k}\,\omega) - i\,\frac{\omega}{c}\,\phi(\underset{\sim}{k}\,\omega) = 0$$

2.19

$$i\,\underset{\sim}{k}\cdot\underset{\sim}{A}_t(\underset{\sim}{k}\,\omega) = 0$$

2.20

A brief glance at the previous expressions shows that the spectral amplitudes for the longitudinal fields (as well as that for the scalar potential) always have $1/\varepsilon(\omega)$ which is the factor responsible for the plasma waves, whereas those for the transverse fields always have the factor $1/k^2 - \frac{\omega^2}{c^2}\varepsilon(\omega)$ which is responsible for the appearance of radiation fields.

Expressions (2.9) to (2.16) are completely general. However, one should not be misled into believing that the previous equations will provide a neat separation of the two types of excitation for every possible source. Using these equations it is possible, for example, to treat highly idealized sources (e.g. sudden changes in either a charge density or in a solenoidal current, satisfying either equation (2.7) or (2.8), which do produce one definite type of excitation. But, a moving point charge as a source of both types of excitation, does not allow such a neat separation due to the existence of field components, not immediately related to either the radiation field or the plasma wave.

§ 3. - A COMMENT ON THE METHOD OF SEPARATION EMPLOYED.

Before applying the previous results to a particular case, it is worthwhile noting that the method we have used to separate the inhomogeneous coupled system is not the only possible one. The separation may also be carried out through direct comparison of the solutions of the system in Lorentz gauge and Coulomb gauge.

In this last gauge the system is given by :

$$\frac{\partial^2 \underset{\sim}{P}}{\partial t^2} + \omega_0^2 \underset{\sim}{P} = \frac{\eta^2}{4\pi}\left(-\operatorname{grad}\phi^{(c)} - \frac{1}{c}\frac{\partial \underset{\sim}{A}^{(c)}}{\partial t}\right) \qquad 3.1$$

$$\Box \underset{\sim}{A}^{(c)} = -\frac{4\pi}{c}\underset{\sim}{j} - \frac{4\pi}{c}\frac{\partial \underset{\sim}{P}}{\partial t} + \operatorname{grad}\frac{1}{c}\frac{\partial \phi^{(c)}}{\partial t} \qquad 3.2$$

$$\nabla^2 \phi^{(c)} = -4\pi\rho + 4\pi \operatorname{div} \underset{\sim}{P} \qquad 3.3$$

together with the equation of charge conservation and :

$$\nabla . \underset{\sim}{A}^{(c)} = 0 \qquad 3.4$$

where the superindex (C) indicates that we are now dealing with potentials in Coulomb gauge.

Still assuming that the current may be separated into :

$$\underset{\sim}{j} = \underset{\sim}{j}_l + \underset{\sim}{j}_t$$

and carrying out a Fourier analysis of the system we obtain :

$$\phi^{(c)}(\mathbf{k}\,\omega) = \frac{4\pi\,\rho(\mathbf{k}\,\omega)}{k^2\,\varepsilon(\omega)} \tag{3.5}$$

$$\mathbf{A}^{(c)}(\mathbf{k}\,\omega) = \frac{4\pi}{c}\,\frac{\mathbf{j}_t(\mathbf{k}\,\omega)}{\left(k^2 - \frac{\omega^2}{c^2}\,\varepsilon(\omega)\right)} \tag{3.6}$$

Therefore :

$$\mathbf{A}^{(c)}(\mathbf{k}\,\omega) = \mathbf{A}_t(\mathbf{k}\,\omega) \tag{3.7}$$

while it is easily proved that :

$$\begin{aligned} \mathbf{A}_l(\mathbf{k}\,\omega) &= \mathbf{A}(\mathbf{k}\,\omega) - \mathbf{A}^{(c)}(\mathbf{k}\,\omega) \\ &= \frac{4\pi}{c}\,\frac{\mathbf{j}_l(\mathbf{k}\,\omega)}{\left(k^2 - \frac{\omega^2}{c^2}\right)\varepsilon(\omega)} \end{aligned} \tag{3.8}$$

where $\mathbf{A}(\mathbf{k}\,\omega)$ is the spectral amplitude for the <u>total</u> vector potential in Lorentz gauge.

The expressions for the electric field, polarization and magnetic induction separated into transverse and longitudinal parts, remain obviously the same, being now :

$$\mathbf{E}_l(\mathbf{k}\,\omega) = -i\,\mathbf{k}\,\phi^{(c)}(\mathbf{k}\,\omega) = -i\,\mathbf{k}\,\phi(\mathbf{k}\,\omega) + i\,\frac{\omega}{c}\,\mathbf{A}_l(\mathbf{k}\,\omega) \tag{3.9}$$

$$\mathbf{E}_t(\mathbf{k}\,\omega) = i\,\frac{\omega}{c}\,\mathbf{A}^{(c)}(\mathbf{k}\,\omega) = i\,\frac{\omega}{c}\,\mathbf{A}_t(\mathbf{k}\,\omega) \tag{3.10}$$

$$\mathbf{B}(\mathbf{k}\,\omega) = i\,\mathbf{k}\times\mathbf{A}^{(c)}(\mathbf{k}\,\omega) = i\,\mathbf{k}\times\mathbf{A}_t(\mathbf{k}\,\omega) \tag{3.11}$$

and corresponding expressions for the polarization.

We conclude from these results that the separation into transverse and longitudinal parts by using vectorial properties of the partial waves, is essentially equivalent to separating from the total fields the instantaneous part. This will be seen more clearly further on.

§ 4. - MOVING CHARGE AS A SOURCE.

We shall consider now the case in which the excitation of the oscillators is produced by a moving charged particle.

We shall assume that the velocity of the particle remains constant, that is, we implicitly admit the existence of an external agent which will compensate the losses due to collisions, radiation reaction, etc...
The charge and current densities to be introduced in the inhomogeneous system are in this case :

$$\rho(\underset{\sim}{x},t) = e\,\delta(\underset{\sim}{x} - \underset{\sim}{V}t) \qquad 4.1$$

$$\underset{\sim}{j}(\underset{\sim}{x}\,t) = e\underset{\sim}{V}\,\delta(\underset{\sim}{x} - \underset{\sim}{V}t) \qquad 4.2$$

whose spectral amplitudes in a Fourier analysis are :

$$\rho(\underset{\sim}{k}\,\omega) = \frac{e}{2\pi}\,\delta(\underset{\sim}{k}.\underset{\sim}{V} - \omega) \qquad 4.3$$

$$\underset{\sim}{j}(\underset{\sim}{k}\,\omega) = \frac{e\underset{\sim}{V}}{2\pi}\,\delta(\underset{\sim}{k}.\underset{\sim}{V} - \omega) \qquad 4.4$$

and separating the current into longitudinal and transverse parts we obtain

$$\underset{\sim}{j}_{l}(\underset{\sim}{k}\,\omega) = \frac{\underset{\sim}{k}.\underset{\sim}{j}(\underset{\sim}{k}\,\omega)}{k}\,\frac{\underset{\sim}{k}}{k} = \frac{e}{2\pi}\,\frac{\underset{\sim}{k}.\underset{\sim}{V}}{k}\,\delta(\underset{\sim}{k}.\underset{\sim}{V} - \omega)\,\frac{\underset{\sim}{k}}{k} \qquad 4.5$$

$$\underset{\sim}{j}_t(\underset{\sim}{k}\,\omega) = \left(\frac{\underset{\sim}{k}}{k} \times \underset{\sim}{j}(\underset{\sim}{k}\,\omega)\right) \times \frac{\underset{\sim}{k}}{k} = \frac{e}{2\pi}\,\delta(\underset{\sim}{k}\cdot\underset{\sim}{V} - \omega)\left\{\underset{\sim}{V} - (\underset{\sim}{k}\cdot\underset{\sim}{V})\,\frac{\underset{\sim}{k}}{k^2}\right\} \qquad 4.6$$

Expressions 2.9 to 2.16 together with 4.3, 4.5 and 4.6 give the separated spectral amplitudes of the fields created by the moving charged particle.

§ 5. - LONGITUDINAL FIELDS OF THE CHARGED PARTICLE.

We shall assume for simplicity that the particle is moving in the positive direction of the z axis and that at $t = 0$ the particle crosses the origin of coordinates. Due to the symmetry of the problem the integrations are carried out in cylindrical coordinates (ρ, z, φ'). Therefore the Fourier integrals for any one of the fields will be given by :

$$\underset{\sim}{F}(\underset{\sim}{r}t) = \frac{1}{(2\pi)^2}\iiiint_{-\infty}^{\infty} \underset{\sim}{F}(\underset{\sim}{k}\omega)\, e^{i(\underset{\sim}{k}\cdot\underset{\sim}{r} - \omega t)}\, d^3k\, d\omega \longrightarrow$$

$$\longrightarrow \frac{1}{(2\pi)^2}\int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk_z \int_0^{\infty} k_\rho\, dk_\rho\, \underset{\sim}{F}(\underset{\sim}{k}\omega)\, e^{i(k_\rho \rho\cos\varphi + k_z z - \omega t)} \tag{5.1}$$

where for reasons of convenience we have assumed the position vector of the observation point contained in the plane (X, Z), i.e. $\varphi' = 0$.

All integrals present singularities. We shall adopt as general method of dealing with the singularities :

1°) Assume that the oscillators have a small absorption term. The equation of motion of the oscillators would be in this case :

$$\ddot{\underset{\sim}{P}} + \gamma \dot{\underset{\sim}{P}} + \omega_0^2 \underset{\sim}{P} = \frac{\eta^2}{4\pi}\left(-\operatorname{grad}\phi - \frac{1}{c}\frac{\partial \underset{\sim}{A}}{\partial t}\right) \tag{5.2}$$

and

$$\varepsilon(\omega) = 1 + \frac{\eta^2}{\omega_0^2 - \omega^2 - i\omega\gamma} \simeq \frac{\left(\omega + i\frac{\gamma}{2}\right)^2 - \omega_p^2}{\left(\omega + i\frac{\gamma}{2}\right)^2 - \omega_0^2} = $$

$$= \frac{(\omega + i\epsilon)^2 - \omega_p^2}{(\omega + i\epsilon)^2 - \omega_0^2} \qquad 5.3$$

where we have neglected quadratic terms in γ, and for brevity have written $\frac{\gamma}{2} = \epsilon$, ϵ a positive infinitesimal.

2°) Once performed the integration, find the limit for ϵ tending to zero. This method is justified by the well known result that a complex dielectric constant with the sign corresponding to a damping of the oscillators in time will assure the correct causal behaviour of the solution. Taking these considerations into account we find after integration :

$\underline{z - vt > 0}$

$$\phi = \frac{e}{\beta} \int_0^\infty \frac{\left(k_\rho^2 + \frac{\omega_0^2}{v^2}\beta^2\right) J_0(k_\rho \rho)}{k_\rho^2 + \frac{\omega_p^2}{v^2}\beta^2} e^{-k_\rho \frac{z - vt}{\beta}} \, dk_\rho \qquad 5.4$$

$$A_{l_z} = -\frac{ec}{v} \int_0^\infty \frac{\left(k_\rho^2 + \frac{\omega_0^2}{v^2}\right) J_0(k_\rho \rho)}{k_\rho^2 + \frac{\omega_p^2}{v^2}} e^{-k(z - vt)} \, dk_\rho \; +$$

$$+ \frac{ec}{v} \frac{1}{\beta} \int_0^\infty \frac{\left(k_\rho^2 + \frac{\omega_0^2}{v^2}\beta^2\right) J_0(k_\rho \rho) \, e^{-k_\rho \frac{z - vt}{\beta}}}{k_\rho^2 + \frac{\omega_p^2}{v^2}\beta^2} \, dk_\rho$$

$$5.5$$

$$A_{l\rho} = -\frac{ec}{v}\int_0^\infty \frac{\left(k_\rho^2+\frac{\omega_0^2}{v^2}\right) J_1(k_\rho \rho)}{k_\rho^2+\frac{\omega_p^2}{v^2}}\, e^{-k_\rho(z-vt)}\, dk_\rho \; +$$

$$+\frac{ec}{v}\int_0^\infty \frac{\left(k_\rho^2+\frac{\omega_0^2}{v^2}\beta^2\right) J_1(k_\rho \rho)}{k_\rho^2+\frac{\omega_p^2}{v^2}\beta^2}\, e^{-k_\rho \frac{z-vt}{\beta}}\, dk_\rho \qquad 5.6$$

$$P_{lz} = \frac{e\eta^2}{4\pi v^2}\int_0^\infty \frac{k_\rho J_0(k_\rho \rho)}{k_\rho^2+\frac{\omega_p^2}{v^2}}\, e^{-k_\rho(z-vt)}\, dk_\rho \qquad 5.7$$

$$P_{l\rho} = \frac{e\eta^2}{4\pi v^2}\int_0^\infty \frac{k_\rho J_1(k_\rho \rho)}{k_\rho^2+\frac{\omega_p^2}{v^2}}\, e^{-k_\rho(z-vt)}\, dk_\rho \qquad 5.8$$

$$E_{lz} = e\int_0^\infty \frac{k_\rho\left(k_\rho^2+\frac{\omega_0^2}{v^2}\right) J_0(k_\rho \rho)}{k_\rho^2+\frac{\omega_p^2}{v^2}}\, e^{-k_\rho(z-vt)}\, dk_\rho \qquad 5.9$$

$$E_{l\rho} = e\int_0^\infty \frac{k_\rho\left(k_\rho^2+\frac{\omega_0^2}{v^2}\right) J_1(k_\rho \rho)}{k_\rho^2+\frac{\omega_p^2}{v^2}}\, e^{-k_\rho(z-vt)}\, dk_\rho \qquad 5.10$$

<u>$z-vt<0$</u>

$$\phi = 2e\frac{\eta^2}{v^2}\frac{\operatorname{sen}\frac{\omega_p}{v}(z-vt)}{\frac{\omega_p}{v}} K_0\!\left(\frac{\omega_p}{v}\beta\rho\right) + \frac{e}{\beta}\int_0^\infty \frac{\left(k_\rho^2+\frac{\omega_0^2}{v^2}\beta^2\right) J_0(k_\rho \rho)}{k_\rho^2+\frac{\omega_p^2}{v^2}\beta^2}\, e^{-k_\rho \frac{|z-vt|}{\beta}}\, dk_\rho \qquad 5.11$$

$$A_{l_z} = \frac{2e\eta^2 c}{v^3} \, \frac{\operatorname{sen} \frac{\omega_p}{v}(z-vt)}{\frac{\omega_p}{v}} \left\{ K_0\left(\frac{\omega_p}{v}\beta\rho\right) - K_0\left(\frac{\omega_p}{v}\rho\right) \right\} -$$

$$- \frac{ec}{v} \int_0^\infty \frac{\left(k_\rho^2 + \frac{\omega_0^2}{v^2}\right) J_0(k_\rho \rho)\, e^{-k_\rho |z-vt|}}{k_\rho^2 + \frac{\omega_p^2}{v^2}} \, dk_\rho + \frac{ec}{v\beta} \int_0^\infty \frac{\left(k_\rho^2 + \frac{\omega_0^2}{v^2}\beta^2\right) J_0(k_\rho \rho)\, e^{-k_\rho \frac{|z-vt|}{\beta}}}{k_\rho^2 + \frac{\omega_p^2}{v^2}\beta^2} \, dk_\rho \tag{5.12}$$

$$A_{l_\rho} = \frac{2e\eta^2 c}{v^3} \, \frac{\cos \frac{\omega_p}{v}(z-vt)}{\frac{\omega_p^2}{v^2}} \left\{ \frac{\omega_p}{v}\beta\, K_1\left(\frac{\omega_p}{v}\beta\rho\right) - \frac{\omega_p}{v} K_1\left(\frac{\omega_p}{v}\rho\right) \right\} +$$

$$+ \frac{ec}{v} \int_0^\infty \frac{\left(k_\rho^2 + \frac{\omega_0^2}{v^2}\right) J_1(k_\rho \rho)}{k_\rho^2 + \frac{\omega_p^2}{v^2}} e^{-k_\rho |z-vt|} \, dk_\rho - \frac{ec}{v} \int_0^\infty \frac{\left(k_\rho^2 + \frac{\omega_0^2}{v^2}\beta^2\right) J_1(k_\rho \rho)}{k_\rho^2 + \frac{\omega_p^2}{v^2}\beta^2} e^{-k_\rho \frac{|z-vt|}{\beta}} \, dk_\rho \tag{5.13}$$

$$P_{l_z} = \frac{e\eta^2}{2\pi v^2} \cos \frac{\omega_p}{v}(z-vt)\, K_0\left(\frac{\omega_p}{v}\rho\right) - \frac{e\eta^2}{4\pi v^2} \int_0^\infty \frac{k_\rho J_0(k_\rho \rho)\, e^{-k_\rho |z-vt|}}{k_\rho^2 + \frac{\omega_p^2}{v^2}} \, dk_\rho \tag{5.14}$$

$$P_{l_\rho} = -\frac{e\eta^2}{2\pi v^2} \operatorname{sen} \frac{\omega_p}{v}(z-vt)\, K_1\left(\frac{\omega_p}{v}\rho\right) + \frac{e\eta^2}{4\pi v^2} \int_0^\infty \frac{k_\rho J_1(k_\rho \rho)\, e^{-k_\rho |z-vt|}}{k_\rho^2 + \frac{\omega_p^2}{v^2}} \, dk_\rho \tag{5.15}$$

$$E_{l_z} = -\frac{2e\eta^2}{V^2}\cos\frac{\omega_p}{V}(z-Vt)\,K_0\left(\frac{\omega_p}{V}\rho\right) - e\int_0^\infty \frac{k_\rho\left(k_\rho^2+\frac{\omega_0^2}{V^2}\right)J_0(k_\rho\rho)\,e^{-k_\rho|z-Vt|}}{k_\rho^2+\frac{\omega_p^2}{V^2}}\,dk_\rho$$

5.16

$$E_{l_\rho} = \frac{2e\eta^2}{V^2}\,\mathrm{sen}\,\frac{\omega_p}{V}(z-Vt)\,K_1\left(\frac{\omega_p}{V}\rho\right) + e\int_0^\infty \frac{k_\rho\left(k_\rho^2+\frac{\omega_0^2}{V^2}\right)J_1(k_\rho\rho)\,e^{-k|z-Vt|}}{k_\rho^2+\frac{\omega_p^2}{V^2}}\,dk_\rho$$

5.17

where:

$J_\nu(x)$ and $K_\nu(x)$ are the Bessel and modified Bessel functions of order ν, and:

$$\beta = \sqrt{1-V^2/c^2}$$

§ 6. - PHYSICAL INTERPRETATION OF THE LONGITUDINAL SOLUTIONS. LIMITING CASES, ASYMPTOTIC BEHAVIOUR.

The longitudinal solutions are divided in two groups, i.e. for $z-vt \gtrless 0$, corresponding to the fields in front and behind the particle. It can be easily proved that the given solutions satisfy the Lorentz condition.

The integrals appearing in expressions 5.4 to 5.17 cannot be evaluated in closed form. Nevertheless, it is possible to understand their physical meaning by studying the limiting values in certain special cases. For instance :

Scalar potential ϕ

$z-vt>0$ From equation (5.4) we obtain :

$$\lim_{v\to 0} \phi(\rho z t) = \frac{e}{\varepsilon_{st}} \frac{1}{\sqrt{\rho^2+z^2}} \qquad 6.1$$

where : $\varepsilon_{st} = \frac{\omega_p^2}{\omega_0^2}$, is the static dielectric constant as given by our model.

And :

$$\lim_{\eta\to 0} \phi(\rho z t) = \frac{e}{\sqrt{1-v^2/c^2}} \frac{1}{\sqrt{\rho^2+\left(\frac{z-vt}{\sqrt{1-v^2/c^2}}\right)^2}} = \frac{1}{\sqrt{1-v^2/c^2}} \phi'(\rho, z') \qquad 6.2$$

In this limit ϕ is the Lorentz transform of the potential measured in a reference system at rest with the particle, i.e. the Lienard Wiechert scalar potential.

$z - vt < 0$

The limits of the second term of the solution (5.11) are similar to those of the previous case.

The first term of the solution may be written :

$$\phi_{osc}(\rho z t) = 2e\frac{\eta^2}{v}\frac{\operatorname{sen}(k_p z - \omega_p t)}{\omega_p} K_0\left(k_p\sqrt{1 - v^2/c^2}\,\rho\right) \qquad 6.3a$$

where :
$$k_p = \frac{\omega_p}{v} \qquad 6.3b$$

This contribution to the scalar potential is a cylindrical wave propagating in the z direction, with an amplitude which depends on the density of oscillators and the velocity of the particle, its physical meaning will be clear when we analyze the analogous term of the longitudinal polarization and the electric field.

Nevertheless, we can already see that this part of the solution corresponds to a wave constructed by the particle with a propagation vector $\underset{\sim}{k}_p$, which is a function of the density of oscillators and the velocity of the particle.

It is easily seen that :

$$\lim_{\eta \to 0} \phi_{osc} = 0 \qquad 6.4$$

On the other hand, taking into account the asymptotic behaviour of the modified Bessel functions :

$$K_\nu(x) \simeq \sqrt{\frac{\pi}{2x}}\, e^{-x} \quad , \text{ for } x \gg 1 \qquad 6.5$$

we have, given a fixed distance ρ from the trajectory :

$$\lim_{V \to 0} \phi_{osc} = 0 \qquad 6.6$$

In all rigorousness the limit $V \to 0$ means

$$k_p \to \infty \qquad i.e. \; \lambda_p \to 0$$

and this takes us beyond the range of validity of the model we have used. Since the velocity of the particle cannot be higher than the velocity of light in vacuum $\underline{c}$, there must also be a mihimum value for k_p. Therefore the possible values of k_p lie in the range :

$$\frac{\omega_p}{c} \ll k_{p_{min}} \longleftrightarrow k_{p_{max.}}$$

such that $\lambda_{min} \gg$ atomic dimensions.

This, of course, assuming that within that range of velocities of the particle it is possible to speak independently of the longitudinal fields. However, the conclusion on the possible values of k_p remains the same in the case of the total fields i.e. in the presence of radiation fields.

<u>Longitudinal vector potential</u>.

From equations (5.5), (5.6), (5.12), (5.13), it can be easily seen that :

$$\lim_{V \to 0} \underset{\sim}{A}_l = 0 \qquad 6.7$$

On the other hand, the limiting value of $\underset{\sim}{A}_l$ when the density of oscillators tends to zero $(\eta \to 0)$ is not what at first sight might be expected, i.e. it is not the Lorentz transform of the scalar potential in a reference system at rest with the particle. What does have the correct transformation properties in the limit $\eta \to 0$, is the total vector potential. It can be shown using the expressions already given that :

$$\lim_{\eta\to 0} \underset{\sim}{A}(\underset{\sim}{k}\omega) = \lim_{\eta\to 0} \underset{\sim}{A}_{\ell}(\underset{\sim}{k}\omega) + \lim_{\eta\to 0} \underset{\sim}{A}_{t}(\underset{\sim}{k}\omega) = \frac{\underset{\sim}{V}}{c} \lim_{\eta\to 0} \phi(\underset{\sim}{k}\omega) \tag{6.8}$$

and integrating this last expression we do obtain the vector potential corresponding to a charge moving in vacuum, i.e. the L.W. potential.

Longitudinal Polarization $\underset{\sim}{P}_{\ell}$.-

To analyze the longitudinal polarization, equations (5.7), (5.8), (5.14), (5.15), we separate it into two parts :

$$\underset{\sim}{P}_{\ell} = \underset{\sim}{P}_{\ell_{osc}} + \underset{\sim}{P}_{\ell_{exp}} \tag{6.9}$$

where $\underset{\sim}{P}_{\ell_{osc}}$ indicates the oscillating term, and $\underset{\sim}{P}_{\ell_{exp}}$ the contribution to the polarization due to the integral. It is easily seen from the solutions :

$$\lim_{\eta\to 0} \underset{\sim}{P}_{\ell_{osc}} = 0 \qquad \lim_{\eta\to 0} \underset{\sim}{P}_{\ell_{exp}} = 0 \tag{6.10}$$

While :

$$\lim_{V\to 0} P_{\ell_{z\,exp}} = \frac{e}{4\pi} \frac{\varepsilon_{st}-1}{\varepsilon_{st}} \frac{z}{\left[z^2+\rho^2\right]^{3/2}} \tag{6.11}$$

$$\lim_{V\to 0} P_{\ell_{\rho\,exp}} = \frac{e}{4\pi} \frac{\varepsilon_{st}-1}{\varepsilon_{st}} \frac{\rho}{\left[z^2+\rho^2\right]^{3/2}} \tag{6.12}$$

where

$$\frac{\varepsilon_{st} - 1}{\varepsilon_{st}} = \frac{\gamma^2}{\omega_p^2} \qquad 6.13$$

This result is valid for $z \gtrless 0$.

In this limit the oscillators behave like a continuous medium of dielectric constant ε_{st}. This last result may also be written :

$$\lim_{V \to 0} \underset{\sim}{P}_{l\,exp} = \frac{1}{4\pi} \frac{\varepsilon_{st} - 1}{\varepsilon_{st}} \underset{\sim}{D} \qquad 6.14$$

where $\vec{D}$ is the displacement vector of a charge at rest in such a medium. Therefore, in the limit $V \to 0$ the contribution to the polarization due to the integrals is a contribution of Coulomb type, equal to the polarization created by a real charge $\underline{e}$ in a medium of $\varepsilon = \varepsilon_{st}$.

For large values of the arguments of the modified Bessel functions, the oscillating term of the solutions may be written as follows :

$$P_{l_{z\,osc}} \simeq \frac{e\gamma^2}{2\pi V^2} \cos\left(k_p z - \omega_p t\right) \sqrt{\frac{\pi}{2 k_p \rho}}\, e^{-k_p \rho} \qquad 6.15$$

$$P_{l_{\rho\,osc}} \simeq -\frac{e\gamma^2}{2\pi V^2} \sin\left(k_p z - \omega_p t\right) \sqrt{\frac{\pi}{2 k_p \rho}}\, e^{-k_p \rho} \qquad 6.16$$

where k_p has already been defined and $k_p \rho \gg 1$.

This part of the solution is a real polarization wave with an amplitude which, in this approximation, decreases exponentially with increasing distance from the trajectory.

The mean range of thoses waves is given by :

$$\rho_{mean} = \frac{V}{\omega_p} \qquad 6.17$$

i.e. proportional to the velocity of the particle, and decreasing (through ω_p), with an increasing density of oscillators. These waves have a group velocity :

$$g_p = \frac{\partial \omega_p}{\partial k_p} = V \qquad 6.18$$

i.e. equal to the velocity of the particle. It can also be seen from (6.15) and (6.16) that :

$$\lim_{V \to 0} \underset{\sim}{P}_{l\,osc} = 0 \qquad 6.19$$

For reasons which will become apparent further on, we shall call this part of the solutions the <u>Plasma waves</u>.

The results for the electric field are completely analogous to those obtained for the polarization. Thus, from equations (5.9), (5.10), (5.16), (5.17), writing :

$$\underset{\sim}{E}_l = \underset{\sim}{E}_{l\,osc} + \underset{\sim}{E}_{l\,exp} \qquad 6.20$$

we find that, in the limit $\eta \to 0$, $\underset{\sim}{E}_{l\,exp}$ becomes the Coulomb field in vacuum of a moving particle in the approximation $V \ll C$, while in the limit $V \to 0$ we obtain the Coulomb field of a particle at rest in a medium of $\varepsilon = \varepsilon_{st}$. On the other hand, the oscillating term vanishes in both limits.

By straightforward derivation taking due account of the equations satisfied by the Bessel functions J_ν and modified Bessel functions K_ν it can be shown that the curl of $\underset{\sim}{P}_\ell(\underset{\sim}{E}_\ell)$ as given by equations (5.7), (5.8), (5.14), (5.15) [(5.9), (5.10), (5.16), (5.17)] vanishes identically over all space. The solutions have singularities at the position of the particle for $\underset{\sim}{P}_{\ell\,exp}(\underset{\sim}{E}_{\ell\,exp})$, and on the z axis behind the particle for $\underset{\sim}{P}_{\ell\,nc}(\underset{\sim}{E}_{\ell\,nc})$. Therefore the solutions we have obtained are pure longitudinal solutions.

LONGITUDINAL DISPLACEMENT VECTOR.-

We shall now define a longitudinal displacement vector by the usual expression :

$$\underset{\sim}{D}_\ell(\underset{\sim}{r}\,t) = \underset{\sim}{E}_\ell(\underset{\sim}{r}\,t) + 4\pi\,\underset{\sim}{P}_\ell(\underset{\sim}{r}\,t) \qquad 6.21$$

or as a function of the spectral amplitudes :

$$\underset{\sim}{D}_\ell(\underset{\sim}{k}\,\omega) = \underset{\sim}{E}_\ell(\underset{\sim}{k}\,\omega) + 4\pi\,\underset{\sim}{P}_\ell(\underset{\sim}{k}\,\omega) \qquad 6.22$$

Using expressions (2.12), (2.14) and (4.3), we find :

$$\underset{\sim}{D}_\ell(\underset{\sim}{k}\,\omega) = -\frac{2e\,\delta(\underset{\sim}{k}.\underset{\sim}{v}-\omega)}{k^2}\,i\,\underset{\sim}{k} \qquad 6.23$$

This spectral amplitude can be integrated exactly with the following result:

$$D_{\ell z} = e\frac{z-vt}{\left[(z-vt)^2+\rho^2\right]^{3/2}} \qquad 6.24$$

$$D_{\ell\rho} = e\frac{\rho}{\left[(z-vt)^2+\rho^2\right]^{3/2}} \qquad 6.25$$

Therefore the longitudinal displacement vector we have defined is such that it satisfies the equation :

$$\nabla . \underset{\sim}{D}_\ell = 4\pi e\, \delta(\underset{\sim}{r} - \underset{\sim}{v}t) = \frac{2\pi e}{\rho}\, \delta(\rho)\, \delta(z - vt) \qquad 6.26$$

and consequently may be defined in terms of a potential :

$$\underset{\sim}{D}_\ell = - \operatorname{grad} \Phi_\ell \qquad 6.27$$

where :

$$\Phi_\ell = \frac{e}{\sqrt{(z - vt)^2 + \rho^2}} \qquad 6.28$$

From a physical point of view $\underset{\sim}{D}_\ell$ corresponds to the instantaneous Coulomb field carried convectively by the real charge <u>e</u>.

The potential Φ_ℓ <u>does not</u> correspond to $\lim_{\eta \to 0} \phi$ (6.2) which we have already analyzed, but to that one which results from integrating the spectral amplitude of the potential in Coulomb gauge (3.5) in the limit $\eta \to 0$ .This may be easily seen by direct comparison of (3.5) and (6.23).

From the previous results it is quite clear that the longitudinal displacement vector <u>does not</u> depend on the properties of the medium and what is more important, it does not present the cylindrical wave part.

A brief glance at the solutions for $\underset{\sim}{P}_\ell$ and $\underset{\sim}{E}_\ell$ (5.7 to 5.10 and 5.14 to 5.17) shows that, from a formal point of view we may also write :

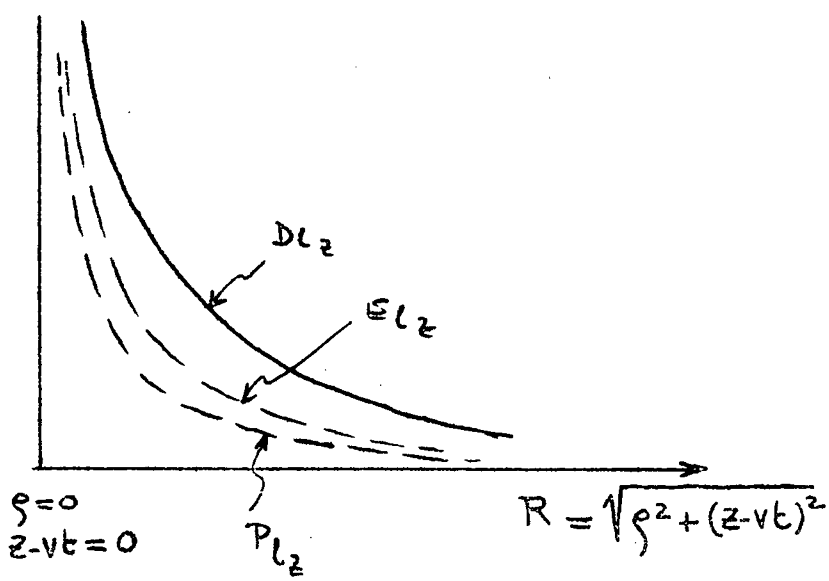
$$\underset{\sim}{D}_\ell = \underset{\sim}{D}_{\ell\, osc} + \underset{\sim}{D}_{\ell\, exp} \qquad 6.29$$

where : $\underset{\sim}{D}_{l\,exp} = \underset{\sim}{E}_{l\,exp} + 4\pi \underset{\sim}{P}_{l\,exp} = -\,\mathrm{grad}\,\Phi_l$ 6.30

$$\underset{\sim}{D}_{l\,osc} = \underset{\sim}{E}_{l\,osc} + 4\pi \underset{\sim}{P}_{l\,osc} = 0 \qquad 6.31$$

This last expression is in agreement with the fact that the cylindrical waves must be identified with <u>oscillator plasma waves</u>, and therefore have a null displacement vector at a frequency corresponding to a zero of $\mathcal{E}(\omega)$ Moreover, the fact that the displacement vector associated to this part of the solutions vahishes, implies that the sources corresponding to $\underset{\sim}{E}_{l\,osc}$, $\underset{\sim}{P}_{l\,osc}$ (i.e. $\nabla.\underset{\sim}{E}_{l\,osc}$, $\nabla.\underset{\sim}{P}_{l\,osc}$) are not in the real charge. This is precisely the characteristic feature of the plasma waves. The same conclusion has been reached by A. Bohr [3], although by a different line of reasoning.

It is interesting to notice that despite the fact that we cannot carry out the integrations for $\underset{\sim}{E}_{l\,exp}$ and $\underset{\sim}{P}_{l\,exp}$ (5.7 to 5.10 and 5.14 to 5.17), the displacement vector $\underset{\sim}{D}_l$ gives us the means to study the general behaviour of these contributions and their relative importance. Since the expression for $\underset{\sim}{D}_l$ is known exactly, it can be seen from the analysis of the solutions that the approximate behaviour of that part of the longitudinal polarization and the electric field depending on the integrals, is the following :



The relative importance of $\underset{\sim}{P}_{l\,exp}$ and $\underset{\sim}{E}_{l\,exp}$ is a function of the proper frequency of the oscillators and their density.

In summary we may say :

The longitudinal polarization and electric fields created by a moving charge in a medium represented by oscillators, have contributions of two different types :

a) a symmetrical field carried convectively by the particle, of Coulomb nature. The source for these fields is in the particle. The polarization associated to this contribution acts producing a <u>screening</u> of the real charge.

b) a cylindrical wave, defined in all the semispace behind the particle bound by the plane $Z - Vt = 0$, with a frequency equal to the plasma frequency ω_p , and a group velocity equal to the velocity of the particle. This wave is an excitation constructed by the particle which tends to disappear when $V \to 0$, although strictly speaking a proper understanding of what happens in this limit requires a quantum treatment. The source for this contribution is not in the real charge. This part of the fields corresponds to the longitudinal part predicted by A. Bohr (3), although in all rigorousness it cannot be said to be left "in the wake" of the particle. This is an excitation which, as already mentioned, appears in principle in a whole semispace although its importance decreases exponentially with the distance ρ from the path of the particle.

It is important to notice that the screening and the plasma wave are not independent of each other. The intimate relation between the two fields can be understood if we compare the present results with the response of an oscillator to the force exerted by a moving charged particle. The details of this calculation will be published in a separate paper (14).

§ 7. - TRANSVERSE SOLUTIONS.

From the analysis of 2.11, 2.13, 2.15, 2.16 and 4.4, 4.5, 4.6, it becomes obvious that the result of integrating the transverse spectral amplitudes cannot be anything else than the difference between the solutions of the total inhomogeneous coupled system (2.1, 2.2, 2.3) and the longitudinal fields that we have already found. The integration is carried out in cylindrical coordinates. The singularities appearing in the integrals are dealt with by assuming as in the case of the longitudinal fields the existence of an absorption term and therefore replacing $\mathcal{E}(\omega)$ in the spectral amplitudes by expression 5.3, and after the integration finding the limit when the absorption tends to zero. We obtain :

$$E_{t\rho} = \frac{e}{\pi V}\int_{-\infty}^{\infty} d\omega \, \frac{1}{\mathcal{E}(\omega)} \, \Lambda(\omega) \, K_1(\Lambda(\omega)\rho) \, e^{i\omega\left(\frac{z}{V}-t\right)} - E_{\ell\rho} \qquad 7.1$$

where :

$$E_{\ell\rho} = \frac{e}{\pi V}\int_{-\infty}^{\infty} d\omega \, \frac{1}{\mathcal{E}(\omega)} \left|\frac{\omega}{V}\right| K_1\left(\left|\frac{\omega}{V}\right|\rho\right) e^{i\omega\left(\frac{z}{V}-t\right)} \qquad 7.1\,a$$

$$E_{t_z} = \frac{ei}{\pi V^2}\int_{-\infty}^{\infty} d\omega \left(\frac{V^2}{C^2} - \frac{1}{\mathcal{E}(\omega)}\right) \omega \, K_0(\Lambda(\omega)\rho) \, e^{i\omega\left(\frac{z}{V}-t\right)} - E_{\ell_z} \qquad 7.2$$

where :

$$E_{l_z} = -\frac{ei}{\pi V^2}\int_{-\infty}^{\infty} d\omega \, \frac{\omega}{\varepsilon(\omega)} \, K_0\left(\left|\frac{\omega}{V}\right|\rho\right) e^{i\omega\left(\frac{z}{V}-t\right)} \tag{7.2 a}$$

$$B_\varphi = H_\varphi = \frac{e}{\pi c}\int_{-\infty}^{\infty} d\omega \, \Lambda(\omega) \, K_1(\Lambda(\omega)\rho) \, e^{i\omega\left(\frac{z}{V}-t\right)} \tag{7.3}$$

$$P_{t_\rho} = -\frac{e}{(2\pi)^2 V}\int_{-\infty}^{\infty} d\omega \, \frac{\eta^2}{\omega^2-\omega_p^2} \, \Lambda(\omega) \, K_1(\Lambda(\omega)\rho) \, e^{i\omega\left(\frac{z}{V}-t\right)} - P_{l_\rho} \tag{7.4}$$

$$P_{t_z} = \frac{ie}{(2\pi)^2 V^2}\int_{-\infty}^{\infty} d\omega \, \frac{\eta^2}{\omega^2-\omega_p^2} \left\{1-\frac{V^2}{c^2}\varepsilon(\omega)\right\} \omega \, K_0(\Lambda(\omega)\rho) \, e^{i\omega\left(\frac{z}{V}-t\right)} - P_{l_z} \tag{7.5}$$

$$D_{t_\rho} = \frac{e}{\pi V}\int_{-\infty}^{\infty} d\omega \, \Lambda(\omega) \, K_1(\Lambda(\omega)\rho) \, e^{i\omega\left(\frac{z}{V}-t\right)} - \frac{e\rho}{\left[(z-Vt)^2+\rho^2\right]^{3/2}} \tag{7.6}$$

$$D_{t_z} = -\frac{e i}{\pi V^2}\int_{-\infty}^{\infty} d\omega \left(1 - \frac{V^2}{c^2}\mathcal{E}(\omega)\right) \cdot \omega \, K_0(\Lambda(\omega)\rho)\, e^{i\omega\left(\frac{z}{v} - t\right)} -$$

$$- \frac{e\,(z - vt)}{\left[(z - vt)^2 + \rho^2\right]^{3/2}} \qquad 7.7$$

Where K_ν are the modified Bessel functions and:

$$\Lambda^2(\omega) = \frac{\omega^2}{V^2} - \frac{\omega^2}{c^2}\mathcal{E}(\omega) \qquad 7.8$$

The transverse solutions, except for the magnetic induction, are given by two terms :

- the first term corresponds to the <u>total fields</u> whose expressions for $\vec{E}$, $\vec{H}$ ($\vec{B}$) and $\vec{P}$ were already given by Fermi [(2)].
- the second term gives the longitudinal fields that we have already found.

The total fields are given as integrals over all the frequency spectrum. The function $\Lambda^2(\omega)$ requires some discussion. For low velocities of the particle $\left(V < c/\sqrt{\mathcal{E}_{st}}\right)$ there may be a zone from $\omega = 0$ to a certain $\omega = \omega_c$, where $\Lambda^2(\omega) \geqslant 0$.
This frequency range diminishes as the velocity of the particle increases. From $\omega_c \to \omega_0$, $\Lambda^2(\omega)$ varies in the range $0 \geqslant \Lambda^2(\omega) \geqslant -\infty$
From $\omega_0 \to \omega = \infty$, it remains always positive. We define the critical frequency ω_c as the frequency at which, in this range of velocities, $\Lambda^2(\omega)$ changes sign and remains finite, i.e. :

$$\Lambda^2(\omega_c) = 0$$

As is well known this is precisely the frequency at which Cherenkov radiation begins, since for :

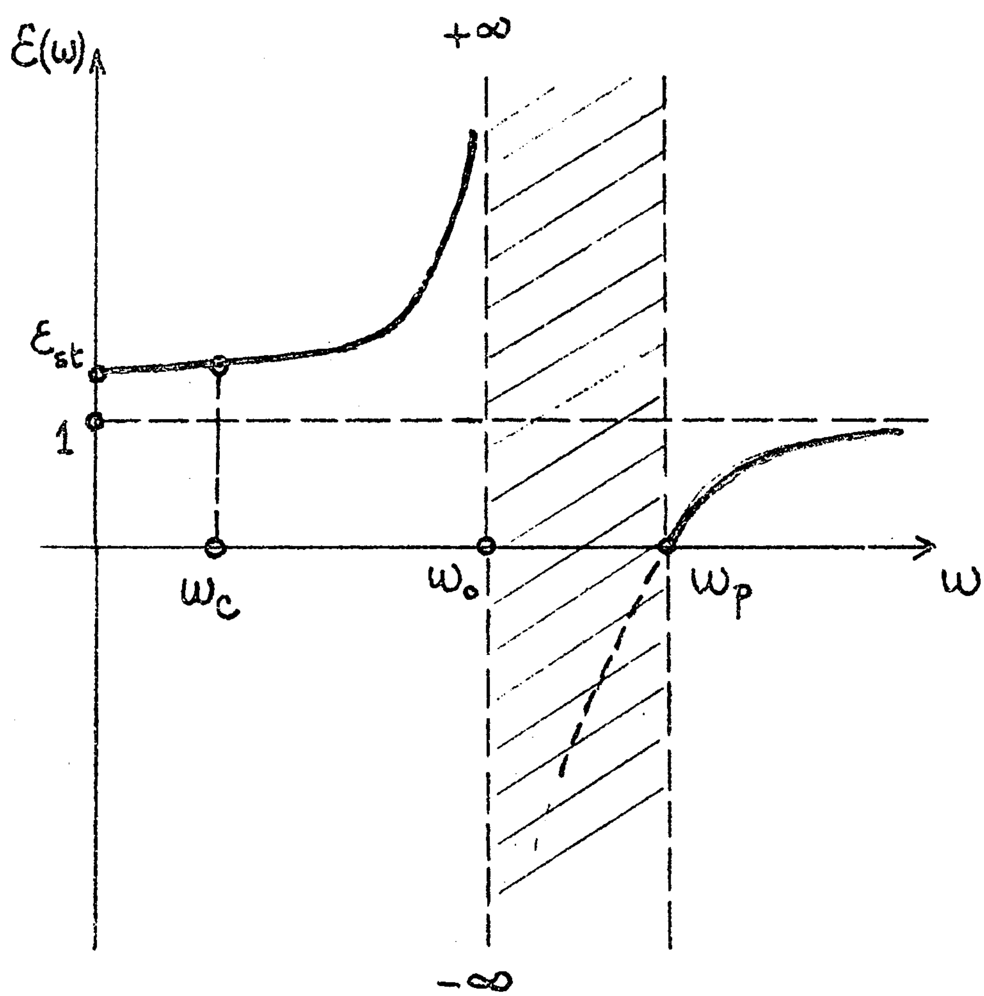
$$\omega > \omega_c \ , \quad \frac{\sqrt{\varepsilon(\omega)}}{c} = \frac{1}{u_\omega} > \frac{1}{V}$$

i.e. the phase velocity of a light wave u_ω is smaller than the velocity of the particle.

For the case we are considering of oscillators of only one resonance frequency this condition may only be fulfilled in the range :

$$\omega_c < \omega < \omega_0$$

This is clearly seen in the approximate diagram of $\varepsilon(\omega)$:



Taking into account the expression for $\mathcal{E}(\omega)$ we may also write ω_c as :

$$\omega_c^2 = \frac{\omega_o^2\left(1 - \frac{V^2}{c^2}\mathcal{E}_{st}\right)}{1 - \frac{V^2}{c^2}} \tag{7.9}$$

and $\Lambda(\omega)$ as :

$$\Lambda(\omega) = \sqrt{\frac{\omega^2}{V^2}\left\{\frac{\omega^2 - \omega_c^2}{\omega^2 - \omega_o^2}\right\}(1 - V^2/c^2)} \tag{7.10}$$

It should be noticed that for sufficiently high velocities $\left(V > \frac{c}{\sqrt{\mathcal{E}_{st}}}\right)$, ω_c may become pure imaginary.

For the moment and unless otherwise stated we shall assume that the velocity of the particle is such that ω_c is real.

If we assume the existence of a small absorption term in the oscillators, i.e. $\mathcal{E}(\omega)$ is replaced by (5.3), $\Lambda(\omega)$ may also be written :

$$\Lambda(\omega,\gamma) = \sqrt{\frac{\omega^2}{V^2}\left\{\frac{\omega^2 + i\omega\gamma - \omega_c^2}{\omega^2 + i\omega\gamma - \omega_o^2}\right\}(1 - V^2/c^2)}$$

$$\simeq \sqrt{\frac{\omega^2}{V^2}\left\{\frac{(\omega + i\epsilon)^2 - \omega_c^2}{(\omega + i\epsilon)^2 - \omega_o^2}\right\}(1 - V^2/c^2)} = \Lambda(\omega,\epsilon) \tag{7.11}$$

where we have neglected quadratic terms in γ and have replaced : $\gamma = 2\epsilon$, ϵ positive infinitesimal. ω_c remains unchanged. In what follows Λ will be used in either one of the three forms given (i.e. 7.8, 7.10 or 7.11).

It becomes apparent while integrating the k_ξ variable, that expressions 7.1 to 7.7 are valid provided Re $\Lambda(\omega,\epsilon) > 0$ in all the range of frequencies. In order to fulfill this condition we must take $\Lambda(\omega, \epsilon)$ in the fourth quadrant for positive frequencies, while for the negative frequencies it follows from (7.11) :

$$\Lambda(-\omega,\epsilon) = \Lambda^*(\omega,\epsilon) \qquad 7.12$$

§ 8. - PROPERTIES OF THE TRANSVERSE AND TOTAL FIELDS. ASYMPTOTIC BEHAVIOUR.

While in the case of the longitudinal solutions it has been possible to separate completely their contribution, such a neat separation cannot be effected for the transverse fields. This is due partly to the physical implications of the separation as shown in paragraph (3), partly to the nature of the source. Nevertheless, even if not as satisfactorily as in the case of the longitudinal fields, the difference between the total fields and longitudinal fields do give the transverse contribution.

One may now pose the question of whether it is possible to identify the longitudinal part as originating in a definite zone of the frequency spectrum or, in other words, carry out a classification of the spectrum in terms of longitudinal and transverse contributions. In fact, and according to A. Bohr [3], the contribution from $\omega_c \to \omega_o$ should yield a purely transverse field.

However, the vectorial properties of the fields do not provide a proper basis for the classification of the spectrum, since the partial waves in all the frequency range have properties characteristic of both types of fields. A consistent classification of the spectrum may best be accomplished in terms of radiation part (which is not necessarily transverse in the sense we have used), screening part and plasma contribution. In order to see it we shall now analyze the displacement vector, whose longitudinal part has a particularly simple expression, by using the conventional approach. In the next paragraph we shall carry out an analysis of the different contributions by a method which provides a clearer understanding of their character. Taking into account the conditions $\Lambda(\omega)$ must satisfy as stated in the previous paragraph, and the equations defining the modified Bessel functions in terms of the Hankel functions :

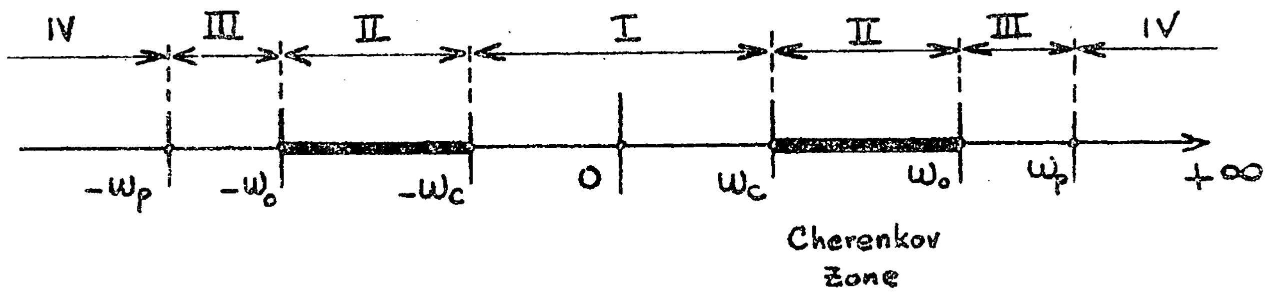
$$K_\nu(z) = \frac{\pi i}{2} e^{\nu\frac{\pi}{2}i} H_\nu^{(1)}(z e^{i\frac{\pi}{2}}) = -\frac{\pi i}{2} e^{-\nu\frac{\pi}{2}i} H_\nu^{(2)}(z e^{-i\frac{\pi}{2}}) \qquad 8.1$$

we may separate the solutions for the transverse displacement vector as a sum of integrals over different frequency ranges, writing :

$$D_{t\rho} = \frac{e}{\pi v}\int_{\mathrm{I,III,IV}} d\omega\, \Lambda K_1(\Lambda\rho)\, e^{i\omega(\frac{z}{v}-t)} + \frac{ei}{2v}\int_{\mathrm{II}} d\omega\, \sigma H_1^{(1)}(\rho\sigma)\, e^{i\omega(\frac{z}{v}-t)} - \frac{e\rho}{\left[(z-vt)^2+\rho^2\right]^{3/2}} \qquad 8.2$$

$$D_{tz} = -\frac{ei}{\pi}\int_{\mathrm{I,III,IV}} d\omega\, \frac{\Lambda^2}{\omega} K_0(\Lambda\rho)\, e^{i\omega(\frac{z}{v}-t)} - \frac{e}{2}\int_{\mathrm{II}} d\omega\, \frac{\sigma^2}{\omega} H_0^{(1)}(\rho\sigma)\, e^{i\omega(\frac{z}{v}-t)} - \frac{e(z-vt)}{\left[(z-vt)^2+\rho^2\right]^{3/2}} \qquad 8.3$$

where the integration ranges are defined by the following diagram :



and :

$$\sigma(\omega) = \frac{\omega}{v}\sqrt{\frac{v^2}{c^2}\,\varepsilon(\omega) - 1}$$

In the present form of equations (7.1) to (7.7), it is not at all obvious which are the space regions where the different contributions may be observed. Therefore we must assume that, in principle, equations (8.2) and (8.3) are valid for $z - vt \gtrless 0$

Taking the curl and the divergence of (8.2), (8.3) we obtain for the contributions in the different ranges :

$$\nabla \times \underset{\sim}{D}_{\text{II}} = -\frac{e}{2c^2}\int_{\text{II}} d\omega\; \sigma(\omega)\, \varepsilon(\omega)\, \omega\, H_1^{(1)}(\rho\, \sigma(\omega))\, e^{i\omega\left(\frac{z}{v} - t\right)}$$

$$\nabla \cdot \underset{\sim}{D}_{\text{II}} = 0 \qquad 8.5$$

$$\nabla \times \underset{\sim}{D}_{I,III,IV} = \frac{e i}{\pi c^2} \int_{I,III,IV} d\omega \, \Lambda(\omega) \, \omega \, \mathcal{E}(\omega) \, K_1(\Lambda(\omega)\rho) \, e^{i\omega\left(\frac{z}{v} - t\right)}$$

$$\nabla \cdot \underset{\sim}{D}_{I,III,IV} = 0 \tag{8.6}$$

The curl of the partial waves is different from 0 in all the frequency range, exception made of the points $\omega = \omega_p$, $\omega = \omega_c$. The first point corresponds to the longitudinal solutions for which $\underset{\sim}{D}_{\ell}(\omega_p) = 0$ The second one is a trivial case without meaning in the present context. Therefore, zones I, III, IV have transverse contributions. In all frequency ranges the divergence vanishes as a direct consequence of the equations satisfied by the Bessel functions, but without additional calculations nothing can be said about the behaviour for $\rho = 0$, $z - vt = 0$. Despite this fact it is obvious there must be at least a singularity at the position of the particle which we cannot see at first sight how is going to manifest itself through the partial waves.

However, some more information can be obtained if we analyze the well known (8) asymptotic behaviour of the partial waves. We have for instance :

<u>Zone II</u> - <u>Cherenkov zone</u>

For $\sigma(\omega) \neq 0$ and ρ sufficiently large the asymptotic expressions of the Hankel functions are the following :

$$H_{\nu}^{(1)}(z) \simeq \sqrt{\frac{2}{\pi z}} \, e^{i\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)} S_{\nu}(-2iz),$$

$(-\pi < \arg z < 2\pi)$

where :

$$S_\nu(z) = 1 + \frac{(\nu,1)}{z} + \frac{(\nu,2)}{z^2} + \cdots\cdots$$

and :

$$(\nu,0) = 1$$

$$(\nu,m) = \frac{(4\nu^2-1^2)(4\nu^2-3^2)\ldots\ldots(4\nu^2-(2m-1)^2)}{2^{2m}\;m!}$$

$$\text{for } m = 1,2,3,\ldots \qquad 8.7$$

In first order of approximation the partial waves in this zone may be written :

$$D_{II\rho} \simeq D^{(1)}_{II\rho} = \frac{e}{v}\sqrt{\frac{\sigma}{2\pi\rho}}\; e^{i(\rho\sigma + \frac{\omega}{v}z - \omega t)}\, e^{-i\frac{\pi}{4}} \qquad 8.8$$

$$D_{II z} \simeq D^{(1)}_{II z} = -\frac{e}{\omega}\sqrt{\frac{\sigma^3}{2\pi\rho}}\; e^{i(\rho\sigma + \frac{\omega}{v}z - \omega t)}\, e^{-i\frac{\pi}{4}} \qquad 8.9$$

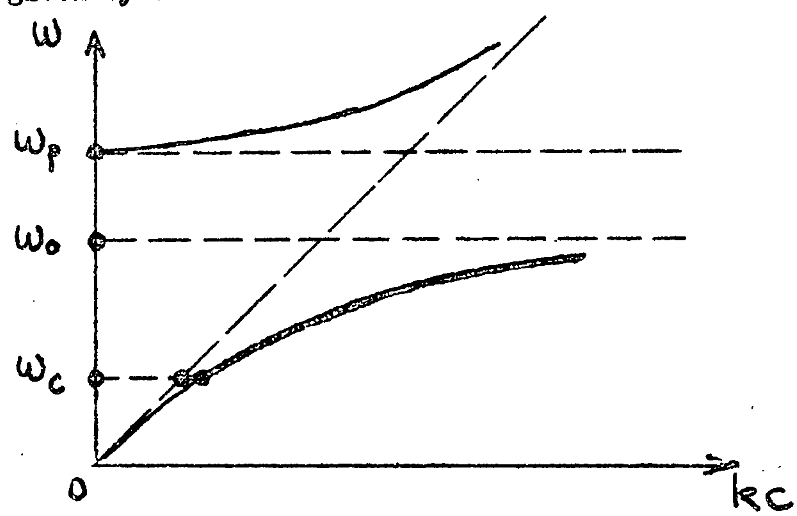
That is, the integral over the Cherenkov zone is for large distances from the trajectory a superposition of waves with wave vector $\underset{\sim}{k}$:

$$\underset{\sim}{k}_\omega = \{k_{\omega\rho}, k_{\omega z}\} = \left\{\sigma(\omega), \frac{\omega}{v}\right\} \qquad 8.10$$

the angle formed by $\underset{\sim}{k}_\omega$ with the Z axis being the Cherenkov angle :

$$\cos\theta_\omega = \frac{c}{v\sqrt{\varepsilon(\omega)}} \qquad 8.11$$

In an (ω, kc) diagram the frequency range in which these waves may exist is given by :



These waves satisfy the condition :

$$\underset{\sim}{D}^{(1)}_{II}(\omega) \cdot \underset{\sim}{k}_\omega = 0 \qquad 8.12$$

Despite this result $\underset{\sim}{D}^{(1)}_{II}(\omega)$ is not a transverse field according to the conditions we have imposed since :

$$\nabla \times \underset{\sim}{D}^{(1)}_{II}(\omega) \neq 0$$

$$\nabla \cdot \underset{\sim}{D}^{(1)}_{II}(\omega) = \frac{e}{2v\rho}\sqrt{\frac{\sigma(\omega)}{2\pi\rho}}\; e^{i\left(\rho\sigma(\omega) + \frac{\omega}{v}z - \omega t\right)}\, e^{-i\frac{\pi}{4}}$$

$$\neq 0 \qquad 8.13$$

This does not mean to say that there is a contradiction with equations (8.5). What was proved there is that for each frequency (and $\rho \neq 0$, $Z - Vt \neq 0$), the divergence of the exact solutions vanishes.
Taking the asymptotic expressions of the Hankel function in first order of approximation is completely analogous to taking the radiation part of the fields of an electric dipole in vacuum. In the case of the dipole the fields may be separated into three different parts according to their $(1/R)$ dependence. In the case of the fields created in the medium by the moving charge there is an infinite number of such zones.
It can be easily shown taking the asymptotic expressions of the Hankel functions up to the second order, that $\underset{\sim}{D}_{II}(\omega)$ may be separated in two parts :

$$\underset{\sim}{D}_{II}(\omega) \simeq \underset{\sim}{D}^{(1)}_{II}(\omega) + \underset{\sim}{D}^{(2)}_{II}(\omega) \tag{8.14}$$

The first one corresponds to the expressions already given, (8.8), (8.9), and the second one is a vector which is not normal to $\underset{\sim}{k}_{\omega}$. Their behaviour with respect to the divergence is also similar to that of a dipole (i.e. the divergence of $\underset{\sim}{D}^{(1)}_{II}(\omega)$ is compensated by part of the divergence of $\underset{\sim}{D}^{(2)}_{II}(\omega)$). As we go to higher orders of approximation of the Hankel functions the successive contributions become more longitudinal, while their relative importance increases as we get nearer the trajectory.

Zones I, III, IV.-

The asymptotic expressions for the modified Bessel functions in the first order of approximation are :

$$K_{\nu}(z) \simeq \sqrt{\frac{\pi}{2z}}\, e^{-z} \tag{8.15}$$

Using this expression the first terms of (8.2), (8.3) may be written :

$$D_{I,III,IV_\rho} \simeq D^{(1)}_{I,III,IV_\rho}(\omega) = \frac{e}{v}\sqrt{\frac{\Lambda(\omega)}{2\pi\rho}}\, e^{-\Lambda(\omega)\rho}\, e^{i\omega\left(\frac{z}{v}-t\right)}$$

$$D_{I,III,IV_z} \simeq D^{(1)}_{I,III,IV_z}(\omega) = -\frac{ei}{\omega}\,\frac{\Lambda(\omega)}{\sqrt{2\pi\rho}}\, e^{-\Lambda(\omega)\rho}\, e^{i\omega\left(\frac{z}{v}-t\right)}$$

8.16

Thus, the contribution to the fields given by the integrals in the regions I, III, IV, correspond in this approximation to a superposition of waves with an amplitude which shows an exponential dependence on distance from the trajectory, and a phase velocity equal to the velocity of the particle. While these contributions have transverse components, they correspond to fields "stuck" to the charge and therefore must be connected in a way which is not apparent from the present form of the equations to the screening of the particle which was found when studying the longitudinal solutions. We shall return to this point in the next paragraph.

From the preceding results it becomes evident that it is not possible to divide the frequency spectrum into zones yielding transverse fields or longitudinal fields (except for the contribution at $\omega = \omega_p$ which must correspond to the plasma waves), every zone being a mixture of both types of fields. While the Cherenkov fields, as an approximation, may be considered as transverse in A. Bohr's sense for $\rho \rightarrow \infty$, this is not true near the trajectory where they show large longitudinal components.

Moreover, a proper description of the perturbations created by the particle should not neglect the intimate relation between the three types of fields which only get sorted out as we move away from the particle in different directions. See also [14].

Limit of the Transverse Displacement Vector for $\eta \to 0$.-

It is interesting to notice that (7.6) and (7.7) may be integrated exactly in the limit when the density of oscillators tends to 0. What we have called the transverse displacement vector becomes in this limit :

$$\lim_{\eta \to 0} D_{t_\rho} = e \frac{\rho(1-v^2/c^2)}{\left[(z-vt)^2+\rho^2(1-v^2/c^2)\right]^{3/2}} - e \frac{\rho}{\left[(z-vt)^2+\rho^2\right]^{3/2}}$$

$$\lim_{\eta \to 0} D_{t_z} = e \frac{(z-vt)(1-v^2/c^2)}{\left[(z-vt)^2+\rho^2(1-v^2/c^2)\right]^{3/2}} - e \frac{(z-vt)}{\left[(z-vt)^2+\rho^2\right]^{3/2}} \qquad 8.17$$

That is, the difference between the vacuum electric fields of a moving charge calculated through the Lienard Wiechert potentials and the instantaneous field. This is a clear example of the conclusions we arrived at in § 3.

§ 9. - EVALUATION OF THE INTEGRAL OVER THE FREQUENCIES FOR THE TOTAL FIELDS.

We shall evaluate now the total solutions in order to show clearly the nature of the different contributions to the fields created by the particle in the medium, as well as their relation to the longitudinal solutions that we have calculated. We shall only consider the Z component of the electric field, since this is sufficient for our purposes. We recall that as a function of real ω , $\Lambda(\omega,\epsilon)$ defined by (7.11) must be chosen in such a way that :

$$\mathrm{Re}\,\Lambda(\omega,\epsilon) > 0 \quad \text{for} \quad \epsilon > 0$$

and :

$$\Lambda(-\omega,\epsilon) = \Lambda^{*}(\omega,\epsilon)$$

for all the range of frequencies.

Taking this into account, the first right hand term of equation (7.2), (Fermi's solution), may be written :

$$E_z = \lim_{\epsilon\to 0} \frac{e}{\pi V^2}\, 2\,\mathrm{Re}\int_0^{\infty} d\omega \left(\frac{V^2}{C^2} - \frac{1}{\mathcal{E}(\omega,\epsilon)}\right) i\,\omega\, K_0\big(\Lambda(\omega,\epsilon)\,\rho\big)\, e^{i\frac{\omega}{V}(z-Vt)} \tag{9.1}$$

The integrations will be carried out by analytic continuation in the complex frequency plane. The singularities of the integrand are :

1°) Pole of $\dfrac{1}{\mathcal{E}(\omega,\epsilon)}$ for $\omega = \pm\,\omega_p - i\epsilon$

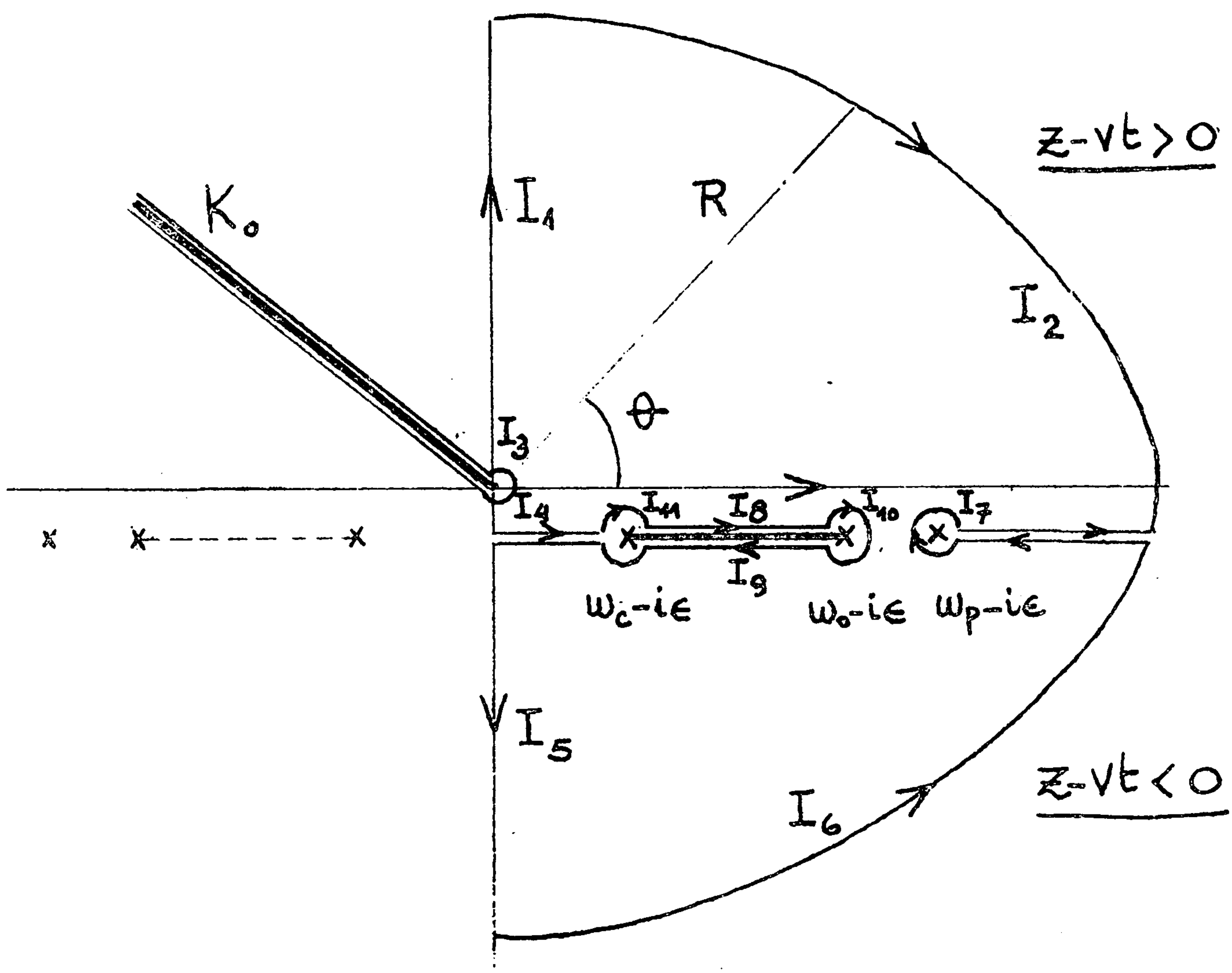
2°) The modified Bessel function K_ν is a function of a function of ω. As such it will show the singularities of the argument and the singularities proper of the K_ν. The singularities of the argument $\Lambda(\omega,\epsilon)\rho$ in the complex ω plane are the Branching points of the function $\Lambda(\omega,\epsilon)$:

$$\omega = \pm\,\omega_c - i\epsilon$$

$$\omega = \pm\,\omega_0 - i\epsilon$$

Due to the existence of the absorption term, the cut and the pole appear in the lower half plane. The singularity proper of the modified Bessel function is a branching point for : $\omega = 0$.

The integrations will be carried out in the first and fourth quadrants and the paths of integration are defined according to the following diagram :



The details of this rather interesting calculation will be published separately. In the present paper we shall confine ourselves to the actual results and their physical interpretation.
We shall assume in the first place :

$$1 - \frac{V^2}{c^2}\varepsilon_{st} > 0 \qquad \text{i.e. } \omega_c \text{ real}$$

ω_c defined by equation (7.9).
We distinguish two different cases :

1°) Z - Vt > 0

The integration path is around a quarter circle centered at the origin of radius $r \to 0$, the positive imaginary axis, and a quarter circle of radius $R \to \infty$, in the sense prescribed in the diagram. We obtain in the limit $r \to 0$, $R \to \infty$ and $\epsilon \to 0$:

$$E_z = E_1 = 2\,R_e\,I_1$$

$$= e\int_0^\infty dk\left(\frac{k^2+\omega_0^2/V^2}{k^2+\omega_p^2/V^2} - \frac{V^2}{c^2}\right)k\,J_0\left(k\sqrt{1-\frac{V^2}{c^2}\frac{k^2+\omega_p^2/V^2}{k^2+\omega_0^2/V^2}}\,\rho\right)e^{-k(z-vt)} \tag{9.2}$$

valid for :

$$\rho\sqrt{1-V^2/c^2} \neq 0$$

$$z - vt \neq 0 \qquad \text{, and} \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

In this expression the quantity under the radical sign in the argument of the Bessel function is always positive for ω_c real.
k is a new variable of integration with the dimensions of a wave vector.

2°) $\underline{z - Vt < 0}$

The integration is along a quarter circle af radius $r \to 0$ centered at the origin, the negative imaginary axis, a quarter circle of radius $R \to \infty$, and around the singularities in the prescribed sense, taking due account of the contributions round the branching points themselves.
In the limit $r \to 0$, $R \to \infty$, there are only three non vanishing contributions. That is :

$$E_z = E_5 + E_p + E_{ch} \tag{9.3}$$

where :

$$\begin{aligned} E_5 &= 2\,\mathrm{Re}\, I_5 \\ &= -e\int_0^\infty dk \left(\frac{k^2 + \omega_o^2/V^2}{k^2 + \omega_p^2/V^2} - \frac{V^2}{c^2}\right) k\, J_o\left(k\sqrt{1 - \frac{V^2}{c^2}\frac{k^2 + \omega_p^2/V^2}{k^2 + \omega_o^2/V^2}}\,\rho\right) e^{-k|z-Vt|} \end{aligned} \tag{9.4}$$

$$\begin{aligned} E_p &= 2\,\mathrm{Re}\, I_7 \\ &= -\frac{2e}{V^2}\eta^2 K_o\left(\frac{\omega_p}{V}\rho\right)\cos\frac{\omega_p}{V}(z - Vt) \end{aligned} \tag{9.5}$$

These two results given in the limit $\epsilon \to 0$.

The third contribution E_{ch} is given by the integral around the cut.

Writing for ω along the cut :

$$\omega = \omega' + i\,\omega'' = \omega' - i\,\epsilon \qquad 9.6$$

we obtain :

$$E_{ch} = 2\,\mathrm{Re} \lim_{\epsilon \to 0} \left(I_8 + I_9 \right)$$

$$= 2\,\mathrm{Re} \lim_{\epsilon \to 0} -\frac{e}{v^2} \int_{\omega_c - i\epsilon}^{\omega_0 - i\epsilon} d\omega' \left(\frac{v^2}{c^2} - \frac{\omega'^2 - \omega_0^2}{\omega'^2 - \omega_p^2} \right) (\omega' - i\epsilon)\, I_0 \left(i\, \frac{\omega' - i\epsilon}{v} \sqrt{\frac{\omega'^2 - \omega_c^2}{\omega_0^2 - \omega'^2}} \sqrt{1 - \frac{v^2}{c^2}}\, \rho \right) \times$$

$$\times\, e^{-i \frac{\omega' - i\epsilon}{v} |z - vt|} \qquad 9.7$$

where I_0 is the modified Bessel function. This expression is exact. The asymptotic expression of the modified Bessel function $I_\nu(Z)$ for large values of $|z|$ is given by :

$$I_\nu(z) \simeq \frac{e^z}{\sqrt{2\pi z}} \left\{ 1 - \frac{4\nu^2 - 1}{8z} + \cdots \right\}, \quad \text{for } |\arg z| < \frac{1}{2}\pi \qquad 9.8$$

which is our case due to the existence of $\epsilon > 0$. Making use of this last equation and the definitions of $\Lambda(\omega)$ and $\sigma(\omega)$, we obtain for the E_{ch} contribution in the asymptotic approximation :

$$E_{ch} = 2\,\mathrm{Re} \lim_{\epsilon\to 0} (I_8 + I_9)$$

$$\simeq -\frac{2e}{v^2}\int_{\omega_c}^{\omega_0} d\omega \left(\frac{v^2}{c^2} - \frac{1}{\varepsilon(\omega)}\right)\omega\, \frac{\cos\left(\sigma(\omega)\rho + \frac{\omega}{v}z - \omega t - \frac{\pi}{4}\right)}{\sqrt{2\pi\,\sigma(\omega)\,\rho}} \tag{9.9}$$

$\sigma(\omega)$ positive and real in all the integration range. The expressions given for E_s, E_p and E_{ch} are valid for :

$$\left.\begin{array}{l} \rho\sqrt{1-v^2/c^2} \neq 0 \\ z - vt \neq 0 \end{array}\right\} \quad \text{and} \quad 0 \leqslant \vartheta \leqslant \frac{\pi}{2}$$

Discussion of Results :

The fields created by the moving charged particle have contributions of three different types. The first of them corresponds to the contributions given by (9.2) and (9.4), which represent a symmetric screening of the charge. The final integrations cannot be carried out in closed form, but nevertheless the physical meaning of these fields can be made clear by the study of certain limiting cases. For example :

Limit for $C \to \infty$

$$\lim_{c\to\infty} E_1 = e\int_0^\infty dk\, \frac{k^2 + \omega_0^2/v^2}{k^2 + \omega_p^2/v^2}\, k\, J_0(k\rho)\, e^{-k(z-vt)} \tag{9.10}$$

This represents a purely longitudinal field and in fact is is identical to the longitudinal field $E_{\ell z}$ (5.9) that we have already analized.

Limit for $V \rightarrow 0$.

$$\lim_{V\to 0} E_1 = \frac{e}{\varepsilon_{st}} \int_0^\infty dk\, k\, J_0(k\rho)\, e^{-kz} = \frac{e}{\varepsilon_{st}} \frac{z}{\left[z^2+\rho^2\right]^{3/2}} \qquad 9.11$$

That is, the Z component of the Coulomb field created by a charge at rest in a medium of dielectric constant ε_{st}.

Limit for $V \rightarrow C$.

The evaluation of this limit cannot be carried out in the same way because for a velocity higher than a certain value the branching point ω_c becomes pure imaginary and the character of the solutions changes. This case will be treated further on.

The behaviour of E_5 in the limiting cases is similar to that corresponding to E_1 except for an obvious change in sign.

E_1 and E_5 are carried convectively by the particle. They exist in front of it and behind it, and in fact, must correspond to that part of the fields which, in the case of the displacement vector and in the asymptotic approximation, was given by (8.16).

The second contribution is given by E_p (9.5). It represents the plasma excitation that we have already seen, defined in all the semispace behind the particle and vanishing in the $\lim_{V\to 0}$. Finally E_{ch}, (9.9), represents the Cherenkov radiation field.

This field is defined for $Z - Vt < 0$, and represents an outgoing field existing behind the particle, each of its frequency components forming with the trajectory the angle characteristic of Cherenkov radiation. When the velocity of the particle tends to zero we have :

$$\lim_{V \to 0} \omega_c = \omega_0 \qquad \text{i.e.:} \int_{\omega_c}^{\omega_0} \dots \, d\omega \Rightarrow 0$$

Therefore in this limit the Cherenkov fields and as seen previously the plasma excitation also, vanish. We are left only with contributions E_1 and E_5 which give in this case the Coulomb field of a particle at rest in a medium of dielectric constant $\mathcal{E} = \mathcal{E}_{st}$.

When the density of oscillators tends to zero we have :

$$\lim_{\eta \to 0} \omega_c = \omega_0$$

and thus the Cherenkov fields (9.9) disappear. Since the plasma excitation (9.5) also disappears in this limit, we are left once again with contributions E_1 and E_5 which become in this case :

$$\lim_{\eta \to 0} E_1 = e(1 - V^2/c^2) \int_0^\infty dk \, k \, J_0\left(k\sqrt{1 - V^2/c^2}\,\rho\right) e^{-k(z - vt)}$$

$$= e \frac{\dfrac{z - vt}{\sqrt{1 - V^2/c^2}}}{\left[\left(\dfrac{z - vt}{\sqrt{1 - V^2/c^2}}\right)^2 + \rho^2\right]^{3/2}} = e \frac{z'}{\left[z'^2 + \rho^2\right]^{3/2}} \qquad 9.12$$

and a similar result for E_5.

That is, the fields of a moving charge in vacuum either as seen from a reference system in which the charge is at rest or from a reference system in which the charge is moving with a velocity V, result which is in agreement with the Lorentz transformation laws for electromagnetic fields. The results in the present paragraph have so far been given under the assumption ω_c real.

The critical frequency ω_c is a function of :

a) The properties of the medium (and also the model assumed) through ε_{st}.

b) The velocity of the incident particle.

For a velocity V satisfying the condition :

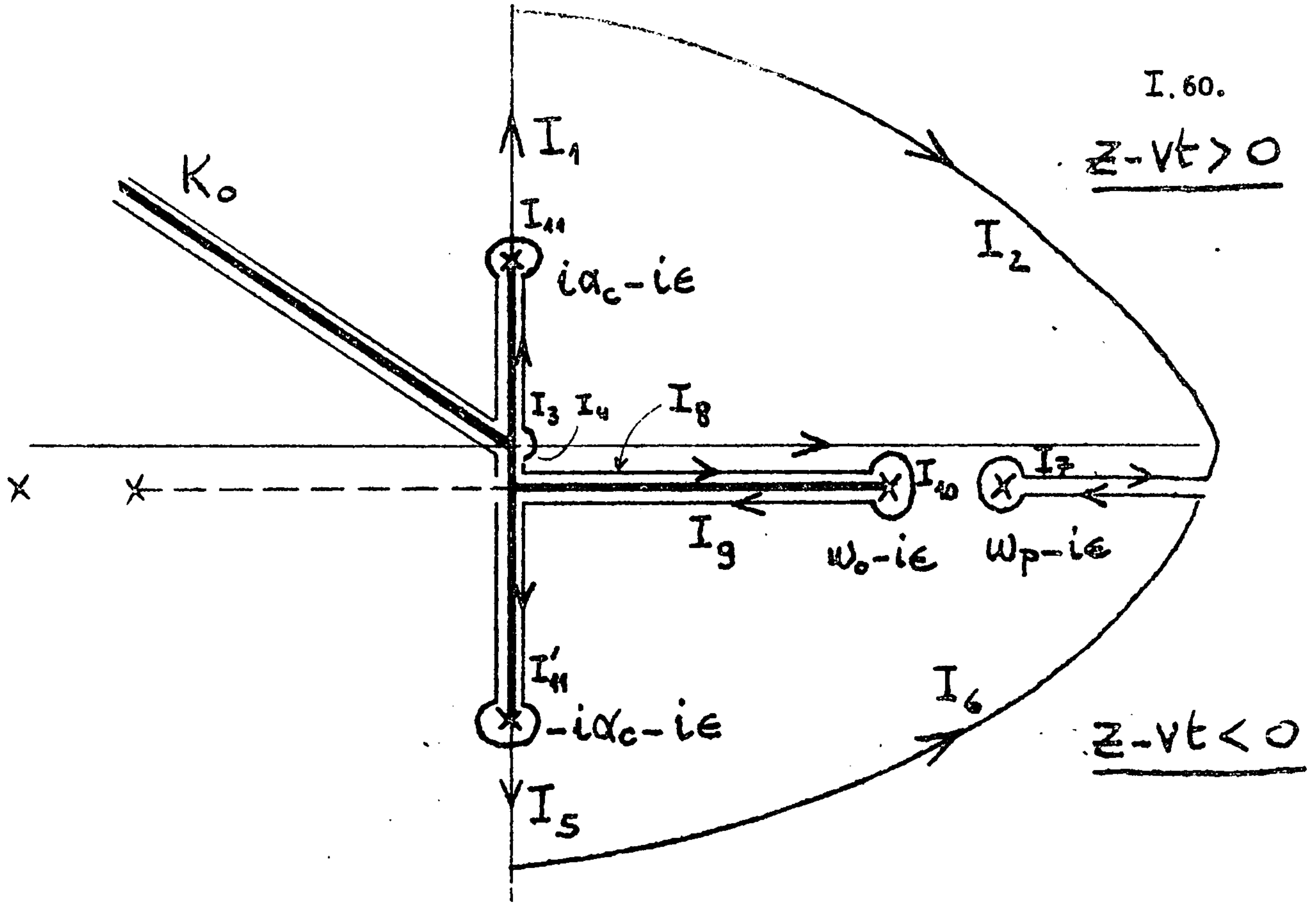
$$V > c/\sqrt{\varepsilon_{st}}$$

the critical frequency becomes :

$$\omega_c = \pm i\alpha_c \qquad , \quad \alpha_c = \frac{\omega_0\sqrt{\frac{V^2}{c^2}\varepsilon_{st} - 1}}{\sqrt{1 - V^2/c^2}}$$

Therefore the branching points of $\Lambda(\omega,\epsilon)$ corresponding to the critical frequency move over to the imaginary axis.

The integration of equation (9.1) is carried out as in the previous case, the paths of integration being now defined by the following diagram :



where the same notation previously employed has been used to designate the different parts of the integration circuits.

Taking due account of the phase relations which must be satisfied by $\Lambda(\omega,\epsilon)$ we find :

a) The plasma excitation is given by the same equation (9.5).

b) The Cherenkov fields are also given by (9.7) or (9.9) except for the fact that the emitted frequencies range now from $0 \to \omega_0$.

c) The charge screening is now given by :

$\underline{z-vt>0}$

$$E_1 = e\int_{k_c}^{\infty} dk\, k\left(\frac{k^2+\omega_0^2/v^2}{k^2+\omega_p^2/v^2} - \frac{v^2}{c^2}\right) J_0\left(k\sqrt{1-\frac{v^2}{c^2}\frac{k^2+\omega_p^2/v^2}{k^2+\omega_0^2/v^2}}\,\rho\right) e^{-k(z-vt)}$$

$\underline{z - vt < 0}$

$$E_5 = -e\int_{k_c}^{\infty} dk\, k\left(\frac{k^2+\omega_0^2/v^2}{k^2+\omega_p^2/v^2} - \frac{v^2}{c^2}\right) J_0\left(k\sqrt{1-\frac{v^2}{c^2}\frac{k^2+\omega_p^2/v^2}{k^2+\omega_0^2/v^2}}\,\rho\right) e^{-k|z-vt|}$$

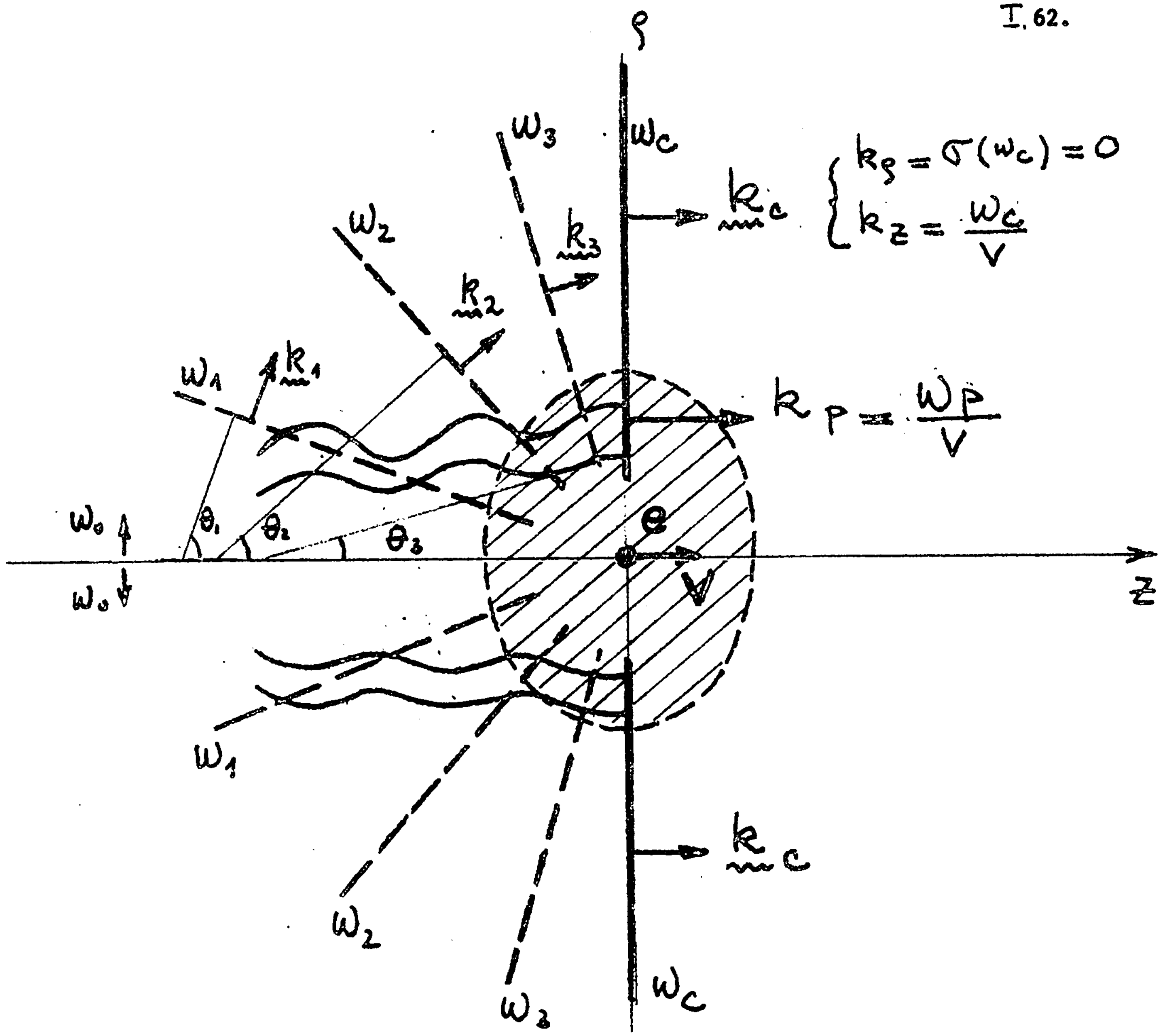
where : $k_c = \dfrac{\alpha_c}{v}$

i.e. the range of integration is not the same as in the previous case or, in other words, the screening starts losing the components of longer wavelength.

Since : $\lim\limits_{v\to\infty} k_c = \lim\limits_{v\to\infty} \dfrac{\alpha_c}{v} = \infty$, it is clear that the charge screening tends to disappear for ultrarelativistic velocities, and we are left only with the Cherenkov fields and the plasma excitation. We have analyzed only one component of the fields, and have seen that the character of the fields is essentially determined by the singularities of the integrands in the complex frequency plane. Since all the first terms of equations (7.1) to (7.7) giving the total fields present the same type of singularities, except for $\vec{B}$ ($\vec{H}$) and $\vec{D}$ which have no pole for the plasma frequency (i.e. no plasma excitation), we conclude that our results must remain qualitatively the same for all the different components whether we speak of electromagnetic fields or polarization. The distribution of fields surrounding the particle may be represented schematically by the following diagrams :

1°) $v < c/\sqrt{\varepsilon_{st}}$

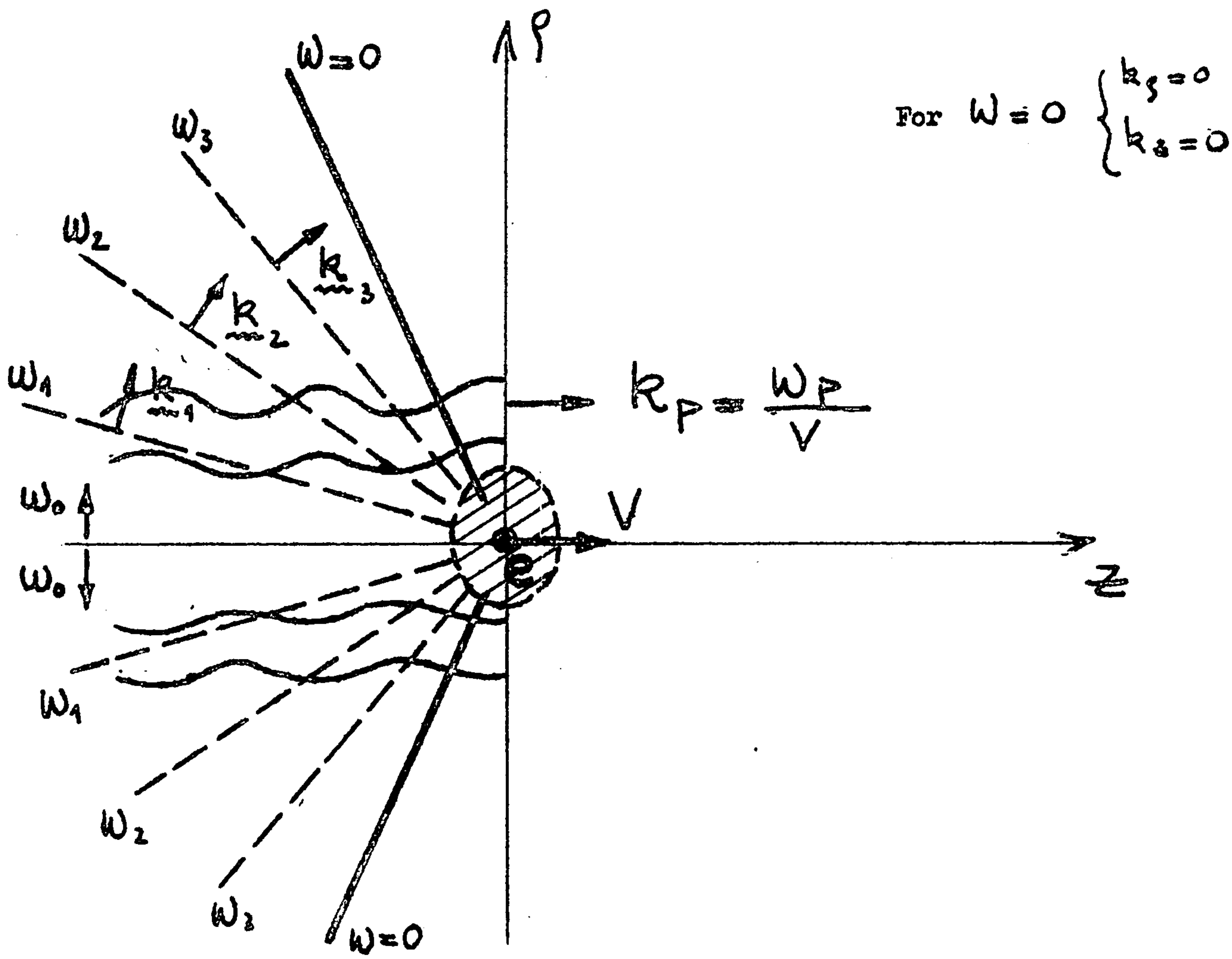
at ω_c we have : $\cos\theta_c = \dfrac{u_c}{v} = \dfrac{c}{v\sqrt{\varepsilon(\omega_c)}} = 1$



In this range of velocities the wavevector $\underset{\sim}{k}_c$ associated to the partial wave of frequency ω_c is parallel to the wavevector $\underset{\sim}{k}_p$ corresponding to the plasma waves.

2°) $V > c/\sqrt{\varepsilon_{st}}$

This corresponds to the case ω_c imaginary. The emitted frequencies range from $\omega = \omega_0 \rightarrow \omega = 0$



In this range of velocities the screening tends to disappear, all the wavefronts move backwards but the plasma waves are still defined in all the semispace behind the particle.

It is worth noting that most of the conclusions we have arrived at concerning the symmetric fields surrounding the particle and the plasma excitation, can all be seen to have their origin in the ordinary response of one single oscillator to the field of a moving charged particle, as is shown in a separate paper [14].

§ 10. - ENERGY LOSS OF THE CHARGED PARTICLE. SEPARATION INTO T-T AND L-T MODES.

The work done upon the electrons has also been calculated by Schönberg [5]. However, the main interest of the present paragraph is to show, by using a different method the connection between Fermi's result and A. Bohr's conclusions. In order to establish this comparison, we shall calculate the energy lost by the charged particle applying Fermi's method. That is : the energy lost by the particle per unit time at distances greater than a certain ρ is given by the flux of Poynting's vector through a cylinder of radius ρ and infinite length enclosing the trajectory. As in Fermi's paper the present classical theory will be assumed to be valid for ρ larger than a certain $\rho_{min} \sim 1$ Å. Therefore :

$$-\frac{dW}{dt} = \int \underset{\sim}{S} \cdot d\underset{\sim}{\sigma} = \frac{c\rho}{2} \int_{-\infty}^{\infty} (\underset{\sim}{E} \times \underset{\sim}{H})_n \, dz \qquad 10.1$$

and the loss per unit path :

$$-\frac{dW}{dz} = \frac{c\rho}{2V} \int_{-\infty}^{\infty} (\underset{\sim}{E} \times \underset{\sim}{H})_n \, dz \qquad 10.2$$

Separating the electric field into T and L parts we may write :

$$-\frac{dW}{dz} = \frac{c\rho}{2V} \int_{-\infty}^{\infty} (\underset{\sim}{E}_t \times \underset{\sim}{H})_n \, dz + \frac{c\rho}{2V} \int_{-\infty}^{\infty} (\underset{\sim}{E}_\ell \times \underset{\sim}{H})_n \, dz \qquad 10.3$$

$$= -\frac{dW_{tt}}{dz} - \frac{dW_{\ell t}}{dz}$$

Using the expressions already given for the fields equations (7.2) (7.2a) (7.3), we obtain :

$$-\frac{dW_{tt}}{dz} = -\frac{c\rho}{2V}\int_{-\infty}^{\infty} E_{t_z} H_\varphi \, dz$$

i.e.

$$-\frac{dW_{tt}}{dz} = \frac{e^2\rho}{2V^3\pi^2}\int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \left(\frac{1}{\varepsilon(\omega)} - \frac{V^2}{c^2}\right) i\omega \Big\{$$

$$K_0(\Lambda(\omega)\rho)\,\Lambda(\omega')\,K_1(\Lambda(\omega')\rho)\, e^{i(\omega+\omega')\frac{z}{V}}\Big\} + \frac{c\rho}{2V}\int_{-\infty}^{\infty} E_{l_z} H_\varphi \, dz$$

$$-\frac{dW_{lt}}{dz} = -\frac{c\rho}{2V}\int_{-\infty}^{\infty} E_{l_z} H_\varphi \, dz = \qquad 10.4$$

$$= \frac{e^2\rho}{2V^3\pi^2}\int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \, \frac{i\omega}{\varepsilon(\omega)} K_0\left(\left|\frac{\omega}{V}\right|\rho\right) \Lambda(\omega')\, K_1(\Lambda(\omega')\rho)\, e^{i(\omega+\omega')\frac{z}{V}} \qquad 10.5$$

where for reasons of simplicity we have set t = 0 (particle at the origin of coordinates). The first right hand term of equation 10.4 corresponds to Fermi's expression for the energy loss.

We may write then the T-T energy loss as :

$$-\frac{dW_{tt}}{dz} = -\frac{dW_{FERMI}}{dz} + \frac{dW_{lt}}{dz} \qquad 10.6$$

where after exact analytical integration :

$$-\frac{dW_{FERMI}}{dz} = \frac{e^2}{V^2}\int_{\omega_c}^{\omega_0}\left(\frac{V^2}{C^2} - \frac{1}{\varepsilon(\omega)}\right)\omega\, d\omega + \frac{e^2 \varrho\, \eta^2}{V^2}\frac{\omega_p}{V} K_0\left(\frac{\omega_p}{V}\varrho\right) K_1\left(\frac{\omega_p}{V}\varrho\right) \qquad 10.7$$

This gives the total energy loss as calculated by Fermi in the limit when the absorption tends to zero, written in our notation. As is well known the first term of equation (10.7) corresponds to the energy lost as Cherenkov radiation. The second term corresponds to energy which remains stored in the medium.

We shall prove now that this energy which is lost in exciting the oscillators is given, except for a small term, by $\left(-\frac{dW_{lt}}{dz}\right)$

ENERGY LOSS DUE TO THE L-T MODE

From equation (10.5), integrating over Z and ω', and taking into account the properties of $\Lambda(\omega)$, we obtain :

$$-\frac{dW_{lt}}{dz} = \frac{e^2\varrho}{V^2\pi}\int_{-\infty}^{\infty} d\omega\, \frac{i\omega}{\varepsilon(\omega)} K_0\left(\left|\frac{\omega}{v}\right|\varrho\right)\Lambda^*(\omega)\, K_1\left(\Lambda^*(\omega)\varrho\right)$$

$$= \frac{e^2\varrho}{V^2\pi}\, 2\,\mathrm{R_e}\int_0^{\infty} d\omega\, \frac{i\omega}{\varepsilon(\omega)} K_0\left(\frac{\omega}{v}\varrho\right)\Lambda^*(\omega)\, K_1\left(\Lambda^*(\omega)\varrho\right)$$

10.8

In this case the singularities of K_0 and K_1 have no bearing on the results. We integrate along the upper border of the cut. However, we must deal with the singularity coming from $\mathcal{E}(\omega)$. Instead of assuming the existence of the absorption term and the limiting process, we shall deform the integration path by the introduction in the upper border of the real axis of a semicircle of radius $r \to 0$, centered at the point $\omega = \omega_p$, and therefore replacing :

$$\int_0^\infty \dots d\omega \longrightarrow P.V.\int_0^\infty \dots d\omega + \int_{\frown} \dots d\omega$$

Equation (10.8) may then be written :

$$-\frac{dW_{lt}}{dz} = \frac{e^2\rho}{v^2\pi} 2\,\mathrm{Re}\, P.V.\int_0^\infty d\omega \frac{i\omega}{\mathcal{E}(\omega)} K_0\left(\frac{\omega}{v}\rho\right) \Lambda^*(\omega) K_1\left(\Lambda^*(\omega)\rho\right)$$

$$- 2\,\mathrm{Re}\, \pi i\, R(\omega_p) \qquad 10.9$$

Outside the Cherenkov zone the first right hand term becomes pure imaginary i.e. except for the pole there is no contribution outside that range. We obtain then after evaluation of the residue :

$$-\frac{dW_{lt}}{dz} = \frac{e^2\rho}{\pi v^2} 2\,\mathrm{Re}\int_{\omega_c}^{\omega_0} d\omega \frac{i\omega}{\mathcal{E}(\omega)} K_0\left(\frac{\omega}{v}\rho\right) \Lambda^*(\omega) K_1\left(\Lambda^*(\omega)\rho\right) +$$

$$+ \frac{e^2\rho\,\eta^2}{v^2} \frac{\omega_p}{v} K_0\left(\frac{\omega_p}{v}\rho\right) K_1\left(\frac{\omega_p}{v}\rho\right) \qquad 10.10$$

In the range of integration :

$$\Lambda(\omega) = -i\,\sigma(\omega) = -i\sqrt{\frac{\omega^2}{c^2}\varepsilon(\omega) - \frac{v^2}{V^2}}$$

Taking into account the defining relations for the modified Bessel functions and the Hankel functions :

$$K_\nu(z) = \frac{\pi i}{2} e^{\nu\frac{\pi}{2}i} H_\nu^{(1)}\left(z e^{i\frac{\pi}{2}}\right) = -\frac{\pi i}{2} e^{-\nu\frac{\pi}{2}i} H_\nu^{(2)}\left(z e^{-i\frac{\pi}{2}}\right) \qquad 10.11$$

and : $$H_1^{(2)}(z) = J_1(z) - i\,N_1(z) \qquad 10.12$$

we obtain from equation (10.10) after taking the real part :

$$-\frac{dW_{lt}}{dz} = -\frac{dW_{lt}^{(1)}}{dz} - \frac{dW_{lt}^{(2)}}{dz} =$$

$$= \frac{e^2\rho}{V^2}\int_{\omega_c}^{\omega_0} d\omega\, \frac{\omega\,\sigma(\omega)}{\varepsilon(\omega)} K_0\left(\frac{\omega}{V}\rho\right) J_1(\sigma\rho) + \qquad 10.13$$

$$+ \frac{e^2\rho\,\eta^2}{V^2}\,\frac{\omega_p}{V}\,K_0\left(\frac{\omega_p}{V}\rho\right) K_1\left(\frac{\omega_p}{V}\rho\right)$$

This expression is exact. It gives the energy loss of the L-T mode as the sum of two contributions :

a) a contribution provided by the integral in the Cherenkov zone. This contribution has its origin in the longitudinal part of the fields, which as we have seen, is present even in the radiation range;

b) a contribution which is a function of the plasma frequency, the distance from the trajectory and the density of oscillators. This contribution is the same as the second term in Fermi's expression for the energy loss.

It should be noticed that this contribution is due to a coupling of the transverse mode of the magnetic field with the purely longitudinal field due to the plasma excitation :

$$E_{z\,osc}(\omega_p) \cdot H_\varphi(\omega_p)$$

In the case of the homogeneous coupled system (no external charge), the longitudinal solutions (plasma solutions) cannot combine with the transverse solutions (magnetic field) to produce a net flow of energy through their associated Poynting's vector.

APPROXIMATE CALCULATION OF L T CONTRIBUTION TO THE ENERGY LOSS IN THE CHERENKOV ZONE.

We shall estimate now the importance of the first right hand term of equation (10.13). From the analysis of the integrand in the first term of this equation, it is immediately evident that, except for $\rho = 0$ (which is outside the range of validity of our formulae), there are no singularities in the interval of integration.

We distinguish two cases :

1st case : small values of the arguments of the Bessel functions i.e. small distances.

In this approximation :

$$K_0(x) \sim \ln \frac{2}{\gamma x} \quad , \quad 0 < x \ll 1 \quad , \quad \gamma = \text{Euler constant}$$

$$J_1(x) \sim \frac{1}{\Gamma(2)} \cdot \frac{x}{2} = \frac{x}{2} \quad , \quad x \ll 1 \tag{10.14}$$

Using this expression we obtain :

$$-\frac{dW_{tt}^{(1)}}{dz} \simeq \frac{e^2}{4v^2} \int_{\omega_c}^{\omega_0} d\omega \left(\frac{v^2}{c^2} - \frac{1}{\varepsilon(\omega)} \right) \omega \, \frac{\omega^2 \rho^2}{v^2} \ln \frac{4}{\gamma^2 \, \frac{\omega^2 \rho^2}{v^2}} \tag{10.15}$$

valid for :

$$0 < \frac{\omega}{v} \rho \ll 1 \quad , \quad \sigma(\omega) \rho \ll 1$$

This expression is formally very similar to the first term of the equation giving Fermi's total energy loss, although there is a dependence on ρ which does not exist in Fermi's result. This represents a very small part of the total energy loss in the Cherenkov range.

Note : It is worth while mentioning that, if the approximation of small arguments had been carried out in equation (10.10), we obtain :

$$z \to 0 \qquad K_\nu(z) \sim \frac{1}{2} \Gamma(\nu) \left(\frac{z}{2} \right)^{-\nu} \quad , \quad \mathrm{Re}\, \nu > 0$$

and :

$$K_1(\Lambda^* \rho) \simeq \frac{1}{\Lambda^* \rho}$$

Therefore :

$$-\frac{dW_{lt}^{(1)}}{dz} = 0$$

since the integral becomes pure imaginary. However, this is due to the approximation used. The contribution is small, but it exists.

2nd case : Large values of the arguments. The asymptotic approximations are in this case :

$$J_1(\sigma\rho) \simeq \sqrt{\frac{2}{\pi\sigma\rho}}\,\cos\left(\sigma\rho - \tfrac{3}{4}\pi\right) \qquad , \sigma\rho \gg 1$$

$$K_0\left(\frac{\omega}{v}\rho\right) \simeq \sqrt{\frac{\pi v}{2\omega\rho}}\,e^{-\frac{\omega}{v}\rho} \qquad , \frac{\omega}{v}\rho \gg 1 \qquad (10.16)$$

using these expressions in the first term of (10.13) we obtain :

$$-\frac{dW_{lt}^{(4)}}{dz} \simeq \frac{e^2}{v^2}\int_{\omega_c}^{\omega_0} d\omega\,\frac{\omega}{\varepsilon(\omega)^{3/4}}\left(\frac{v^2}{c^2} - \frac{1}{\varepsilon(\omega)}\right)^{1/4} e^{-\frac{\omega}{v}\rho}\cos\left(\sigma\rho - \tfrac{3}{4}\pi\right) \qquad 10.17$$

The L-T contribution to the energy loss in the Cherenkov range shows in this approximation an exponential decrease with distance from the trajectory.

APPROXIMATE CALCULATION OF THE T T CONTRIBUTION TO THE ENERGY LOSS

We give the expression in the approximation of small arguments only. The energy loss due to the T-T mode exclusively is given by equation (10.6). Using equations (10.7), (10.13) and (10.15), we obtain :

$$-\frac{dW_{tt}}{dz} \simeq \frac{e^2}{v^2}\int_{\omega_c}^{\omega_0} d\omega\left(\frac{v^2}{c^2} - \frac{1}{\varepsilon(\omega)}\right)\omega\left(1 - \frac{1}{4}\,\frac{\omega^2\rho^2}{v^2}\ln\frac{4}{\gamma^2\frac{\omega^2\rho^2}{v^2}}\right) \qquad 10.19$$

From this result we conclude that very near the trajectory the energy loss in the Cherenkov zone is almost completely transverse.

§ 11. - SUMMARY OF RESULTS AND CONCLUSIONS

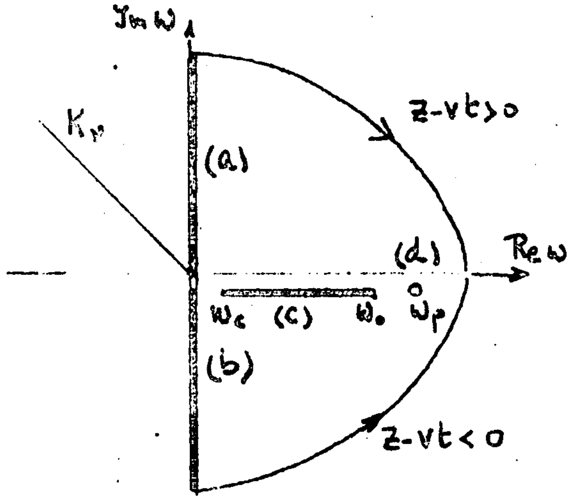
In the absence of boundaries the homogeneous coupled system admits two types of solutions corresponding to two possible excitations of the system. In the case of the transverse excitations these may be produced by a light wave. But for the longitudinal excitations it is always necessary to have a time dependent source. The present work has been basically the classical study of a moving charged particle as a source of both types of excitations. In order to achieve this end we have separated the system expressed initially in Lorentz gauge into transverse and longitudinal parts, and have shown by comparison with the corresponding expressions in Coulomb gauge that the method of separation we have employed is essentially equivalent to isolating the instantaneous Coulomb contribution from the total system.

We have analyzed in detail the properties and physical interpretation of the longitudinal solutions and have seen that in general they show two well defined contributions :

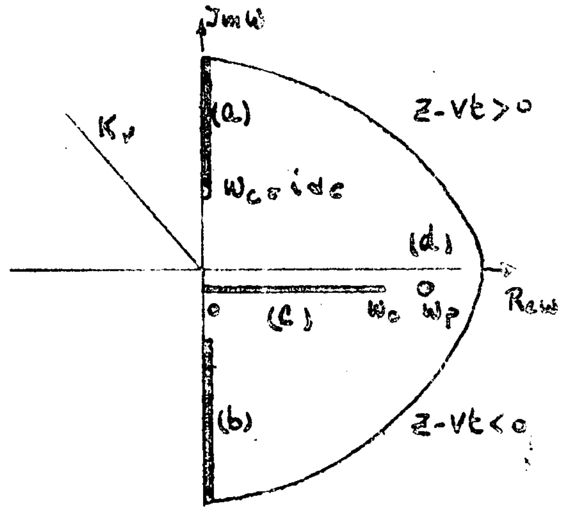
a) a field surrounding the particle and carried convectively which we have identified as a screening field,

b) an oscillator plasma wave. Part (b) represents a wave constructed by the particle, with a group velocity equal to the velocity of the particle and a limited range of possible wave vectors. This part corresponds to the oscillating part predicted by A. Bohr, although contrary to his belief this excitation is not left "in the wake" of the particle, being in principle the excitation of a whole semispace although in fact, due to its exponential behaviour, it is only important near the trajectory.

The transverse solutions have appeared as a difference between Fermi's solution and the purely longitudinal solutions. The separation is not as clear-cut as might have been expected due to the presence of spurious transverse or longitudinal components in either the screening part of the fields or the radiation fields respectively. This represents a minor contradiction with A. Bohr's conclusions who assumed the radiation part should be divergence free. By evaluating the total fields by analytic continuation in the complex frequency plane, with suitable paths of integration, a deeper insight into the nature of the perturbations created by the moving charge has been obtained. We have found that, in general, the fields may be separated in a consistent way, into four different contributions (for $\vec{E}$, $\vec{P}$) as shown in diagram (1) and (2) :



(1) $v < c/\sqrt{\varepsilon_{st}}$



(2) $v > \dfrac{c}{\sqrt{\varepsilon_{st}}}$

a) a contribution carried convectively by the particle and existing in from of it, which for very low velocities, coincides with the longitudinal solutions;

b) a similar contribution behind the particle;

c) the Cherenkov fields;

d) a plasma excitation.

For $\vec{D}$ and $\vec{B}$ ($\vec{H}$) this last contribution does not exist. For $V > c/\sqrt{\varepsilon_{st}}$ the contributions given by (a) and (b) change and tend to disappear for $V \to c$. It should be noted that this change in the character of the screening begins at the onset of the saturation effect predicted by Fermi's theory ($V = c/\sqrt{\varepsilon_{st}}$). The interpretation of the total solutions that we have given differs from the usual solutions in the case $\varepsilon = ct.$ (which can be obtained by the exact analytical integration of the first right-hand terms of (7.1), (7.2) for example), in that neither the screening nor the excitation are taken account of in the last ones.

The energy loss of the particle as calculated by Fermi is given by two terms, one which corresponds to Cherenkov loss and the other which represents energy lost to the medium. By calculating the flux of Poynting's vector associated to the purely longitudinal solutions and the transverse magnetic field, we have seen that there is an energy loss associated to the L-T mode given by two contributions (10.13) :

a) A loss given by an integral in the Cherenkov zone whose origin lies in the longitudinal part of the radiation fields. This term, calculated in the approximation of small arguments, i.e. very near the trajectory, is shown to be very small compared to the first term of Fermi's expression (10.7). For large distances it shows a wave-like behaviour whose importance decreases with an exponential factor.

b) A term identical to the second term of Fermi's expression, which represents the energy delivered to the oscillators (and is stored in them), and is due to a coupling of the purely longitudinal electric field associated to the plasma excitation with the purely transverse magnetic field. This term is in agreement with A. Bohr's predictions.

So far all the results apply to the problem of the excitations created in an ensemble of oscillators by an external source. However, in the limit whan ω_0 becomes vanishingly small the initial homogeneous system may be thought of as describing a classical electron gas, i.e. a smeared out negative charge distribution over a positiva background. (In all rigour we should say $\omega'_0 \to 0$, since when the electrons are not bound to fixed positions $\vec{E}^* = \vec{E}$, where $\vec{E}$ is the average macroscopic valua(13). Formally this does not change our conclusions, but nevertheless it should be kept in mind). In this limit :

$$\omega_p^2 \longrightarrow \eta^2 = \frac{4\pi N e^2}{m} = \Omega_p^2$$

i.e. the usual electron plasma frequency. There is an obvious change in the dispersion relations which become :
for transverse excitation : $\omega^2 = k^2c^2 + \Omega_p^2$
for longitudinal excitation : $\omega^2 = \Omega_p^2$

The resulting system would represent a very poor approximation to physical reality if we try to describe with it the perturbations and energy loss of an external particla whose velocity is comparable to the average thermal velocity of the electrons in an electron gas. However, in the case of a relativistic particle traversing the electron gas, the error incurred in using the present results in the limit $\omega_0 \to 0$, which is equivalent to neglecting random thermal motions of the electrons, should be very small.

Some modifications appear in our results when we try to describe with them a classical electron gas : a) The longitudinal solutions remain as they are, i.e. from a physical point of view there is still a screening and an excitation. b) In the total fields as evaluated in § 9, the Cherenkov part of the fields disappears in agreement with well known results, but we are left with the general screening and the excitation created by the charged particle. c) The first term of equation 10.7 vanishes, and the energy loss is given exclusively by the second term, a result which was found by A. Bohr, while in equation 10.13 this energy loss is proved to be that given to create the purely longitudinal excitation. It is interesting to notice that the results concerning the mechanism of creation of the excitation and the energy loss are quite similar to those derived by D. Pines and D. Bohm[10] [11] [12] although by a far more complete treatment.

The author wishes to express his deep gratitude to Prof. G. Beck for having introduced him to the subject, as well as for his friendly guidance, particularly during the earlier stages of this work. He is grateful to Dr. J. Agudin and J. Heinrichs for stimulating discussions. He would also like to acknowledge the financial support received from the Université de Liège during the completion of this work as well as the hospitality shown by Prof. J. Pirenne and all members of his department.

REFERENCES

1) E. Fermi, Phys. Rev. 56 (1939), 1242.
2) E. Fermi, Phys. Rev. 57 (1940), 485.
3) A. Bohr, Det. Kgl. Danske Videnskabernes Selskab - Mat. Fys. Medd. 24, n° 19 (1948).
4) N. Bohr, Det. Kgl. Danske Videnskabernes Selskab - Mat. Fys. Medd. XVIII, n° 8 (1948).
5) M. Schönberg - Centre de Physique Nucléaire - Université Libre de Bruxelles, Note n° 20, 1950.
6) P. Budini, Il Nuovo Cimento, X (1953), 236.
7) U. Fano, Phys. Rev. 103 (1956), 1202.
8) J.V. Jelley, Cherenkov Radiation and its Applications, Pergamon Press (1958)
9) B.M. Bolotovskii, Sov. Phys. Uspekhi, 4 (1962), 781.
10) D. Pines and D. Bohm, Phys. Rev. 85 (1952), 338.
11) D. Pines, Phys. Rev. 85 (1952), 931.
12) D. Pines, Phys. Rev. 92 (1953), 626.
13) C.G. Darwin, Proc. Roy. Soc. London A146 (1934), 17.
14) J.G. de Vooght, to be submitted for publication.

The mathematical tables extensively used in the present work are :

- Tafeln höherer Funktionen, Jahnke-Emde-Lösch, Teubner Verlagsgesellschaft (1960)
- Handbook of Mathematical Functions, M. Abramowitz and L.A. Stegun, National Bureau of Standards (1964).

ANALOGIES BETWEEN THE RESPONSES OF AN OSCILLATOR AND A DISPERSIVE MEDIUM TO THE FIELDS OF A MOVING POINT CHARGE.

J.G. de VOOGHT*,
Institut de Physique - Thysique Théorique,
Université de Liège - Sart Tilman - Liège,
Belgium.

ABSTRACT :

It is shown how the essential features of the perturbations created in a dispersive medium by an external charge (namely screening, excitation and their respective behaviour in different velocity ranges), can be seen to have their origin in the ordinary response of one single oscillator to the fields of a moving point charge.

* Permanent address : Centro Atomico Bariloche - San Carlos de Bariloche - Rio Negro - Argentina.

INTRODUCTION

As far as the present author knows the response of an oscillator to the force exerted by a moving charged particle was analyzed for the first time by N. BOHR[1], his interest in that opportunity being mainly in the energy transferred to the oscillator and in the subsequent calculation of the energy loss of a charge moving through a medium. The present work takes up the same problem but analyzed from a different point of view, and although it is connected with the problem of energy losses, having in fact had its origin in the attempt to understand the mechanism responsible for part of the energy loss as given by E. Fermi's theory[2] and a subsequent interpretation of the mechanism of this loss by A. BOHR[3], its aim is to study the cinematical behaviour of the oscillator and its relation with the perturvations created by a moving charge in a dispersive medium. By comparing this behaviour with the solutions of a coupled system of an ensemble of oscillators and electromagnetic fields, in the presence of an external moving charge, it is shown :

a) how the properties associated to the response of the coupled system can be clearly seen to arise from the characteristic response of one single oscillator.

b) how the interaction between the oscillators affects that response.

The outline of the present work is as follows : in § 1 the solutions corresponding to the oscillator are given and analyzed; § 2 deals with the solutions of the coupled system and their properties, while finally § 3 is devoted to a discussion and comparison of results.

§ 1. OSCILLATOR SOLUTIONS.

The fields of a charged particle moving with a constant velocity V in the direction of the Z axis, expressed in terms of the present position of the particle, are given by the well known expressions :

$$E_z = e\,\frac{(z-z')(1-V^2/c^2)}{\left[(z-z')^2+\rho^2(1-V^2/c^2)\right]^{3/2}} \tag{1}$$

$$E_\rho = e\,\frac{\rho\,(1-V^2/c^2)}{\left[(z-z')^2+\rho^2(1-V^2/c^2)\right]^{3/2}} \tag{2}$$

$$\underset{\sim}{B} = \frac{\underset{\sim}{V} \times \underset{\sim}{E}}{c}$$

where (z,ρ) and (z',ρ') designate, in cylindrical coordinates, the observation point and the present position of the particle respectively.

We shall make the following simplifying assumptions :

a) The force acting on the oscillator is practically constant within a region covering the possible displacements of the bound charge, and therefore may be approximated by the force acting at the point $\rho=\rho_0$, Z = 0, taken as the mean position of the electron. For ρ_0 sufficiently large compared to atomic dimensions this approximation is valid.

b) We neglect magnetic effects, that is, the force acting on the oscillator is assumed to be due to the electric field only.

As shown in fig.(1) the time scale is fixed so that the particle crosses the origin of coordinates at $t = 0$.

The motion of the oscillator is decomposed into its components parallel and normal to the trajectory, the corresponding equations being the following :

$$\ddot{z} + \gamma \dot{z} + \omega_0'^2 z = \frac{e}{m} E_z = -\frac{e^2}{m} \frac{vt(1 - v^2/c^2)}{\left[v^2 t^2 + \rho^2 (1 - v^2/c^2)\right]^{3/2}} \tag{3}$$

$$\ddot{\rho} + \gamma \dot{\rho} + \omega_0'^2 \rho = \frac{e}{m} E_\rho = \frac{e^2}{m} \frac{\rho_0 (1 - v^2/c^2)}{\left[v^2 t^2 + \rho_0^2 (1 - v^2/c^2)\right]^{3/2}} \tag{4}$$

where $z(t)$ and $\rho(t)$ designate the displacement of the bound charge from its equilibrium position, and the damping term is introduced with the sole aim of fixing integration paths. Solutions are given in the limit $\gamma \to 0$. The use if the damping term is equivalent to assuming as initial conditions for the oscillator :

$$z = \dot{z} = 0 \;, \quad \rho = \dot{\rho} = 0 \qquad \text{for } t = -\infty$$

The initial atom is neutral, and will be turned into a dipole by the action of the particle. The equations of motion are solved by a Fourier analysis in time.

Writing :

$$z(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z(\omega)\, e^{-i\omega t}\, d\omega \tag{5}$$

$$\rho(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \rho(\omega)\, e^{-i\omega t}\, d\omega \tag{6}$$

$$F_{z,\rho}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_{z,\rho}(\omega)\, e^{-i\omega t}\, d\omega \tag{7}$$

replacing these expressions in eq.(3) and (4) and performing the integrations we obtain :

$$z(t) = \frac{e^2}{mv^2\pi} \int_{-\infty}^{\infty} d\omega\, \frac{i\omega(1-v^2/c^2)}{\omega^2 + i\omega\gamma - \omega_0^2}\, K_0\left(\frac{|\omega|}{v}\sqrt{1-v^2/c^2}\,\rho_0\right) e^{-i\omega t} =$$

$$= \frac{e^2}{mv^2\pi}\, 2R_e \int_0^{\infty} d\omega\, \frac{i\omega(1-v^2/c^2)}{\omega^2 + i\omega\gamma - \omega_0^2}\, K_0\left(\frac{\omega}{v}\sqrt{1-v^2/c^2}\,\rho_0\right) e^{-i\omega t} \tag{8}$$

$$\rho(t) = -\frac{e^2}{mv^2\pi} \int_{-\infty}^{\infty} d\omega\, \frac{|\omega|\sqrt{1-v^2/c^2}}{\omega^2 + i\omega\gamma - \omega_0^2}\, K_1\left(\frac{|\omega|}{v}\sqrt{1-v^2/c^2}\,\rho_0\right) e^{-i\omega t} =$$

$$= -\frac{e^2}{mv^2\pi}\, 2R_e \int_0^{\infty} d\omega\, \frac{\omega\sqrt{1-v^2/c^2}}{\omega^2 + i\omega\gamma - \omega_0^2}\, K_1\left(\frac{\omega}{v}\sqrt{1-v^2/c^2}\,\rho_0\right) e^{-i\omega t} \tag{9}$$

where K_ν are the modified Bessel functions. The integration of these expressions will be performed by analytic continuation in the complex frequency plane. The singularities of the integrands of eq.(8) and (9) are the branching point of the K_ν for $\omega = 0$ and, neglecting quadratic terms in γ, the pole for $\omega = \omega_0 - i\frac{\gamma}{2}$.
The integration paths will be defined as follows (fig. 2).
For negative times we integrate along the positive imaginary axis and a quarter circle of radius $R \to \infty$. For positive times the integration is along the negative imaginary axis and a quarter circle of radius $R \to \infty$, taking due account of the singularities. There is no contribution due to the branching point itself. For t and $\rho_0\sqrt{1-V^2/c^2}$ different from 0, integrals I_2 and I_4 vanish for $R \to \infty$. Therefore the only non vanishing contributions are those given by I_1 for $t < 0$, and I_3, I_5 for $t > 0$. Making use of the defining equations of the modified Bessel functions and Hankel functions :

$$K_\nu(z) = \frac{\pi i}{2} e^{\nu\frac{\pi}{2}i} H_\nu^{(1)}(z e^{i\frac{\pi}{2}}) = -\frac{\pi i}{2} e^{-\nu\frac{\pi}{2}i} H_\nu^{(2)}(z e^{-\frac{\pi}{2}i}) \tag{10}$$

$$\left.\begin{aligned} H_\nu^{(1)}(z) &= J_\nu(z) + i N_\nu(z) \\ H_\nu^{(2)}(z) &= J_\nu(z) - i N_\nu(z) \end{aligned}\right\} \tag{11}$$

we obtain in the limit $\gamma \to 0$, and in the cases $V \sim c$, $V \ll c$:

a) $V \sim c$

<u>$t < 0$</u> :

Position of the particle $z' = Vt = -V|t| < 0$

$$z(t) = \frac{e^2}{mV^2}\left(1 - V^2/c^2\right)\int_0^\infty dk \, \frac{k\, J_0\left(k\sqrt{1-V^2/c^2}\,\rho_0\right)}{k^2 + \omega_0'^2/V^2}\, e^{-k|vt|}$$

(12)

$$\rho(t) = \frac{e^2}{mV^2}\sqrt{1 - V^2/c^2}\int_0^\infty dk \, \frac{k\, J_1\left(k\sqrt{1-V^2/c^2}\,\rho_0\right)}{k^2 + \omega_0'^2/V^2}\, e^{-k|vt|}$$

(13)

$\underline{t > 0}$:

Position of the particle $z' = vt > 0$

$$z(t) = \frac{2e^2}{mV^2}\left(1 - V^2/c^2\right)\cos(\omega_0' t)\, K_0\left(\frac{\omega_0'}{V}\sqrt{1-V^2/c^2}\,\rho_0\right) -$$

$$- \frac{e^2}{mV^2}\left(1 - V^2/c^2\right)\int_0^\infty dk \, \frac{k\, J_0\left(k\sqrt{1-V^2/c^2}\,\rho_0\right)}{k^2 + \omega_0'^2/V^2}\, e^{-kvt}$$

(14)

$$\rho(t) = \frac{2e^2}{mV^2}\sqrt{1 - V^2/c^2}\, K_1\left(\frac{\omega_0'}{V}\sqrt{1-V^2/c^2}\,\rho_0\right) \operatorname{sen} \omega_0' t \; +$$

$$+ \frac{e^2}{mV^2}\sqrt{1 - V^2/c^2}\int_0^\infty dk \, \frac{k\, J_1\left(k\sqrt{1-V^2/c^2}\,\rho_0\right)}{k^2 + \omega_0'^2/V^2}\, e^{-kvt}$$

(15)

In these expressions K is a new integration variable with the dimensions of a wave vector, which appears through the integration in the complex frequency plane.

b) $V \ll c$

$\underline{t<0}$:

$$Z(t) = \frac{e^2}{mV^2}\int_0^\infty dk\, \frac{k\, J_0(k\rho_0)\, e^{-k|vt|}}{k^2 + \omega_0'^2/V^2} \tag{16}$$

$$\rho(t) = \frac{e^2}{mV^2}\int_0^\infty dk\, \frac{k\, J_1(k\rho_0)\, e^{-k|vt|}}{k^2 + \omega_0'^2/V^2} \tag{17}$$

$\underline{t>0}$:

$$Z(t) = \frac{2e^2}{mV^2}\cos(\omega_0' t)\, K_0\left(\frac{\omega_0'}{V}\rho_0\right) - \frac{e^2}{mV^2}\int_0^\infty \frac{k\, J_0(k\rho_0)\, e^{-kvt}}{k^2 + \omega_0'^2/V^2}\, dk \tag{18}$$

$$\rho(t) = \frac{2e^2}{mV^2}\sin(\omega_0' t)\, K_1\left(\frac{\omega_0'}{V}\rho_0\right) + \frac{e^2}{mV^2}\int_0^\infty dk\, \frac{k\, J_1(k\rho_0)\, e^{-kvt}}{k^2 + \omega_0'^2/V^2} \tag{19}$$

In both cases ($V \sim C$, $V \ll C$), the solutions show the same characteristic behaviour : before the particle crosses a plane normal to the trajectory at the point $Z = 0$, the response of the oscillator is an increasing distension. When the particle crosses this plane, an oscillating motion of frequency ω_0' starts around an equilibrium position varying in time, which in fact corresponds to a gradual restitution of the equilibrium position of the electron to its initial state.
The integrals appearing in the solutions cannot be carried out analytically. Nevertheless it can be seen that the contribution given by these terms vanishes for $t \to \pm\infty$. If we extrapolate the solutions obtained to values of $t = 0$ we obtain (in the case $V \sim C$ for example) :

a) for the ρ component :

$$\lim_{t \to +0} \rho(t) = \lim_{-|t| \to -0} \rho(t) \tag{20}$$

b) for the Z component :

$$\lim_{t \to +0} Z(t) = \frac{2e^2}{mV^2}\left(1 - V^2/c^2\right) K_0\left(\frac{\omega_0'}{V}\sqrt{1 - V^2/c^2}\,\rho_0\right) - $$
$$- \frac{e^2}{mV^2}\left(1 - V^2/c^2\right) K_0\left(\frac{\omega_0'}{V}\sqrt{1 - V^2/c^2}\,\rho_0\right) = \lim_{-|t| \to -0} Z(t) \tag{21}$$

Therefore, it is reasonable to assume that, at a certain point of coordinates $\rho = \rho_0$, $Z = 0$, the displacement of the electron as a function of time, in the ρ and Z directions, will have the following qualitative behaviour (Fig. 3).

From equations (12) to (15) it can easily be seen that, in the limit $V \to C$, the contributions to the displacement of the bound charge given by the integral terms vanish. Whereas for the oscillating part of the motion, using the approximate expressions of the modified Bessel functions :

$$\left.\begin{aligned} K_0(z) &\approx \ln \frac{2}{\gamma z} \\ K_1(z) &\approx \frac{1}{z} \end{aligned}\right\} \text{ for } |z| < 1 \qquad (22)$$

we obtain for $t > 0$:

$$\lim_{V\to C} z(t)_{osc} \approx \lim_{V \to C} \frac{2e^2}{mV^2}\left(1 - V^2/C^2\right) \cos(\omega_0' t) \ln \frac{2}{\gamma \frac{\omega_0'}{V} \rho_0 \sqrt{1 - V^2/C^2}} = 0 \qquad (23)$$

$$\lim_{V\to C} \rho(t)_{osc} \approx \frac{2e^2}{mV\omega_0\rho_0} \operatorname{sen}(\omega_0' t) \qquad (24)$$

Therefore, in this limit, and within the framework of the assumptions introduced at the beginning, there is no previous distension of the oscillator and the response of the bound electron tends to be an oscillating motion in the ξ direction only, which we may represent schematically as follows (Fig. 4).

This is in agreement with the fact that, in the limit $V \to C$, the fields of the particle tend to behave like a plane wave, and thus its effect on the oscillator is that of a sudden pulse at $t = 0$, in a direction normal to the trajectory.

§ 2. THE INHOMOGENEOUS COUPLED SYSTEM.

A coupled system of oscillators and electromagnetic fields excited by an external point charge moving with a constant velocity, may be represented by the following set of equations :

$$\frac{\partial^2 \underset{\sim}{P}}{\partial t^2} + \omega_0^2 \underset{\sim}{P} = \frac{\eta^2}{4\pi}\left(-\operatorname{grad}\phi - \frac{1}{c}\frac{\partial \underset{\sim}{A}}{\partial t}\right) \tag{25}$$

$$\Box \underset{\sim}{A} = -\frac{4\pi}{c} e \underset{\sim}{v}\, \delta(\underset{\sim}{x} - \underset{\sim}{v} t) - \frac{4\pi}{c}\frac{\partial \underset{\sim}{P}}{\partial t} \tag{26}$$

$$\Box \phi = -4\pi e\, \delta(\underset{\sim}{x} - \underset{\sim}{v} t) + 4\pi \operatorname{div} \underset{\sim}{P} \tag{27}$$

together with the defining equations for the fields :

$$\underset{\sim}{E} = -\operatorname{grad}\phi - \frac{1}{c}\frac{\partial \underset{\sim}{A}}{\partial t} \qquad \underset{\sim}{B} = \nabla \times \underset{\sim}{A} \tag{28}$$

and the Lorentz condition :

$$\nabla . \underset{\sim}{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0 \tag{29}$$

where : $\eta^2 = \frac{4\pi N e^2}{m}$ N = N° oscillators per unit volume; m = mass of the bound charges (30)

and :

$$\omega_0^2 = \omega_0'^2 - \frac{1}{3}\eta^2 \tag{31}$$

is the proper frequency of the oscillators corrected for the coupling with the electromagnetic field (i.e. the Lorentz correction). $\vec{P}$ represents the polarization per unit volume due to the presence of sufficiently densely packed oscillators. For an infinite medium (i.e. no boundaries), and in the absence of external sources, the system admits two types of solutions :

a) longitudinal plane wave solutions of frequency :

$$\omega_p^2 = \omega_0^2 + \eta^2 \tag{32}$$

which correspond to an oscillator plasma.

b) transverse plane wave solutions satisfying the dispersion relation :

$$k^2 = \frac{\omega^2}{c^2}\left(1 + \frac{\eta^2}{\omega_0^2 - \omega^2}\right) = \frac{\omega^2}{c^2}\,\varepsilon(\omega) \tag{33}$$

where $\varepsilon(\omega)$ represents the dielectric constant of the medium as given by the model.

A more detailed study of this system is given in a separate paper[4], where it is also shown that, for a particle moving along the positive direction of the Z axis and crossing the origin of coordinates at $t = 0$, the longitudinal solutions for the polarization are the following :

$z - Vt > 0$ (In front of the particle)

$$P_{l_z}(\rho z t) = \frac{e\eta^2}{4\pi V^2}\int_0^\infty \frac{k\, J_0(k\rho)\, e^{-k(z-vt)}}{k^2 + \omega_p^2/V^2}\, dk \tag{34}$$

$$P_{l_\rho}(\rho z t) = \frac{e\eta^2}{4\pi V^2}\int_0^\infty \frac{k\, J_1(k\rho)\, e^{-k(z-vt)}}{k^2 + \omega_p^2/V^2}\, dk \tag{35}$$

$z - vt < 0$ (Behind the particle)

$$P_{l_z}(\rho z t) = \frac{e\eta^2}{2\pi V^2}\cos\frac{\omega_p}{V}(z-vt)\, K_0\left(\frac{\omega_p}{V}\rho\right) - \frac{e\eta^2}{4\pi V^2}\int_0^\infty \frac{k\, J_0(k\rho)\, e^{-k|z-vt|}}{k^2 + \omega_p^2/V^2}\, dk \tag{36}$$

$$P_{l_\rho}(\rho z t) = -\frac{e\eta^2}{2\pi V^2}\,\mathrm{sen}\,\frac{\omega_p}{V}(z-vt)\, K_1\left(\frac{\omega_p}{V}\rho\right) + \frac{e\eta^2}{4\pi V^2}\int_0^\infty \frac{k\, J_1(k\rho)\, e^{-k|z-vt|}}{k^2 + \omega_p^2/V^2}\, dk \tag{37}$$

The total solutions for the polarization of the coupled system (i.e. without separation into transverse and longitudinal parts), given as integrals over all the frequency spectrum are the following :

$$P_z(\rho z t) = \frac{ie}{(2\pi)^2 v^2}\int_{-\infty}^{\infty} d\omega \, \frac{\eta^2}{\omega^2 - \omega_p^2}\left\{1 - \frac{v^2}{c^2}\varepsilon(\omega)\right\}\omega\, K_0\left(\sqrt{\frac{\omega^2}{v^2}\left(1 - \frac{v^2}{c^2}\varepsilon(\omega)\right)}\,\rho\right) e^{i\omega\left(\frac{z}{v} - t\right)}$$

(38)

$$P_\rho(\rho z t) = -\frac{e}{(2\pi)^2 v}\int_{-\infty}^{\infty} d\omega \, \frac{\eta^2}{\omega^2 - \omega_p^2}\sqrt{\frac{\omega^2}{v^2}\left(1 - \frac{v^2}{c^2}\varepsilon(\omega)\right)}\, K_1\left(\sqrt{\frac{\omega^2}{v^2}\left(1 - \frac{v^2}{c^2}\varepsilon(\omega)\right)}\,\rho\right) e^{i\omega\left(\frac{z}{v} - t\right)}$$

(39)

The properties of these solutions, physical interpretation and asymptotic expressions are fully discussed in the above mentioned paper. We summarize the conclusions which are relevent in the present context.

1°) The longitudinal solutions show two well defined contributions :
a) a symmetrical screening field surrounding the particle in its motion, given by eq.(34), (35) and the second terms of eq. (36), (37). b) a wave given by the first terms of eq. (36), (37), defined in all the semiespace behind the particle bound by the plane normal to the trajectory and containing the particle. Due to the exponential behaviour of the modified Bessel functions for large values of the arguments, this wave has a mean range given by : $\rho_{mean} = \frac{v}{\omega_p}$

This excitation shows typical oscillator plasma behaviour, and corresponds to the longitudinal oscillations predicted by A. BOHR[3], although they cannot be said to be left "in the wake" of the particle. As shown in[4], the creation by the particle of this part of the fied is responsible for one of the contributions to the energy loss as given by Fermi's theory[2].

2°) The total solutions given by eq. (38), (39), correspond to Fermi's expressions[2] for the polarization written in our notation. In their present form it is not at all obvious which are its properties. However, by performing an integration in the complex frequency plane with suitable paths of integration it is possible to show[4] that the polarization and electromagnetic fields solutions of equations (25) to (29), have three distinct contributions
a) a symmetric screening field defined in front and behind the particle;
b) a plasma excitation defined in all the semiespace behind the particle;
c) the Cherenkov fields. The configuration of these fields changes according to whether $V \gtrless c/\sqrt{\varepsilon_{st}}$, where ε_{st} is the static dielectric constant as given by the present model. This evolution may be represented schematically by the following diagrams (Fig. 5).

For $V < c/\sqrt{\varepsilon_{st}}$, the Cherenkov wavefronts corresponding to frequencies from ω_0 to a certain threshold frequency $0 < \omega_c < \omega_0$, and Cherenkov angles varying in the range $\pi \geqslant \theta_c \geqslant 0$, together with the screening field and the plasma wave, occupy in principle the whole semiespace behind the particle, although their relative importance is a function of the distance from the trajectory.

For $V > c/\sqrt{\varepsilon_{st}}$, the Cherenkov wavefronts corresponding to frequencies from ω_0 to 0 fold backwards into a cone, and the screening begins to disappear and disappears completely in the limit $V \to C$. However, the plasma wave still remains defined in front of the cone. Our present task is now to try to understand these apparently contradictory results in the light of what we know about the response of the single oscillator.

§ 3. DISCUSSION OF THE SOLUTIONS.

The solutions of the equations of motion of the oscillator bear a striking similarity to those corresponding to the polarization of the coupled system. This is quite evident for the solutions of the oscillator in the case $V \ll C$ (eq. 16 to 19), and the solutions for the longitudinal polarization (eq. 34 to 37). However we must bear in mind that the solutions we have found for the oscillator are not the complete solution of the problem since we are not taking account of the radiation field created by the oscillator and its reaction on the moving charge and on itself. This radiation field with suitable corrections due to the interaction among the oscillators is what afterwards must give rise to the Charenkov radiation fields. Nevertheless, the solutions as they are, are sufficient for our present purposes, and the assumptions taken as a basis (i.e. neglect of the magnetic term in the force acting on the oscillator and constant velocity of the particle) put the solutions on the same degree of approximation as those for the coupled system.

The mechanism of generation of the plasma excitation becomes now clear. Let us assume that the particle is moving among an ensemble of oscillators which do not interact between themselves. Within the approximation $V \ll C$ the action of the fields may be considered to be instantaneous. As the particle moves along the Z axis, the oscillators placed on a plane normal to the trajectory and containing the particle become excited. This excitation of one plane, when combined with the response of the oscillators placed on neighbouring parallel planes already crossed by the particle gives rise to the formation of a wave. The Coulomb field of the particle produces the excitation, or, to be more precise, the change in direction of the force acting on the oscillator, even if this change is gradual. Since we assume no interaction between the oscillators, the polarization per unit volume would simply be given by :

$$P'_z(t) = N e\, z(t) \tag{40}$$

$$P'_\varrho(t) = N e\, \varrho(t) \tag{41}$$

Using for $z(t), \varrho(t)$ the expressions given by eq. (18), (19), for example, and taking into account the definition of η^2, we obtain :

$$P'_z(t) = \frac{\eta^2 e}{2\pi V^2} \cos(\omega'_0 t)\, K_0\left(\frac{\omega'_0}{V}\varrho_0\right) - \frac{e\eta^2}{4\pi V^2}\int_0^\infty \frac{k\, J_0(k\varrho_0)\, e^{-kVt}}{k^2 + \omega'^2_0/V^2}\, dk \tag{42}$$

$$P'_\varrho(t) = \frac{\eta^2 e}{2\pi V^2} \sin(\omega'_0 t)\, K_1\left(\frac{\omega'_0}{V}\varrho_0\right) + \frac{e\eta^2}{4\pi V^2}\int_0^\infty \frac{k\, J_1(k\varrho_0)\, e^{-kVt}}{k^2 + \omega'^2_0/V^2}\, dk \tag{43}$$

If we compare these expressions with those given for the longitudinal polarization of the coupled system eq. (36), (37), it is immediately evident that, except for the trivial change in the coordinates of the observation point, the interaction between the oscillators and the electromagnetic field appears through the change of :

$$\omega'_0 \longrightarrow \omega_0 \longrightarrow \omega_p = \sqrt{\omega'^2_0 + \frac{2}{3}\eta^2} \tag{44}$$

Furthermore, it can be seen that, the screening appearing in the longitudinal solutions of the polarization is a direct consequence of the initial distension of the individual oscillators and the gradual return of their equilibrium positions to their original state.

Let us consider now the solutions of the oscillator in the case $V \sim C$ The total polarization, eq. (38) and (39), is given by integrals over all the frequency spectrum. Therefore, in order to establish a comparison with the solutions of the oscillator we must work with eq. (8) and (9). If, as in the previous case, we consider an ensemble of oscillators without any interaction, and we define a polarization per unit volume as given by eq. (40), (41), we obtain using equations (8) and (9) :

$$P'_z = \frac{ie}{(2\pi)^2 V^2} \int_{-\infty}^{\infty} d\omega \, \frac{\gamma^2}{\omega^2 + i\omega\gamma - \omega_0'^2} \left\{ 1 - \frac{V^2}{C^2} \right\} \omega \, K_0\left(\frac{|\omega|}{V} \sqrt{1 - \frac{V^2}{C^2}} \, \rho_0 \right) e^{-i\omega t} \tag{45}$$

$$P'_\rho = -\frac{e}{(2\pi)^2 V^2} \int_{-\infty}^{\infty} d\omega \, \frac{\gamma^2 \, |\omega|}{\omega^2 + i\omega\gamma - \omega_0'^2} \sqrt{1 - \frac{V^2}{C^2}} \cdot K_1\left(\frac{|\omega|}{V} \sqrt{1 - \frac{V^2}{C^2}} \, \rho_0 \right) e^{-i\omega t} \tag{46}$$

A brief comparison of these expressions with eq. (38) and (39) shows[++]

a) The interaction between the oscillators appears as in the previous case, i.e. through the change of $\omega_0' \rightarrow \omega_0 \rightarrow \omega_p$

[++]As already noted, the term $i\omega\gamma$ appearing in the denominators of (45), (46) defines integration paths. An analogous term must also be introduced in (38), (39) to deal with the singularities appearing in these expressions.

b) As already mentioned at the beginning of this section, eq/ (8) and (9) correspond to the response of the oscillator to the field of the charge, but do not give account of the radiation field generated by the oscillator once excited. In order to do so, the system which is implicit in equations (3) and (4) :

$$\ddot{\mathbf{r}} + \gamma \dot{\mathbf{r}} + \omega_0'^2 \mathbf{r} = e\,\mathbf{E}^{(0)} = \left\{ -\operatorname{grad} \phi^{(0)} - \frac{1}{c}\frac{\partial \mathbf{A}^{(0)}}{\partial t} \right\}$$

$$\Box \mathbf{A}^{(0)} = -\frac{4\pi}{c} e \mathbf{V}\, \delta(\mathbf{x} - \mathbf{V}t)$$

$$\Box \boldsymbol{\phi}^{(0)} = -4\pi e\, \delta(\mathbf{x} - \mathbf{V}t)$$

(47)

must be supplemented by the equations :

$$\Box \mathbf{A}^{(1)} = -\frac{4\pi}{c} e \dot{\mathbf{r}}$$

$$\Box \phi^{(1)} = +4\pi e \operatorname{div} \mathbf{r}$$

(48)

changing the force acting on the oscillator from $e\mathbf{E}^{(0)} \rightarrow e\left\{\mathbf{E}^{(0)} + \mathbf{E}^{(1)}\right\}$ and finding afterwards the simultaneous solutions of (47) and (48). However, this is completely equivalent, once we have taken into account the Lorentz correction, to what is done in eq. (25) to (27) when we introduce the densities of current and polarization as sources. The interaction of the fields so created by the ensemble of oscillators, with the "external" electromagnetic fields, brings about a change in the velocity of propagation of light signals thereby changing :

$$\sqrt{1 - V^2/c^2} \longrightarrow \sqrt{1 - V^2/c^2\, \varepsilon(\omega)}$$

(49)

This entails as a direct consequence the possibility for the velocity of the particle of becoming larger than the velocity of light in the medium, in a certain frequency range, which as is well known, is the origin of the Cherenkov effect. Despite the fact that eq. (38), (39), include the effect of the fields generated by the oscillators themselves, they still retain two common characteristics with eq. (45), (46), or for that matter, with eq. (8), (9). As mentioned in § 2, the integration of eq. (38) and (39) in the complex frequency plane by a method which is altogether equivalent to the one used to obtain the solutions of the oscillators, shows that, the screening and the excitation thereby obtained are clearly due to the distension and excitation of the oscillator as given by eq. (12) to (15), once the effects of the interaction of the oscillators and the change in the velocity of propagation of light signals are taken into account. Thus it is now possible to understand why in the limit V C the screening part of the total fields disappear while the plasma wave still remains. However a new feature arises. Equations (14), (15) and (18), (19) show that, the introduction of relativistic effects (i.e. retardation), changes the amplitude of the excitation through the introduction of a dependence on $\sqrt{1-V^2/c^2}$ (and powers of), in such a way that, in the limit $V \to C$, there is an oscillating motion in the ξ direction only. The corresponding effect does not appear on integration of eq. (38), (39). This is due to the fact that the plasma wave as calculated from these last two equations, appears as a pole for $\omega = \omega_p$, for which $\sqrt{1-V^2/c^2\,\varepsilon} = 1$, therefore cancelling retardation effects. We conclude from this result that, even when relativistic effects are taken into account, the plasma waves appear as if they were generated by the instantaneous Coulomb fields. This also means that in the limit $V \to C$ the plasma waves will have components not only normal to the trajectory, as in the case of the oscillator, but also parallel to it.

The author wishes to express his deep gratitude to Prof. G. Beck for having introduced him to the subject and for his interest on the different stages of this work, as well as to Dr. J. AGUDIN for stimulating discussions. He would also like to acknowledge the financial support received from the Université de Liège during the completion of this work as well as the hospitality shown by Prof. J. Pirenne and all members of his department.

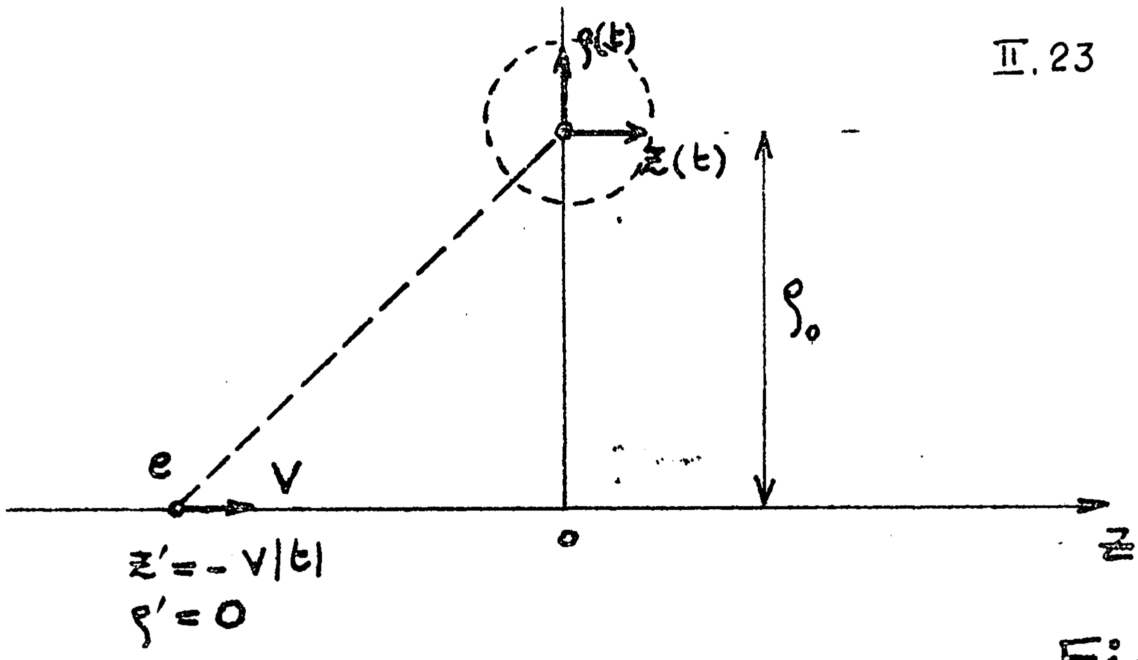
REFERENCES :

(1) N. BOHR (1913), Phil. Mag. (6) 25, 10.

(2) E. FERMI (1940), Phys. Rev. 57, 485.

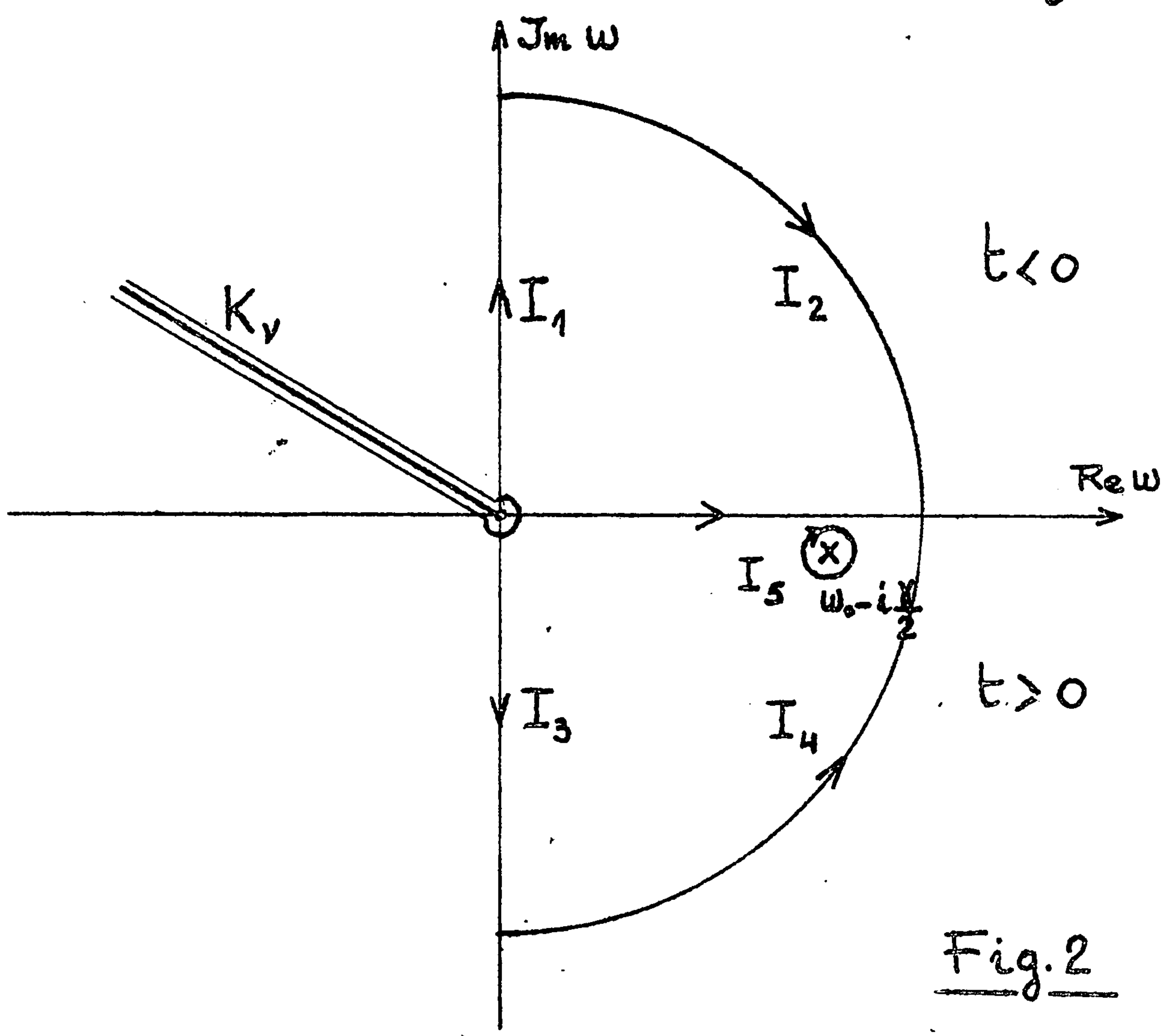
(3) A. BOHR (1948), Det. Kgl. Danske Videnskabernes Selskab, 24, n° 10.

(4) J.G. de VOOGHT - To be submitted for publication.



$\rho(t)$
$z(t)$
ρ_0
e
V
0
z
$z' = -V|t|$
$\rho' = 0$

Fig. 1



$\mathrm{Im}\,\omega$
I_1
I_2
$t < 0$
K_ν
$\mathrm{Re}\,\omega$
I_5
$\omega_0 - i\frac{\gamma}{2}$
I_3
I_4
$t > 0$

Fig. 2

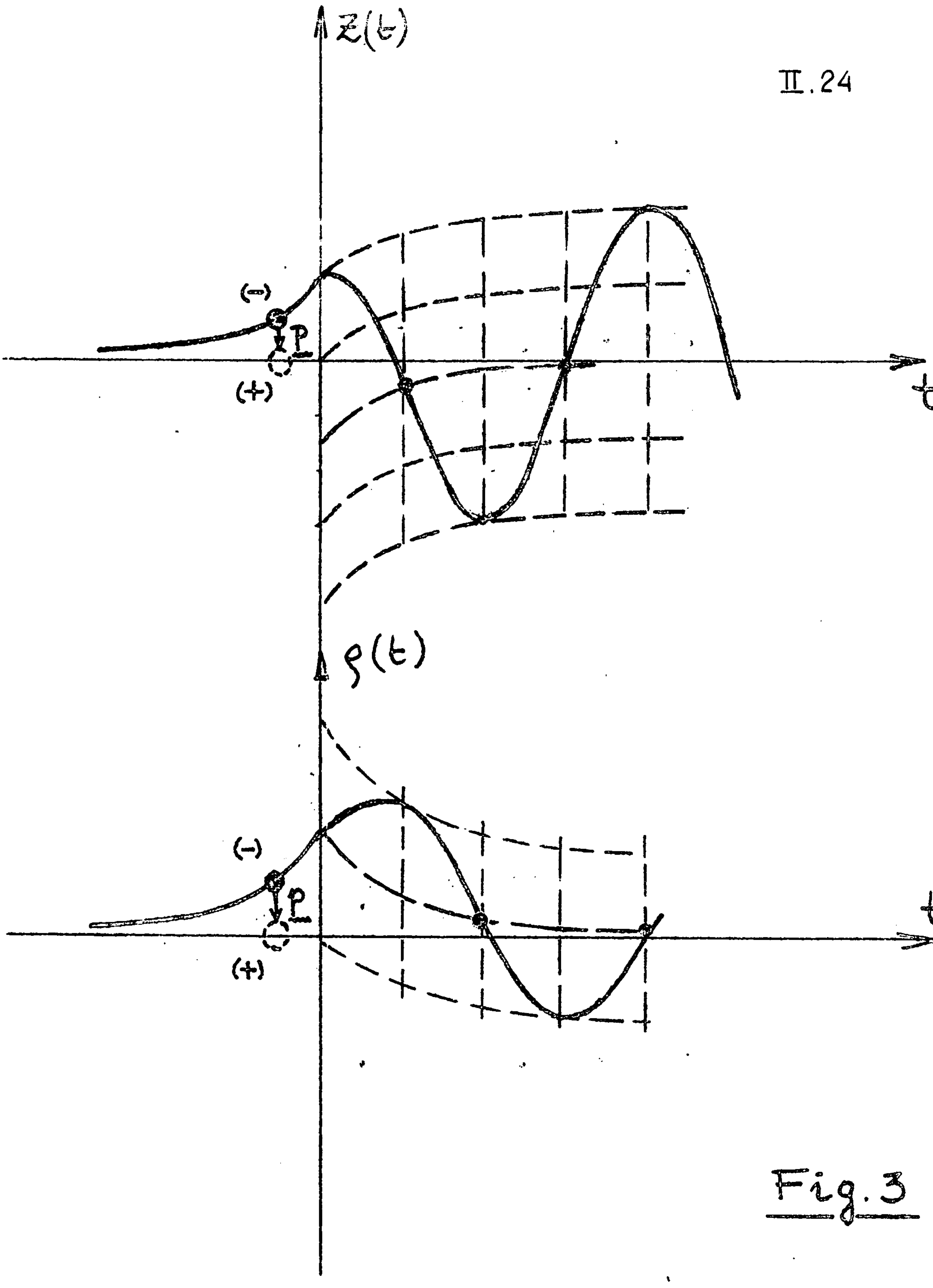


Fig.3

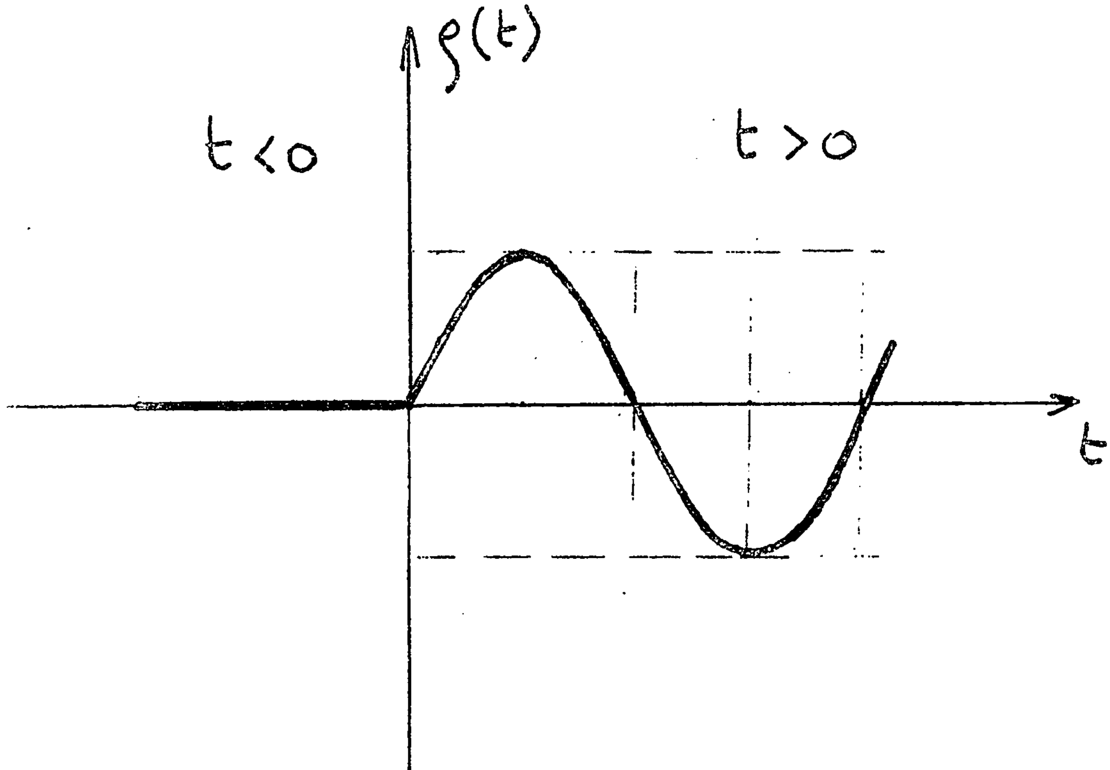


Fig. 4

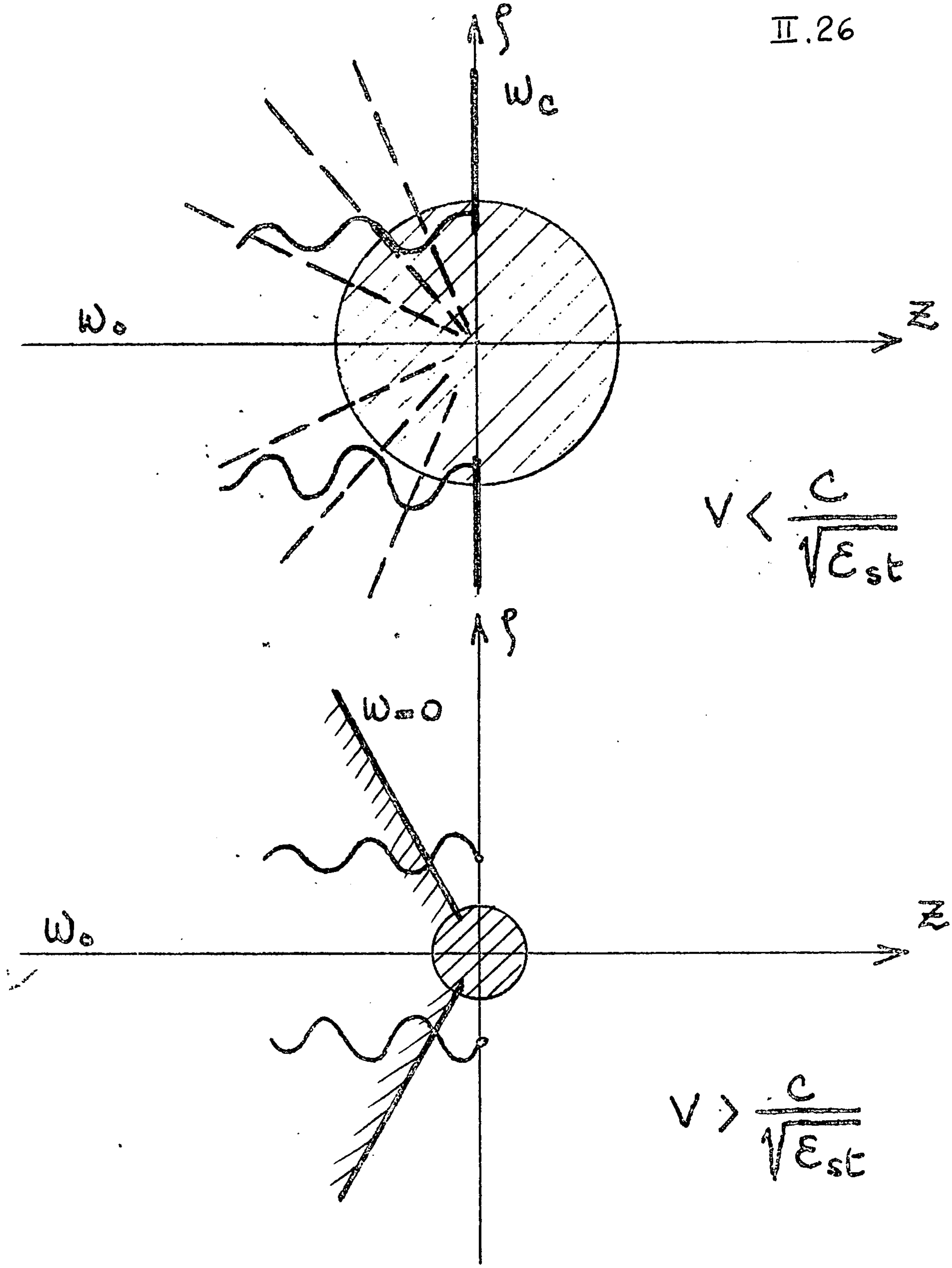

ρ
ω_c
ω_o
z
$v < \frac{c}{\sqrt{\varepsilon_{st}}}$
ρ
$\omega = 0$
ω_o
z
$v > \frac{c}{\sqrt{\varepsilon_{st}}}$

Fig. 5